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ON COMPLETELY POSITIVE MAPS DEFINED BY AN IRREDUCIBLE CORRESPONDENCE

BY

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ABSTRACT. Completely positive maps defined by an irreducible correspondence between two von Neumann algebras M and N are introduced. We give results about their structure and characterize, among them, those which are extreme points in the convex set of all unital completely positive maps from M to N. As particular cases we obtain known results of M. D. Choi [4] on completely positive maps between complex matrices and of J. A. Mingo [8] on inner completely positive maps.

In [4], Choi has described the structure of completely positive linear maps $\Phi: M_m(\mathbf{C}) \longrightarrow M_n(\mathbf{C})$ where, for $k \geq 1$, $M_k(\mathbf{C})$ is the algebra of $k \times k$ complex matrices: there exists an essentially unique set of independent $m \times n$ matrices $v_1, \ldots, v_l$ such that $\Phi(x) = \sum_{i=1}^{l} v_i^* x v_i$ for all $x \in M_m(\mathbf{C})$. On the other hand, Mingo has studied in [8] the completely positive maps Φ from a von Neumann algebra M into itself such that there exists a family $(a_i)_{i \in I}$ in M with $\Phi(x) = \sum_{i \in I} a_i^* x a_i$ for all $x \in M$, where the series converges σ-weakly. Such completely positive maps are called inner. When M is a factor, Mingo has noticed that, although the family $(a_i)_{i \in I}$ is not uniquely determined by Φ, the dimension of the linear span of $\{ a_i, i \in I\}$ only depends on Φ; furthermore there is an independent family $(b_j)_{j \in J}$ with $\Phi(x) = \sum_{j \in J} b_j^* x b_j$ for all $x \in M$.

In this paper, we show that completely positive maps between complex matrix algebras, and inner completely positive maps from a factor into itself are particular cases of completely positive maps defined by an irreducible correspondence between two von Neumann algebras M and N (Def. 1) or, in other words, completely positive maps of multiplicity k (Def. 2). Then we describe the structure of these maps (Th. 1 and 2), and characterize among them those which are extreme points of the convex set of all unital completely positive maps from M into N (Th. 4). Thus we obtain another proof of Choi's results, which are at the same time extended to several other situations (see examples 1, 2, 3), and in particular to inner completely positive maps of a factor.

Throughout this paper M and N are von Neumann algebras. We assume that the reader is acquainted with the notions of correspondence from M to N [5] and of Hilbert N-module [9]. Recall that a *correspondence* from M to N is a Hilbert space H with a pair of commuting normal representations π_M and π_{N^o} of M and N^o (the opposite of N) respectively. As usually, the triple (H, π_M, π_{N^o}) will be simply denoted by H and for $x \in M$, $y \in N$ and $h \in H$, we shall write xhy instead of $\pi_M(x)\pi_{N^o}(y^o)h$. The correspondence H is said to be *irreducible* when the set of operators on H commuting with $\pi_M(M)$ and

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$\pi_{N^o}(N^o)$ is reduced to the scalar ones. The standard form [7] of M (unique up to isomorphism) yields a correspondence from M to M, denoted by $L^2(M)$, and called the identity correspondence of M.

The N-valued inner product of a selfdual (right) Hilbert N-module X [9] is denoted by $\langle\ ,\ \rangle$, and is supposed to be conjugate linear in the first variable. The von Neumann algebra of all N-linear bounded operators from X to X will be denoted by $\mathcal{L}_N(X)$ (or $\mathcal{L}(X)$ in the classical case where $N = \mathbf{C}$).

Given a correspondence H from M to N, the space $\mathrm{Hom}_{N^o}(L^2(N), H)$ of bounded linear operators from $L^2(N)$ into H which commute with the actions of N^o on $L^2(N)$ and H will be denoted by X_H .Then X_H, gifted with a right action of N by composition of operators and with a N-valued inner product by

$$\langle x, y\rangle = x^*y \qquad \text{for } x, y \in X_H,$$

is a selfdual Hilbert N-module ([12], Th. 6.5). Moreover, again by composition of operators, we define a normal homomorphism π from M into $\mathcal{L}_N(X_H)$. We get back the correspondence H from (X_H, π) (up to unitary equivalence) in the following way: H is the Hilbert space $X_H \otimes_N L^2(N)$ obtained by inducing the standard representation of N up to M via X_H ([11], Th. 5.1); the induced representation of M into $X_H \otimes_N L^2(N)$ gives the left action, and the right action of N is the one defined by

$$(\xi \otimes h)y = \xi \otimes hy \qquad \text{for } \xi \in X_H, h \in L^2(N), y \in N.$$

In this way, correspondences from M to N may be identified, up to unitary equivalence, to pairs (X, π) formed by selfdual Hilbert N-modules and normal representations π of M on X, i.e. normal homomorphisms $\pi: M \longrightarrow \mathcal{L}_N(X)$ (see [3], Th. 2.2).

