A NOTE ON VECTOR LATTICES

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1. Definitions and results

Let E be a vector lattice in the sense of Birkhoff [1]. We use the following notations:

$$x^+ = x \cup O, \quad x^- = (-x)^+ \quad \text{and} \quad |x| = x^+ + x^-.$$

A subset I is called *an ideal* if (i) I is a linear subset and (ii) $x \in I$ and $|y| \leqq x$ imply $y \in I$.

An ideal is said to be *maximal* if it is a proper ideal and is not a proper subset of another proper ideal.

E is said to be *semi-simple* if the intersection of all maximal ideals consists of only zero element.

E is said to be *radical* if there exist no maximal ideals.

An element a is said to be *atomic* if

$$|a| = a_1 + a_2 \text{ and } a_1 \cap a_2 = 0 \text{ imply either } a_1 = 0 \text{ or } a_2 = 0.$$

For any $a \in E$, the set

$$I(a) = \{x \in E \mid |x| \cap |a| = 0\}$$

is an ideal. The following theorem can be proved easily.

THEOREM 1. *If the ideal $I(a)$ is maximal, the element a is atomic.*

The converse of this theorem is not true. For example, let us consider the space (C) of all real-valued continuous functions defined on the interval $[0, 1]$. This space (C) is a vector lattice, if we define, for $x(t)$ and $y(t)$ in (C), the vector lattice structure as follows:

(i) $(\alpha x + \beta y)(t) = \alpha x(t) + \beta(y)t$ for every $t \in [0, 1]$;

(ii) $x \geqq y$ if and only if $x(t) \geqq y(t)$ for every $t \in [0, 1]$.

Now, let us take, for example, the following element:

$$a(t) = 0 \quad \text{if} \quad 0 \leqq t \leqq \tfrac{1}{2}; \; = t - \tfrac{1}{2} \quad \text{if} \quad \tfrac{1}{2} < t \leqq 1,$$

then a is atomic and the ideal $I(a)$ is not maximal, because

$$I(a) = \{x \in (C) \mid x(t) = 0 \text{ for } t \in (\tfrac{1}{2}, 1]\} \subset \{x \in (C) \mid x(t) = 0 \text{ for } t \in (\tfrac{2}{3}, 1]\}.$$

(We owe this example to Professor P. Conrad.)

As is well-known, this vector lattice (C) is not conditionally complete. A vector lattice is said to be *conditionally complete* if every subset which is bounded from above has the least upper bound. We can prove the following theorem.

THEOREM 2. *If E is conditionally complete, the ideal $I(a)$ is maximal whenever a is atomic.*

A vector lattice E is said to be *atomic* if the set $A(E)$ of all atomic elements is *dense*: if $x \cap a = 0$ for every $a \in A(E)$, then $x = 0$. E is said to be *non-atomic* if $A(E)$ is empty.

The following theorem follows immediately from Theorem 1 and 2.

THEOREM 3. *If E is conditionally complete,*

1. *E is atomic if and only if the intersection of all closed ideals consists of only zero element.*

2. *E is non-atomic if and only if there are no closed maximal ideals.*

An ideal I is said to be *closed* if, for any increasing set $x_\lambda \in I$ $(\lambda \in \Lambda)$, $x = \bigcup_{\lambda \in \Lambda} x_\lambda$ implies $x \in I$. (cf. [1], p. 232) The ideal $I(a)$ is closed, because, if $x_\lambda \in I(a) (\lambda \in \Lambda)$ and $x = \bigcup_{\lambda \in \Lambda} x_\lambda$, $|x| \cap |a| \leqq \bigcup_{\lambda \in \Lambda} (|x_\lambda| \cap |a|) = 0$. Maximal ideals are not always closed. As an example of vector lattices in which every maximal ideal is closed, we take BK-spaces which have been introduced by [2].

A conditionally complete vector lattice E is said to be a *BK-space* if it is a normed lattice with a norm $||x||$ $(x \in E)$ which satisfies the following two conditions:

(i) $\lim_{n\to\infty} x_n = 0$ *in order convergence implies* $\lim_{n\to\infty} ||x_n|| = 0$;

(ii) *If $\{x_n\}$ is increasing and is not bounded from above then* $\lim_{n\to\infty} ||x_n|| = \infty$.

(The condition (ii) has been studied in detail in [4].) Then, the following theorem, which is the main theorem of this paper, is an easy consequence of Theorem 3.

