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A comparison theorem for functional differential equations

Athanassios G. Kartsatos and Hiroshi Onose

In this paper, we study the oscillation of nth-order differential equations. Recently, Atkinson and the present authors studied (separately) the comparison properties of differential inequalities. Kartsatos treated the nth-order ordinary case and proposed several open problems.

The purpose of this paper is to answer one of them in the affirmative concerning more general functional differential equations. The result is that if under several conditions, the equation

$$x^{(n)}(t) + H_1\big(t, x(g(t))\big) = Q(t) \tag{1}$$

is oscillatory for n even or a solution $x(t)$ of (1) is oscillatory or $\liminf_{t\to\infty} x(t) = 0$ for n odd, then this is also the case for the equation

$$x^{(n)}(t) + H_2\big(t, x(g(t))\big) = Q(t) . \tag{2}$$

1. Introduction

Recently, Atkinson [1] and the present authors [2], [3], [5] studied the comparison properties of differential inequalities. Kartsatos [2], [3], treated the nth-order ordinary case and proposed several open problems.

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The purpose of this paper is to answer one of them in the affirmative concerning more general functional differential equations.

Consider the functional differential equations

$$(*) \qquad x^{(n)}(t) + H_i\big(t, x\big(g(t)\big)\big) = Q(t) \quad (i = 1, 2) .$$

In what follows, $R = (-\infty, \infty)$, $R_+ = [0, \infty)$. The functions $H_i(t, x)$ $(i = 1, 2)$ will be defined and continuous on $R_+ \times R$ with values in R . By a solution of equation (*), we mean any function $x \in C^n[t_x, \infty)$, which satisfies (*) for all $t \in [t_x, \infty)$. Here t_x depends on the solution $x(t)$. A solution $x(t)$ is said to be oscillatory if it has an unbounded set of zeros in its interval of definition $[t_x, \infty)$. If all solutions of (*) are oscillatory, then equation (*) is said to be oscillatory.

2. The result

THEOREM. *Let the functions* $H_i(t, u)$, $i = 1, 2$ *be defined on* $R_+ \times R$, *increasing with respect to* u , *and such that* $uH_i(t, u) > 0$ *for* $u \neq 0$. *Let* $P \in C^n[R_+, R]$, $P^{(n)}(t) \equiv Q(t)$ *for every* $t \in R_+$, *and* $\lim_{t\to\infty} P(t) = 0$. *Let the function* $g(t)$ *be continuous on* R_+ *and such that* $g(t) \leq t$, $\lim_{t\to\infty} g(t) = \infty$. *Then if* $P(t)$ *is oscillatory and*

$$H_1(t, u) \leq H_2(t, u) , \quad t \in R_+ , \quad u \geq 0 ,$$

$$H_1(t, u) \geq H_2(t, u) , \quad t \in R_+ , \quad u < 0 ,$$

and the equation

$$(1) \qquad x^{(n)}(t) + H_1\big(t, x\big(g(t)\big)\big) = Q(t)$$

is oscillatory for n *even or a solution* $x(t)$ *of* (1) *is oscillatory or* $\liminf_{t\to\infty} x(t) = 0$ *for* n *odd, this is also the case for the equation*

$$(2) \qquad x^{(n)}(t) + H_2\big(t, x\big(g(t)\big)\big) = Q(t) .$$

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Proof. Let (2) be nonoscillatory for n even and any solution $z(t)$ of (2) be nonoscillatory and $\liminf_{t\to\infty} |z(t)| > 0$ for n odd. Assume that a solution $z(t)$ of equation (2) is positive for $t \geq T \geq t_z$. Then the function $u(t) \equiv z(t) - P(t)$ is an eventually positive solution of the equation

$$(3) \qquad u^{(n)}(t) + H_2\big(t, u(g(t))+P(g(t))\big) = 0 .$$

In fact, $u(g(t)) + P(g(t)) > 0$ on $[T_1, \infty)$ with $T_1 \geq T$, implies $u^{(n)} < 0$ on $[T_1, \infty)$. Consequently, $u(t)$ has to be eventually of constant sign. If $u(t) < 0$ for all large t, then $P(t) > -u(t) > 0$ for all large t, a contradiction to the oscillatory character of $P(t)$. Let $u(t) > 0$ eventually. By $u^{(n)}(t) < 0$, $u(t) > 0$ and Kiguradge's Lemma [4], there exists an odd (even) integer with $0 \leq l \leq n-1$ for n even (odd) such that

$$u^{(i)}(t) > 0 , \quad i = 0, 1, \ldots, l ,$$

$$(-1)^{l+i}u^{(i)}(t) \geq 0 , \quad i = l+1, \ldots, n , \text{ for } t \geq T_1 .$$

