HEREDITARY SEMISIMPLE CLASSES

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It is well-known (see e.g. [**1**, p. 5]) that a class $\mathcal{M}$ of (not necessarily associative) rings is the semisimple class for some radical class, relative to some universal class $\mathcal{W}$ if and only if it has the following properties:

(*a*) If $R \in \mathcal{M}$, then every non-zero ideal I of R has a non-zero homomorphic image $I/J \in \mathcal{M}$.

(*b*) If $R \in \mathcal{W}$ but $R \notin \mathcal{M}$, then R has a non-zero ideal $I \in \mathcal{U}\mathcal{M}$, where $\mathcal{U}\mathcal{M} = \{K \in \mathcal{W} \mid$ every non-zero $K/H \notin \mathcal{M}\}$. In fact $\mathcal{U}\mathcal{M}$ is the radical class whose semisimple class is $\mathcal{M}$. On the other hand, if $\mathcal{P}$ is a radical class, then $\mathcal{S}\mathcal{P} = \{K \in \mathcal{W} \mid$ if I is a non-zero ideal of K, then $I \notin \mathcal{P}\}$ is its semisimple class. If a class $\mathcal{M}$ is hereditary (that is, when $R \in \mathcal{M}$, then all its ideals are in $\mathcal{M}$), it clearly satisfies (*a*), but there do exist non-hereditary semisimple classes (see [**2**]). The condition (satisfied in all associative or alternative classes) is that $\mathcal{S}\mathcal{P}$ is hereditary for a radical class $\mathcal{P}$ if and only if $\mathcal{P}(I) \subseteq \mathcal{P}(R)$ for all ideals I of all rings $R \in \mathcal{W}$ [**3**, Lemma 2, p. 595].

It is also well-known (see e.g. [**1**, pp. 6–7]) that, if $\mathcal{M}$ satisfies only condition (*a*), then $\mathcal{U}\mathcal{M}$ is a radical class such that $\mathcal{S}\mathcal{U}\mathcal{M}$ is the unique minimal semisimple class containing $\mathcal{M}$. However, for an arbitrary class $\mathcal{M}$ we have:

THEOREM 1. (i) *Any subclass $\mathcal{M}$ of a universal class $\mathcal{W}$ of associative or alternative rings is included in a unique minimal semisimple class.*

(ii) *In the class $\mathcal{W}$ of all not necessarily associative rings there exists a class $\mathcal{M}$ contained in two incomparable minimal semisimple classes.*

Proof. (i) Let $\mathcal{I}\mathcal{M}$ be the hereditary closure of $\mathcal{M}$ [**2**, p. 1114], so that $\mathcal{M} \subseteq \mathcal{I}\mathcal{M} \subseteq \mathcal{S}\mathcal{U}\mathcal{I}\mathcal{M}$. If $\mathcal{M} \subseteq \bar{\mathcal{M}}$, where $\bar{\mathcal{M}}$ is semisimple in some universal class of associative or alternative rings, then $\bar{\mathcal{M}}$ is hereditary [**3**, Corollary 2, p. 597 and Corollary 2, p. 602]; so $\mathcal{I}\mathcal{M} \subseteq \mathcal{I}\bar{\mathcal{M}} = \bar{\mathcal{M}}$. Thus $\mathcal{S}\mathcal{U}\mathcal{I}\mathcal{M} \subseteq \mathcal{S}\mathcal{U}\bar{\mathcal{M}} = \bar{\mathcal{M}}$.

(ii) Call a class $\mathcal{M}' \supseteq \mathcal{M}$ an *s*-completion of $\mathcal{M}$ if $\mathcal{M}'$ satisfies condition (*a*). Now let R be a ring with exactly one proper ideal I, where I in turn has exactly one proper ideal J, such that J is a simple ring chosen so that $K = I/J \not\cong J$. Two examples of such rings are given in [**2**, p. 1116]. In fact, such a ring can be constructed by the method of [**4**] for any simple J which is an algebra over $Z/(2)$ (or, by a similar construction, over any field). Then in the class $\mathcal{W}$ of all not necessarily associative rings the class $\mathcal{M} = \{O, R\}$ has two incomparable *s*-completions $\mathcal{I}\mathcal{M} = \{O, R, I, J\}$ and $\mathcal{M}' = \{O, R, K\}$. The corresponding semisimple classes are also incomparable, for clearly $J \in \mathcal{U}\mathcal{M}'$ and so $J \notin \mathcal{S}\mathcal{U}\mathcal{M}'$. Similarly $K \notin \mathcal{S}\mathcal{U}\mathcal{I}\mathcal{M}$.