THEOREM 4. *Let E be a BK-space. Then,*

1. *E is semi-simple if and only if E is atomic.*

2. *E is radical if and only if E is non-atomic.*

Most of the standard function spaces which appear in Functional Analysis are BK-spaces. For example, the sequence space l_p $(p \geqq 1)$ and the function space $L_p[0, 1]$ $(p \geqq 1)$ are BK-spaces, because they are conditionally complete vector lattices under the usual definitions of vector lattice structure and the norms:

$$||x|| = (\sum_{n=1}^{\infty} |x_n|^p)^{1/p} \text{ for } x = (x_n) \in l_p$$

and

$$||x|| = \left(\int_0^1 |x(t)|^p dt\right)^{1/p} \text{ for } x = x(t) \in L_p$$

satisfy the above conditions (i) and (ii). The space l is atomic, because the elements:

$$e_k = (e_k^n) \text{ where } e_k^n = 0 \text{ if } k \neq n \text{ and } e_k^n = 1 \text{ if } k = n$$

are atomic and $e_k \cap |x| = 0$ $(k = 1, 2, \cdots)$ implies $x = 0$. The space L_p is non-atomic, because, since a function which is not zero only on a set of measure zero is regarded as a zero function, every non-zero function can be expressed as the sum of two non-zero functions which are mutually disjoint.

2. Proof of theorem 1

Assume that a is not atomic, then there exist a pair of positive (non-zero) elements a_1 and a_2 such that

$$|a| = a_1 + a_2 \text{ and } a_1 \cap a_2 = 0.$$

Let us consider the ideal I which is generated by $I(a)$ and a_1. Obviously, $I(a)$ is a proper subset of I, because a_1 is not in $I(a)$. Moreover, I is a proper ideal. In fact, if $a_2 \in I$, then

$$a_2 \leqq x + na_1 \text{ for some } x \in I(a) \text{ and integer } n.$$

Since $a_1 \cap a_2 = 0$, we have $a_2 \leqq x$, from which it follows that $a_2 \in I(a)$. This is a contradiction, because $0 < a_2 < |a|$.

3. Proof of theorem 2

The vector lattice E is assumed to be conditionally complete.

LEMMA 1. (Theorem 19, p. 233, [1]) *Let J be a closed ideal and $J^\perp$ be its orthogonal complement*:

$$J^\perp = \{x \in E \mid |x| \cap |y| = 0 \text{ for every } y \in J\}.$$

Then,

1. $(J^\perp)^\perp = J$;

2. *$E = J + J^\perp$, in other words, for any $x \in E$ there exists uniquely a pair of elements $x(J) \in J$ and $x(J^\perp) \in J^\perp$ such that $x = x(J) + x(J^\perp)$.*

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LEMMA 2. *If a is atomic, for the ideal I generated by $I(a)$ and a, we have $I = E$.*

PROOF. Take an arbitrary element b. Without loss of generality, we can assume that a and b are positive. Let us denote by J_n the ideals $I((b-na)^+)^\perp$. Then,

$$na(J_n) \leqq b(J_n) \leqq b \text{ for every } n = 1, 2, \cdots,$$

because

$$\begin{aligned} &b(J_n) - na(J_n) \\ &= (b-na)(J_n) = (b-na)^+ \geqq 0. \end{aligned}$$

Therefore, since $a(J_n) \leqq b/n$ for every n and E is conditionally complete, the sequence $\{a(J_n)\}$ converges to zero in order convergence. On the other hand, since a is atomic and

$$a = a(J_n) + a(J_n^\perp),$$

we have either $a(J_n) = 0$ or $a(J_n^\perp) = 0$. Assume that

$$a(J_n) > 0$$

for an infinite number of n, then, for such n, we have

$$a = a(J_n) \to 0,$$

which is a contradiction. Therefore, there exists n_0 such that $a(J_{n_0}) = 0$, which means that

$$a = a(J_{n_0}^\perp) \in I((b-n_0 a)^+).$$

Now, let us consider the set

$$J(a) = I(a)^\perp = \{x \in E \mid |x| \cap |y| = 0 \text{ for every } y \in I(a)\}.$$

Then, $J(a)$ is a closed ideal and, since

$$b - n_0 a = (b-n_0 a)^+ - (b-n_0 a)^-$$

and

$$(b-n_0 a)^+ \in I(a) = J(a)^\perp,$$

we have $(b-n_0 a)^+(J(a)) = 0$ and

$$\begin{aligned} &b(J(a)) - n_0 a \\ &= b(J(a)) - n_0 a(J(a)) = (b-n_0 a)(J(a)) \\ &= (b-n_0 a)^+(J(a)) - (b-n_0 a)^-(J(a)) \\ &= -(b-n_0 a)^-(J(a)) \leqq 0, \end{aligned}$$

from which it follows that, for the ideal I which is generated by $I(a)$ and a, $b(J(a)) \leqq n_0 a \in I$. Therefore, $b(J(a)) \in I$. Since $b(J(a)^\perp) \in I(a)$, we have

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$$b = b(J(a)) + b(J(a)^\perp) \in I,$$

hence it follows that $I = E$.