Let H be a correspondence from M to N and I a set of indices. The Hilbert tensor product $l^2(I) \otimes H$ has, in an obvious way, a structure of correspondence from M to N, called a *multiple* of H. Its Hilbert N-module version is the right N-module of all I-uples $(\xi_i)_{i\in I}$ in X_H such that $\sum_{i\in I}\langle \xi_i, \xi_i\rangle$ is σ-weakly convergent, provided with the obvious left action of M. This selfdual Hilbert N-module (see [9], p. 458) is denoted $l^2(I) \otimes X_H$.

As explained by Paschke in ([9],§5) the notion of normal completely positive map from M to N has narrow connections with that of pair (X, π) where π is a normal representation of M on a selfdual Hilbert N-module X (and thus with the concept of correspondence from M to N). Let $\Phi: M \longrightarrow N$ be a normal completely positive map. We denote by X_Φ the selfdual Hilbert N-module that comes from Φ by the Stinespring construction. Recall that X_Φ is obtained by separation and selfdual completion (see [9], Th. 3.2) of the right N-module $M \odot N$ (algebraic tensor product) gifted with the N-valued inner product

$$\langle x \otimes y, x_1 \otimes y_1\rangle = y^*\Phi(x^*x_1)y_1.$$

We define a normal representation $\pi_\Phi: M \longrightarrow \mathcal{L}_N(X_\Phi)$ by

$$\pi_\Phi(m)(x \otimes y) = mx \otimes y, \qquad \text{for } m, x \in M \text{ and } y \in N.$$

Denoting by ξ_Φ the class of $1 \otimes 1$ in X_Φ we have

$$\Phi(x) = \langle \xi_\Phi, \pi_\Phi(x)\xi_\Phi\rangle \qquad \text{for } x \in M.$$

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Note that ξ_Φ is a cyclic vector for (X_Φ, π_Φ), which means that X_Φ is the selfdual Hilbert N-submodule of X_Φ generated by $\pi_\Phi(M)\xi_\Phi$. The associated correspondence H_Φ (defined up to unitary equivalence) may be described in the following way. The algebraic tensor product $M \odot L^2(N)$ is endowed with the inner product

$$\langle x \otimes h, y \otimes k \rangle = \langle h, \Phi(x^* y)k \rangle$$

for $x, y \in M$ and $h, k \in L^2(N)$, and this defines H_Φ by separation and completion. The right N-action and the left M-action are given by

$$m(x \otimes h)n = mx \otimes hn \qquad \text{for } m, x \in M, n \in N, h \in L^2(N).$$

Now let π be a normal representation of M on a selfdual Hilbert N-module X. Then for $\xi \in X$, the map $\Phi: x \longrightarrow \langle \xi, \pi(x)\xi \rangle$ is normal and completely positive. We say that Φ is a *coefficient* of (X, π).

DEFINITION 1. Let H be a correspondence from M to N. The coefficient associated to $a \in X_H = \mathrm{Hom}_{N^o}(L^2(N), H)$ is $x \longrightarrow a^* x a$. Take now $(a_i)_{i\in I} \in l^2(I) \otimes X_H$. The corresponding coefficient is

$$x \longrightarrow \sum_{i \in I} a_i^* x a_i = \langle (a_i)_{i\in I}, x(a_i)_{i\in I} \rangle .$$

We say that such *normal completely positive maps are defined by H*; they form a convex set denoted by $CP_H(M,N)$. Remark that $CP_H(M,N)$ does not depend on the particular choice of $L^2(N)$, and only depends on the unitary equivalence class of H.