Thus, in particular, $u(t) > 0$, $u'(t) > 0$ for every $t \geq T_1$ if n is even or odd, or, possibly, for n odd, $u(t) > 0$, $u'(t) < 0$ for every $t \geq T_1$. Let now T_1 be so large that we also have $|P(t)| < c < u(T_1)$ for all $t \geq T_1$, where we can take c be a positive constant. Then we obtain

$$u^{(n)}(t) + H_1\big(t, u(g(t))+P(g(t))\big) \leq u^{(n)}(t) + H_2\big(t, u(g(t))+P(g(t))\big) = 0$$

for every $t \geq T_1$.

Notice that $u(g(t)) + P(g(t)) > 0$ for $t \geq T_1$. Consequently, the inequality

$$(4) \qquad u^{(n)}(t) + H_1\big(t, u(g(t))+P(g(t))\big) \leq 0$$

has a solution $u(t)$ with the property that $u(t) > 0$, $u'(t) > 0$ (or

$u'(t) < 0$ in some odd case). By repeated integration of (4), we obtain

$$(5)\quad u(t) \geq u(T_1) + \int_{T_1}^{t} \int_{T_1}^{s_{n-1}} \cdots \int_{T_1}^{s_{n-l+1}} \int_{s_{n-l}}^{\infty} \cdots \int_{s_1}^{\infty} H_1(t, u(g(t))+P(g(t)))\,dt\,ds_1 \cdots ds_{n-1}$$

$$\geq c + \Psi(t, u(g(t))+P(g(t))) , \quad (t \geq T_1),$$

where $c = u(T_1)$ in case $u'(t) > 0$ and $c = u(T_1)/2$ in case $u'(t) < 0$.

Now it is easy to show the existence of a positive solution to the integral equation

$$(6)\qquad v(t) = c + \Psi(t, v(g(t))+P(g(t))) , \quad t \geq T_1 .$$

We define $v_n(t)$, $n = 0, 1, \ldots$ such that

$$v_0(t) = u(t) \qquad \text{for} \quad t \geq T ,$$

$$v_{n+1}(t) = \begin{cases} c + \Psi(t, v_n+P) & \text{for} \quad t \geq T_1 \\ c & \text{for} \quad T \leq t \leq T_1 . \end{cases}$$

Then we see that $v_n(t)$ is well-defined and

$$(7)\qquad 0 < v_n(t) < u(t) , \quad c \leq v_{n+1}(t) \leq v_n(t) .$$

If we put

$$(8)\qquad v(t) = \lim_{t\to\infty} v_n(t) \quad \text{for every point} \quad t \geq T_1 ,$$

then by (7), (8), and Lebesgue's Theorem we have

$$v(t) = c + \Psi(t, v(g(t))+P(g(t))) , \text{ for all } t \geq T_1 .$$

If we differentiate (6) n times, we obtain

$$(9)\qquad r^{(n)}(t) + H_1(t, r(g(t))) = Q(t) , \quad t \geq T_1 .$$

Since $v(t) + P(t) > c + P(t) > 0$, (9) has an eventually positive solution or, for n odd, $\liminf_{t\to\infty} r(t) > 0$, a contradiction.

An analogous proof can be given if we start with an eventually negative solution of (2). This completes the proof.

REMARK. The above theorem is not covered by the results of Onose [5].

References

[1] F.V. Atkinson, "On second order differential inequalities", *Proc. Roy. Soc. Edinburgh Sect. A* 72 (1972/1974), 102-127 (1975).

[2] Athanassios G. Kartsatos, "On nth-order differential inequalities", *J. Math. Anal. Appl.* 52 (1975), 1-9.

[3] Athanassios G. Kartsatos, "Recent results on oscillation of solutions of forced and perturbed nonlinear differential equations of even order", *Stability of dynamical systems* (NSF-CMBS Conference, Mississippi State University, 1975, to appear).

[4] И.Т. Кигурадзе [I.T. Kiguradze], "О колеблемости решений уравнения $\frac{d^m u}{dt^m} + a(t)|u|^n \operatorname{sign} u = 0$ " [On the oscillation of solutions of the equation $\frac{d^m u}{dt^m} + a(t)|u|^n \operatorname{sign} u = 0$], *Mat. Sb.* 65 (107) (1964), 172-187.

[5] Hiroshi Onose, "A comparison theorem and the forced oscillation", *Bull. Austral. Math. Soc.* 13 (1975), 13-19.

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