Now, assume that $I(a)$ is not maximal, then there exists a proper ideal I such that $I(a)$ is a proper subset of I. Therefore, we can find a positive element b such that $b \in I$ and $b \notin I(a)$. Then, for $J(b) = I(b)^\perp$, since $a = a(J(b)) + a(J(b)^\perp)$ and a is atomic, we have

$$a = a(J(b))$$

because $b \notin I(a)$.

Next, we consider the set B of the elements $b(J(x))$, for $J(x) = I(x)^\perp$, such that

$$|x| \cap |a| = 0 \quad \text{and} \quad J(x) \subset J(b).$$

Since the set B is bounded from above by b, there exists the least upper bound, which is denoted by c. Obviously, c is orthogonal to a. Now, put

$$b_0 = b - b(J(c)) = b(J(c)^\perp),$$

then, we can prove that b is an atomic element.

Assume that

$$b_0 = b_1 + b_2 \quad \text{and} \quad b_1 \cap b_2 = 0,$$

then

$$a(J(b_0)) = a(J(b)) - (a(J(b)))(J(c)) = a(J(b)) = a.$$

Therefore,

$$a = a(J(b_0)) = a(J(b_1)) + a(J(b_2)).$$

Since a is atomic, either $a(J(b_1))$ or $a(J(b_2))$ is zero. Let us assume that $a(J(b_1)) = 0$. Then, since

$$b_1 \cap a = 0 \quad \text{and} \quad J(b_1) \subset J(b),$$

we have $b(J(b_1)) \leqq c$. On the other hand, since

$$b_1 \leqq b_0 = b(J(c)^\perp),$$

we have $b(J(b_1)) = 0$, from which it follows that $b_1 = 0$, because $b \cap b_1 = 0$ and $0 \leqq b_1 \leqq b$.

Finally, since b is atomic, we can prove that $a \in I$ by the same method as in the proof of Lemma 2, if we denote by J_n the closed ideals $I((a - nb_0)^+)^\perp$.

4. Proof of theorem 3

We need the following lemma.

LEMMA 3. *Let E be conditionally complete and I be a closed maximal ideal. Then, there exists an atomic element a such that $I = I(a)$.*

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PROOF. Since I is closed, by Lemma 1, we have $I = (I^{\perp})^{\perp}$. Since I is a proper ideal, $(I^{\perp})^{\perp} \subsetneqq E$, hence it follows that there exists an element $a > 0$ such that $a \in I^{\perp}$. Therefore, $I \subset I(a)$. The maximality of I implies $I = I(a)$ and hence a is atomic by Theorem 1.

Now, let us prove our theorem.

1. From Theorem 2 and Lemma 3, it follows that

$$\bigcap_{a \in A(E)} I(a)$$

is exactly the intersection of all closed maximal ideals. Moreover,

$$x \in \bigcap_{a \in A(E)} I(a)$$

is equivalent to that

$$|x| \cap |a| = 0 \quad \text{for} \quad \text{every} \quad a \in A(E).$$

Therefore, E is atomic if and only if $\bigcap_{a \in A(E)} I(a) = \{0\}$.

2. If there exists a closed maximal ideal, then $A(E) \neq \phi$ by Lemma 3. If $A(E) \neq \phi$, then there exists a maximal ideal by Theorem 2.

5. Proof of theorem 4

We have only to prove that every maximal ideal is closed. Let I be a maximal ideal. Then, by [3], Proposition 2], I is the kernel of a real-valued function $f(x)$ on E which satisfies the following conditions: (i) $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$; (ii) $x \geqq 0$ implies $f(x) \geqq 0$; (iii) $|x| \cap |y| = 0$ implies $f(x)f(y) = 0$. Therefore, f is a positive linear functional on E. By [Theorem 8, p. 245 and Theorem 10, p. 248 [1]], f is a norm-continuous linear functional. Now, assume that $\{x_\lambda \in I \ (\lambda \in \Lambda)\}$ is an increasing set and $x = \bigcup_{\lambda \in \Lambda} x_\lambda$. By the condition (i) in the definition of BK-spaces, we can select a sequence x_{λ_n} $(n = 1, 2, \cdots)$ such that $x = \bigcup_{n=1}^{\infty} x_{\lambda_n}$. Since f is norm-continuous, we have

$$f(x) = \lim_{n \to \infty} f(x_{\lambda_n}) = 0,$$

which means that I is closed.

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