EXAMPLE 1. Let p be a projection in M and take for N the reduced von Neumann algebra M_p. Denote by Ψ the completely positive map $x \longrightarrow pxp$ from M into N. Then $X_\Psi = Mp = \{mp, m \in M\}$ with its obvious structures of left M-module and right M_p-module, and inner product given by $(a, b) \longrightarrow a^* b$. If H denotes the associated correspondence from M to N, notice that $CP_H(M,N)$ is the set of completely positive maps of the form $x \longrightarrow \sum_{i\in I} p a_i^* x a_i p$, where $(a_i)_{i\in I}$ is a family in M such that $\sum_{i\in I} p a_i^* a_i p$ converges σ-weakly. The correspondence H is irreducible if and only if M_p is a factor. Remark also that in the particular case where p is the unit element of M, $CP_H(M,N)$ is the set of all inner completely positive maps from M to N, in Mingo's terminology [8].

Suppose that M is a factor and that M_p is an injective von Neumann algebra. Since π_Ψ is faithful, it follows from ([1], prop.5.2 and Th. 2.6) that every completely positive map $\Phi: M \longrightarrow N$ can be approximately factored by Ψ. This means that Φ can be approached in the topology of σ-weak pointwise convergence by completely positive maps of the form $x \longrightarrow \sum_{i=1}^n p a_i^* x a_i p$ with $a_1, \ldots, a_n \in M$. However it is not true that we can always find a family $(b_j)_{j\in J}$ with $\Phi(x) = \sum_{j\in J} p b_j^* x b_j p$ for all $x \in M$. For instance, Mingo has proved [8] that an automorphism Θ of a von Neumann algebra M such that there exists a family $(b_j)_{j\in J}$ with $\Theta(x) = \sum_{j\in J} b_j^* x b_j$ for all $x \in M$ is inner. So a counterexample is given in choosing an outer automorphism of an injective factor, p being equal to 1.

EXAMPLE 2. Let K_1, K_2 be two Hilbert spaces and take $M = \mathcal{L}(K_1)$, $N = \mathcal{L}(K_2)$. Denote by $\overline{K}_2$ the Hilbert space conjugate to K_2 and put $H = K_1 \otimes \overline{K}_2$. We define on H a structure of correspondence from M to N, which is irreducible, by

$$x(h_1 \otimes \bar{h}_2)y = xh_1 \otimes \overline{y^* h_2}$$

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for $x \in M, y \in N, h_1 \in \mathcal{L}(K_1), h_2 \in \mathcal{L}(K_2)$ (where $\bar{h}_2$ is the vector h_2 as an element of $\bar{K}_2$). As $L^2(N) = K_2 \otimes \bar{K}_2$ with it obvious structure of N-bimodule, we see at once that X_H is the space $\mathcal{L}(K_2, K_1)$ of all bounded operators from K_2 into K_1, the identification between $\mathcal{L}(K_2, K_1)$ and $\mathrm{Hom}_{N^o}(L^2(N), H)$ being given by $x \longrightarrow x \otimes 1_{\bar{K}_2}$. In this case $CP_H(M,N)$ is the space of all the completely positive maps $\Phi: M \longrightarrow N$ for which there exists a family $(a_i)_{i\in I}$ of elements in $\mathcal{L}(K_2, K_1)$ with $\Phi(x) = \sum_{i\in I} a_i^* x a_i$ for all $x \in M$. In fact, a well known consequence of the Stinespring dilation theorem shows that every normal completely positive map $\Psi: M \longrightarrow N$ belongs to $CP_H(M,N)$. We recall here the proof. There is a normal representation ρ of M in a Hilbert space K and an isometry v from K_2 into K such that $\Psi(x) = v^*\rho(x)v$ for all $x \in M$. As ρ is a normal unital homomorphism from $\mathcal{L}(K_1)$ into $\mathcal{L}(K)$, we may write K as a tensor product $K_1 \otimes R$ and identify ρ to the amplification $x \longrightarrow x \otimes 1_R$. Let $(e_i)_{i\in I}$ be an orthonormal basis of R. Denote by p_i the orthogonal projection from K onto $K_1 \otimes e_i$ and put $a_i = p_i v$. Then $(a_i)_{i\in I}$ is a family of elements in $\mathcal{L}(K_2, K_1)$ with $\Psi(x) = \sum_{i\in I} a_i^* x a_i$ for all $x \in M$.

