



# Constructing skew left braces whose additive group has trivial centre

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**Abstract.** A complete description of all possible multiplicative groups of finite skew left braces whose additive group has trivial centre is given. As a consequence, some earlier results of Tsang can be improved and an answer to an open question set by Tsang at Ischia Group Theory 2024 Conference is provided.

## 1 Introduction

Skew brace structure plays a key role in the combinatorial theory of the Yang–Baxter equation. Skew left braces, introduced in [8], can be regarded as extensions of Jacobson radical rings and show connections with several areas of mathematics such as triply factorized groups and Hopf–Galois structures (see [1, 3, 4])

Skew left braces classify solutions of the Yang–Baxter equation (see [8]). This connection to the Yang–Baxter equation motivates the search for constructions of skew braces and classification results.

Recall that a *skew left brace* is a set endowed with two group structures  $(B, +)$ , not necessarily abelian, and  $(B, \cdot)$  which are linked by the distributive-like law  $a(b + c) = ab - a + ac$  for  $a, b, c \in B$ .

In the sequel, the word *brace* refers to a skew left brace.

Given a brace  $B$ , there is an action of the multiplicative group on the additive group by means of the so-called *lambda map*:

$$\lambda: a \in (B, \cdot) \mapsto \lambda_a \in \text{Aut}(B, +), \quad \lambda_a(b) = -a + ab, \text{ for all } a, b \in B.$$

Braces can be described in terms of regular subgroups of the holomorph of the additive group. Recall that the holomorph of a group  $G$  is the semidirect product  $\text{Hol}(G) = [G] \text{Aut}(G)$ . Let  $B$  be a brace and set  $K = (B, +)$ . Then  $H = \{(a, \lambda_a) \mid a \in B\}$  is a regular subgroup of the holomorph  $\text{Hol}(K)$  isomorphic to  $(B, \cdot)$  (see [8, Theorem 4.2]). If we consider the subgroup  $S = KH \leq \text{Hol}(K)$ , then

$$S = KH = KE = HE,$$

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where  $E = \{(0, \lambda_b) \mid b \in B\}$  and  $C_E(K) = K \cap E = H \cap E = 1$ . We call  $S(B) = (S, K, H, E)$  the *small trifactorized group* associated with  $B$ .

In [9], Tsang showed it is possible to construct finite braces by just looking at the automorphism group of the additive group instead of looking at the whole holomorph. This is a significant improvement both from an algebraic and computational approach.

**Theorem 1** (see [9, Corollary 2.2]) *If the finite group  $G$  is the multiplicative group of a brace with additive group  $K$ , then there exist two subgroups  $X$  and  $Y$  of  $\text{Aut}(K)$  that are quotients of  $G$  satisfying*

$$XY = X\text{Inn}(K) = Y\text{Inn}(K).$$

She looked for a sort of converse of the above theorem in the case of finite braces with an additive group of trivial centre, and proved the following.

**Theorem 2** (see [9, Proposition 2.7]) *Suppose that the centre of a finite group  $(K, +)$  is trivial and let  $P$  be a subgroup of  $\text{Aut}(K)$  containing  $\text{Inn}(K)$ . If  $P = XY$  is a factorization by two subgroups  $X$  and  $Y$  such that  $X \cap Y = 1$ ,  $X\text{Inn}(K) = Y\text{Inn}(K) = P$  and  $X$  splits over  $X \cap \text{Inn}(K)$ , then there exists a brace  $B$  whose additive group is isomorphic to  $(K, +)$  and whose multiplicative group is isomorphic to a semidirect product  $[X \cap \text{Inn}(K)]Y$  for a suitable choice of the action  $\alpha: Y \rightarrow \text{Aut}(X \cap \text{Inn}(K))$ .*

The above two theorems are the key to prove the main results of [9, 10].

In [11], Tsang posed the following question.

**Question 3** *Is it possible to extend Theorem 2 by dropping the assumption that  $X$  splits over  $X \cap \text{Inn}(K)$ ?*

The aim of this article is to give a complete characterization of the multiplicative groups of a brace with additive group of trivial centre. As a consequence, we present an improved version of Theorem 2 (on which the main result of [9] heavily depends), and we give an affirmative answer to Question 3.

**Theorem A** *Let  $K$  be a finite group with trivial centre. For every brace  $B$  with additive group  $K = (B, +)$  and multiplicative group  $C = (B, \cdot)$ , there exist subgroups  $X$  and  $Y$  of  $\text{Aut}(K)$  satisfying the following properties:*

- (a)  $XY = X\text{Inn}(K) = Y\text{Inn}(K)$ ,
- (b) *there are two subgroups  $N$  and  $M$  of  $\text{Inn}(K)$  such that  $N \trianglelefteq X$  and  $M \trianglelefteq Y$ ,*
- (c) *there exists an isomorphism  $\gamma: Y/M \rightarrow X/N$  such that*

$$\text{Inn}(K) = \{xy^{-1} \mid x \in X, y \in Y, \gamma(yM) = xN\},$$

- (d)  $|K| = |X||M| = |Y||N|$ .

*In this case,*

- (e)  *$C$  has two normal subgroups  $T$  and  $V$  with  $T \cap V = 1$ ,  $X \cong C/T$  and  $Y \cong C/V$ , that is,  $C$  is a subdirect product of  $X$  and  $Y$ .*

Conversely, for every pair  $X, Y$  of subgroups of  $\text{Aut}(K)$  satisfying conditions (a)–(d), there exists a brace  $B$  with  $K = (B, +)$  and  $C = (B, \cdot)$  satisfying (e).