To establish the minimality of $\mathcal{S}\mathcal{U}\mathcal{M}'$, let $\mathcal{M} \subseteq \bar{\mathcal{M}} \subseteq \mathcal{S}\mathcal{U}\mathcal{M}'$, where $\bar{\mathcal{M}}$ is semisimple in $\mathcal{W}$. If $\mathcal{M}' \subseteq \bar{\mathcal{M}}$, then $\mathcal{S}\mathcal{U}\mathcal{M}' \subseteq \mathcal{S}\mathcal{U}\bar{\mathcal{M}} = \bar{\mathcal{M}}$ and hence we have equality. Thus suppose that $\mathcal{M}' \nsubseteq \bar{\mathcal{M}}$. Then, since $\mathcal{M} \subseteq \bar{\mathcal{M}}$, it follows that $K \notin \bar{\mathcal{M}}$. But $\bar{\mathcal{M}}$ has property (*a*); so we would have $\mathscr{I}\mathscr{M} \subseteq \bar{\mathscr{M}}$, contradicting $J \notin \mathscr{S}\mathscr{U}\mathscr{M}'$. We conclude that $\mathscr{S}\mathscr{U}\mathscr{M}' = \bar{\mathscr{M}}$. Similarly, let $\mathscr{M} \subseteq \bar{\mathscr{M}} \subseteq \mathscr{S}\mathscr{U}\mathscr{I}\mathscr{M}$. If $\mathscr{I}\mathscr{M} \nsubseteq \bar{\mathscr{M}}$, then clearly $I \notin \bar{\mathscr{M}}$. We would thus have $K \in \bar{\mathscr{M}}$, contradicting $K \notin \mathscr{S}\mathscr{U}\mathscr{I}\mathscr{M}$. Thus again $\mathscr{S}\mathscr{U}\mathscr{I}\mathscr{M}$ is minimal.

It is an open question as to whether all classes have minimal s-completions, or whether the existence of such a completion implies the minimality of its corresponding semisimple class. It is considered likely that the answer to both questions is negative. However, the situation is different if the semisimple class is required to be hereditary:

THEOREM 2. *In any universal class $\mathscr{W}$ every subclass $\mathscr{M}$ is contained in a unique minimal hereditary semisimple class.*

Proof. Let $\mathscr{H}_1 = \mathscr{I}\mathscr{M}$ and, for β an arbitrary limit ordinal, let $\mathscr{H}_\beta = \bigcup_{\alpha<\beta} \mathscr{H}_\alpha$; otherwise let $\mathscr{H}_\beta = \mathscr{I}\mathscr{S}\mathscr{U}\mathscr{H}_{\beta-1}$. Then define $\mathscr{H} = \bigcup \mathscr{H}_\beta$ taken over all ordinals β. It is clear by induction that all $\mathscr{H}_\beta$ are hereditary; so $\mathscr{H}$ is a hereditary class. Since any $\mathscr{H}_\alpha \subseteq \mathscr{S}\mathscr{U}\mathscr{H}_\alpha$, it is also easy to show by induction that $\alpha < \beta$ implies $\mathscr{H}_\alpha \subseteq \mathscr{H}_\beta$. We already have $\mathscr{H} \subseteq \mathscr{S}\mathscr{U}\mathscr{H}$; so to show equality let $R \in \mathscr{S}\mathscr{U}\mathscr{H}$. Then every non-zero ideal I of R has a non-zero image $I/J \in \mathscr{H}$. Thus $I/J \in \mathscr{H}_\alpha$ for some α and, since the collection of all ideals of R is a set, there must be a largest such α, say γ. Then all $I/J \in \mathscr{H}_\gamma$ and so $R \in \mathscr{S}\mathscr{U}\mathscr{H}_\gamma$. Thus $R \in \mathscr{I}\mathscr{S}\mathscr{U}\mathscr{H}_\gamma = \mathscr{H}_{\gamma+1}$ and hence $R \in \mathscr{H}$. It follows that $\mathscr{H} = \mathscr{S}\mathscr{U}\mathscr{H}$ and so $\mathscr{H}$ is a hereditary semisimple class.

If $\mathscr{M} \subseteq \bar{\mathscr{M}}$, where $\bar{\mathscr{M}}$ is some hereditary semisimple class, then $\mathscr{H}_1 = \mathscr{I}\mathscr{M} \subseteq \mathscr{I}\bar{\mathscr{M}} = \bar{\mathscr{M}}$. Thus suppose that, for a given ordinal β, we have $\mathscr{H}_\alpha \subseteq \bar{\mathscr{M}}$ for all $\alpha < \beta$. Then $\mathscr{H}_\beta \subseteq \bar{\mathscr{M}}$ when β is a limit ordinal, or else from $\mathscr{H}_{\beta-1} \subseteq \bar{\mathscr{M}}$ it follows that $\mathscr{H}_\beta = \mathscr{I}\mathscr{S}\mathscr{U}\mathscr{H}_{\beta-1} \subseteq \mathscr{I}\mathscr{S}\mathscr{U}\bar{\mathscr{M}} = \mathscr{I}\bar{\mathscr{M}} = \bar{\mathscr{M}}$. By induction, $\mathscr{H} \subseteq \bar{\mathscr{M}}$ and thus $\mathscr{H}$ is the unique minimal hereditary semisimple class containing $\bar{\mathscr{M}}$.

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