EXAMPLE 3. Let N be a von Neumann subalgebra of M such that there exists a faithful normal conditional expectation E from M onto N. Choose a faithful normal semifinite weight φ on N and put $\Psi = \varphi \circ E$. We consider the identity correspondences $L^2(N)$ and $L^2(M)$ of N and M respectively, determined by those choices of weights. Denote by H the correspondence from M to N obtained by restricting to N on the right the identity correspondence $L^2(M)$ of M. Note that H is irreducible if and only if $N' \cap M = \mathbf{C}$. We may identify M to a subspace of $X_H = \mathrm{Hom}_{N^o}(L^2(N), L^2(M))$ by considering $m \in M$ as the operator $h \longrightarrow mh$ from $L^2(N)$ into $L^2(M)$. The N-valued inner product on M induced by the one on X_H is given by $\langle a, b\rangle = E(a^*b)$ for $a, b \in M$, and it is easily checked that X_H is the selfdual completion of the right N-module M gifted with this inner product.

Any completely positive map $\Phi: M \longrightarrow N$ of the form $x \longrightarrow \sum_{i\in I} E(a_i^* x a_i)$, where $(a_i)_{i\in I}$ is a family of elements in M such that $\sum_{i\in I} E(a_i^* a_i)$ converges σ-weakly, belongs to $CP_H(M,N)$. In fact Φ is the coefficient of $l^2(I) \otimes X_H$ defined by $(a_i)_{i\in I}$ after having identified M to a subset of X_H.

In the following , **we fix an irreducible correspondence** H **from** M **to** N.

LEMMA 1.1. *Let I be a set and $(a_i)_{i\in I} \in l^2(I) \otimes X_H$. Denote by K the closed subspace of $l^2(I) \otimes H$ generated by $\{(ma_i h)_{i\in I}, m \in M, h \in L^2(N)\}$. The orthogonal projection onto K has the form $p \otimes 1$ with $p \in \mathcal{L}(l^2(I))$, and the rank of p is equal to the dimension of the linear span of $\{a_i, i \in I\}$ in X_H. If p is the identity, then $(a_i)_{i\in I}$ is a family of linearly independent vectors; the converse is true when I is finite.*

Proof. Since K is invariant by the left action of M and the right action of N , and since H is irreducible we see that the projection onto K belongs to $\mathcal{L}(l^2(I)) \otimes 1$. Denote by $r \in \mathbf{N} \cup \{\infty\}$ the dimension of the linear span of $\{a_i, i \in I\}$. Suppose that r is finite and that $a_{i_1}, \dots, a_{i_r}$ are independent. Then we have

$$(1) \qquad a_i = \sum_{k=1}^{r} \lambda_i^k a_{i_k} \quad \text{for } i \notin \{i_1, \dots, i_r\}$$

and the range of $p \otimes 1$ is contained in the set of all $(\eta_i)_{i\in I} \in l^2(I) \otimes H$ which satisfy (1). It follows that rank $p \leq r$.

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Conversely suppose that p has a finite rank s and let $\xi^1, \ldots, \xi^s$ be a basis of the range of p. We write $\xi^j = (\xi_i^j)_{i \in I}$ and we suppose that $i_1, \ldots, i_s$ are indices such that the $s \times s$ complex matrix $(\xi_{i_k}^j)$ is non singular. Then for $i \notin \{i_1, \ldots, i_s\}$ there exist λ_i^k, $k = 1, \ldots, s$ with

$$\xi_i^j = \sum_{k=1}^{s} \lambda_i^k \xi_{i_k}^j, \quad j = 1, \ldots, s.$$

Consider now an element $(\eta_i)_{i \in I}$ in the range of $p \otimes 1$. We get

$$\eta_i = \sum_{k=1}^{s} \lambda_i^k \eta_{i_k} \quad \text{if } i \notin \{i_1, \ldots, i_s\}.$$

Hence

$$a_i h = \sum_{k=1}^{s} \lambda_i^k a_{i_k} h \quad \text{for } i \notin \{i_1, \ldots, i_s\} \quad \text{and all } h \in L^2(N),$$

from which it follows that $r \leq s$.

The second assertion of the lemma is then obvious when I is finite. Let us prove now that $(a_i)_{i \in I}$ is a family of independent vectors if p is the identity. Consider a finite set of indices $i_1, \ldots, i_k$ in I and denote by q the associated projection in $\mathcal{L}(l^2(I))$. Then

$$l^2(\{i_1, \ldots, i_k\}) \otimes H = (q \otimes 1)(K)$$

is the closed space generated by $\{(ma_{i_1}h, \ldots, ma_{i_k}h), m \in M, h \in L^2(N)\}$, and by the first part of the proof $a_{i_1}, \ldots, a_{i_k}$ are independent. ■

THEOREM 1. *Let Φ be a normal completely positive map from M into N. Then H_Φ is (unitary equivalent to) a multiple of H if and only if there is a family $(a_i)_{i \in I}$ in X_H with $\sum_{i \in I} a_i^* a_i$ σ-weakly convergent and $\Phi(x) = \sum_{i \in I} a_i^* x a_i$ for all $x \in M$. In this case, the dimension of the linear span of $\{a_i, i \in I\}$ is equal to the multiplicity of H in H_Φ, and thus depends only on Φ. Moreover, the family $(a_i)_{i \in I}$ may be chosen such that its elements are linearly independent.*

Proof. Suppose first that H_Φ is a multiple of H. There exists an isomorphism U from the Hilbert N-module X_Φ onto a multiple $l^2(I) \otimes X_H$ of X_H which intertwines the left M-actions. Put $(a_i)_{i \in I} = U\xi_\Phi$. Then we have for $x \in M$

$$\begin{aligned} \Phi(x) &= \langle \xi_\Phi, x\xi_\Phi \rangle \\ &= \langle U\xi_\Phi, xU\xi_\Phi \rangle = \sum_{i \in I} a_i^* x a_i. \end{aligned}$$

Notice that $(a_i)_{i \in I}$ is a cyclic vector in $l^2(I) \otimes X_H$ and thus the closed subspace spanned by $\{(ma_i h)_{i \in I}, m \in M, h \in L^2(N)\}$ is $l^2(I) \otimes H$. Then by lemma 1 the family $(a_i)_{i \in I}$ is made of independent vectors.

Conversely, suppose that $\Phi(x) = \sum_{i \in I} a_i^* x a_i$ for all $x \in M$. Then it is easily seen that $U: M \odot L^2(N) \longrightarrow l^2(I) \otimes H$ defined by

$$U(m \otimes h) = (ma_i h)_{i \in I} \quad \text{for } m \in M, h \in L^2(N)$$

gives rise to an equivalence between H_Φ and a subcorrespondence of $l^2(I)\otimes H$. As H is irreducible this subcorrespondence is determined by a projection $p\otimes 1 \in \mathcal{L}(l^2(I))\otimes 1_H$ and thus is a multiple of H. Moreover, by lemma 1 the multiplicity of H in H_Φ is equal to the dimension of the linear span of $\{a_i, i\in I\}$. ∎

DEFINITION 2. We say that a normal completely positive map $\Phi: M \longrightarrow N$ has *multiplicity* $k \in \mathbf{N}\cup\{\infty\}$ if the commutant in $\mathcal{L}(H_\Phi)$ of the von Neumann algebra generated by the left action of M and the right action of N is a factor of type I_k.

This amounts to say that the correspondence H_Φ is the direct sum of k equivalent irreducible correspondences, the class of which is well determined by Φ (see [6], § 5.4). Thus, these completely positive maps are exactly our object of study in this paper.

REMARK 1. When M and N are II_1 factors, one can prove that the index of a completely positive map $\Phi \in CP_H(M,N)$ is equal to the product of k^2 by the index of the correspondence H, where k is the multiplicity of Φ. These two notions of index have been introduced by Popa in ([10],§ 1.4). Recall that the index of a correspondence H is the number $\dim_M H \dim_{N^o} H$, and the index of Φ is, by definition, the index of the correspondence H_Φ. It follows from the theorem 1 that $\{k^2, 0\le k\le mn\}$ is the set of indices of all the completely positive maps from $M_m(\mathbf{C})$ into $M_n(\mathbf{C})$.

Let us return now to the general situation. Although the family $(a_i)_{i\in I}$ to which $\Phi \in CP_H(M,N)$ is associated is not unique, the next result shows that, when H_Φ is a finite multiple of H, there is an essentially unique independent family (b_j) such that

$$\Phi(x) = \sum_j b_j^* x b_j \quad \text{for all } x\in M.$$

THEOREM 2. *Let $\Phi \in CP_H(M,N)$ of multiplicity $k < +\infty$, and let $b_1,\dots,b_k$ be k independent elements of X_H such that $\Phi(x) = \sum_{j=1}^k b_j^* x b_j$ for all $x\in M$. Then a family $(a_i)_{i\in I}$ in X_H satisfies*

$$\sum_{i\in I} a_i^* x a_i = \sum_{j=1}^k b_j^* x b_j \quad \textit{for all } x\in M,$$

if only and only if there exists an isometry $u=(u_{ij})$ from $\mathbf{C}^k$ onto $l^2(I)$ such that

$$a_i = \sum_{j=1}^k u_{ij} b_j \quad \textit{for } i\in I.$$

Proof. Suppose first that $\sum_{i\in I} a_i^* x a_i = \sum_{j=1}^k b_j^* x b_j$ for all $x\in M$. Then

$$U: (mb_jh)_{j=1,\dots,k} \longrightarrow (ma_ih)_{i\in I} \quad \text{for } m\in M, h\in L^2(N)$$

induces an isometry from the closed linear span K of

$$\{(mb_jh)_{j=1,\dots,k}, m\in M, h\in L^2(N)\}$$

into $l^2(I)\otimes H$. By lemma 1, $K = \mathbf{C}^k\otimes H$ since $b_1,\dots,b_k$ are independent. Moreover, U intertwines the left actions of M and the right actions of N. Thanks to the irreducibility of

H we see that $U = u \otimes 1$ where u is an isometry from $\mathbf{C}^k$ into $l^2(I)$. As $U(b_1 h, \ldots, b_k h) = (a_i h)_{i \in I}$ for all $h \in L^2(N)$, it is clear that $a_i = \sum_{j=1}^k u_{ij} b_j$.

The converse is easily checked. ∎

REMARK 2. If k is infinite, the above result remains true provided that we take a family $(b_j)_{j \in J}$ such that the closed linear span of

$$\{(mb_j h)_{j=1,\ldots,k}, m \in M, h \in L^2(N)\}$$

is equal to $l^2(J) \otimes H$. This hypothesis is stronger that the independence of (b_j).

Let now $c \in N, c \geq 0$, and denote by $CP(M, N, c)$ the convex set of completely positive maps $\Phi: M \longrightarrow N$ with $\Phi(1) = c$. Using the Arveson-Paschke characterization of the set of extreme points of $CP(M, N, c)$, that we recall below, it will be easy to find the extreme elements of $CP(M, N, c)$ belonging to $CP_H(M, N)$.

THEOREM 3. ([9], Th. 5.4) *Let $\Phi \in CP_H(M, N)$ with $\Phi(1) = c$. Then Φ is an extreme point of $CP(M, N, c)$ if and only if the map $x \longrightarrow \langle \xi_\Phi, x\xi_\Phi \rangle$ from $\mathcal{L}_N(X_\Phi)$ into N is injective when restricted to the commutant of $\pi_\Phi(M)$.*

THEOREM 4. *Let $\Phi \in CP_H(M, N)$ of multiplicity $k < +\infty$, with $\Phi(1) = c$. Let $b_1, \ldots, b_k$ independent elements of X_H such that $\Phi(x) = \sum_{j=1}^k b_j^* x b_j$ for $x \in M$. Then Φ is an extreme point of $CP(M, N, c)$ if and only if $\{\langle b_i, b_j \rangle, i, j = 1, \ldots, k\}$ is a set of linearly independent elements in N.*

Proof. We have $X_\Phi = \mathbf{C}^k \otimes X_H$ and $\mathcal{L}_N(X_\Phi) = M_k(\mathbf{C}) \otimes \mathcal{L}_N(X_H)$. By the irreducibility of H, we see that the commutant of $\pi_\Phi(M)$ in $\mathcal{L}_N(X_\Phi)$ is $M_k(\mathbf{C}) \otimes 1$. Then, for $x = (x_{ij}) \in M_k(\mathbf{C})$, we have

$$\langle \xi_\Phi, (x \otimes 1)\xi_\Phi \rangle = \sum_{i,j=1}^k b_i^* b_j x_{ij}$$

and the conclusion follows immediately from the Arveson-Paschke result. ∎

REMARK 3. There is an analogue of this theorem when k is infinite, but with a more complicated expression.

In the case $M = M_m(\mathbf{C})$ and $N = M_n(\mathbf{C})$, the above result is exactly the theorem 5 of [4].

When applied to example 3 with $N' \cap M = \mathbf{C}$, our theorem 4 has for instance the following consequence. Let $b_1, \ldots, b_k$ be independent elements in M with $E(\sum_{j=1}^k b_j^* b_j) = 1$. Then the completely positive map $x \longrightarrow E(\sum_{j=1}^k b_j^* x b_j)$ is an extreme point of the convex set of all unital completely positive maps from M into N if and only if the elements $E(b_i^* b_j), i, j = 1, \ldots, k$, are independent.

To conclude, let us remark that the case where H is not irreducible is much more difficult to work out, as we can see it in [8] for the case of inner completely positive maps.

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