**Corollary 4** Let  $K$  be a finite group with trivial centre. Suppose that there exist subgroups  $X, Y$  of  $\text{Aut}(K)$  such that  $X \cap Y = 1$  and  $XY = X\text{Inn}(K) = Y\text{Inn}(K)$ . Then there exists a brace with additive group  $K$  and a multiplicative group that is isomorphic to a subdirect product of  $X$  and  $Y$ .

**Proof** Assume that  $X \cap Y = 1$ . Consider  $N = X \cap \text{Inn}(K)$ ,  $M = Y \cap \text{Inn}(K)$ . Then  $|X||M| = |K|$  as  $|X||Y| = |\text{Inn}(K)||Y|/|Y \cap \text{Inn}(K)|$ . Analogously,  $|Y||N| = |K|$ . Moreover, since

$$Y/M \cong Y\text{Inn}(K)/\text{Inn}(K) = X\text{Inn}(K)/\text{Inn}(K) \cong X/N,$$

we have an isomorphism  $\gamma: Y/M \rightarrow X/N$  given by  $\gamma(bM) = aN$ , where  $b \in Y, a \in X$  such that  $ab^{-1} \in \text{Inn}(K)$ . Since  $X \cap Y = 1$ , for each  $k \in K$ , conjugation by  $k$  can be expressed as  $ab^{-1}$ , for a unique  $a \in X$  and  $b \in Y$ . Then, the groups  $X$  and  $Y$  satisfy Statements (a)–(d) of Theorem A, and therefore, there exists a brace whose additive group is  $K$  and whose multiplicative group is isomorphic to a subdirect product of  $X$  and  $Y$ . ■

Corollary 4 also allows to give a considerably shorter proof of the main results of [9, 10] about the almost simple groups  $K$  that can appear as additive groups of braces with soluble multiplicative group. By Corollary 4, it is enough to find two subgroups  $X$  and  $Y$  of  $\text{Aut}(K)$  such that  $X \cap Y = 1$  and  $XY = X\text{Inn}(K) = Y\text{Inn}(K)$ . Therefore, Codes 2, 3, and 4 in the proof of [9, Theorem 1.3] can be avoided, as well as checking in every case that the subgroup  $X$  splits over  $X \cap \text{Inn}(K)$ .

In Section 3, we present a worked example of a construction of a brace with additive group  $K = \text{PSL}_2(25)$  by means of subgroups  $X$  and  $Y$  of  $\text{Aut}(K)$  satisfying all conditions of Theorem A but  $X \cap Y \neq 1$ .

## 2 Proof of Theorem A

**Proof of Theorem A** Suppose that  $B$  is a brace with additive group  $K$  and lambda map  $\lambda$ . Let  $H = \{(b, \lambda_b) \mid b \in B\}$  be the regular subgroup of  $\text{Hol}(K)$  appearing in the small trifactorized group  $S(B) = (S, K, H, E)$  associated with  $B$ . Recall that  $H$  is isomorphic to the multiplicative group  $(C, \cdot)$  of  $B$ ,  $E = \{(0, \lambda_b) \mid b \in B\} \leq \text{Hol}(K)$ , and  $S = KH = KE = HE$  with  $K \cap E = H \cap E = 1$ .

Observe that  $S$  acts on  $K$  by means of the homomorphism  $\pi: (b, \omega) \in S \mapsto \omega \in \text{Aut}(K)$ . On the other hand,  $S$  also acts on  $K$  by conjugation. In fact, this action naturally induces a homomorphism  $\alpha: S \rightarrow \text{Aut}(K)$ . In particular, for every  $b \in B$  and every  $k \in K$ ,  $(0, \lambda_b)(k, 1)(0, \lambda_b)^{-1} = (\lambda_b(k), 1)$ , that is,  $\alpha(0, \lambda_b) = \lambda_b = \pi(0, \lambda_b)$ . Thus,  $\alpha(E) = \pi(E) = \pi(H)$ .

The restrictions of  $\pi$  and  $\alpha$  to  $H$  induce two actions of  $H$  on  $K$ , with respective kernels  $\text{Ker } \pi|_H = K \cap H \trianglelefteq H$  and  $\text{Ker } \alpha|_H = C_H(K) \trianglelefteq H$ . Moreover, it holds that

$$\begin{aligned} \text{Ker } \pi|_H \cap \text{Ker } \alpha|_H &= K \cap H \cap C_H(K) = K \cap H \cap C_S(K) \\ &= H \cap C_K(K) = H \cap Z(K) = 1 \quad (\text{see Figure 1}). \end{aligned}$$

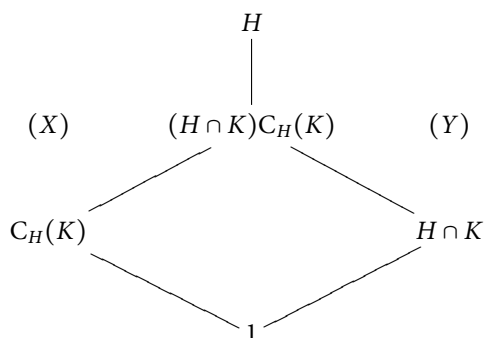


Figure 1: Structure of the multiplicative group in Theorem A.

Let  $X := \alpha(H)$  and  $Y := \pi(H) = \alpha(E) = \{\lambda_b \mid b \in B\}$  such that  $X \cong H/C_H(K)$  and  $Y \cong H/(K \cap H)$ . Since  $\alpha(K) = \text{Inn}(K)$ , we have that

$$\begin{aligned} \alpha(S) &= \alpha(HE) = \alpha(KH) = \alpha(KE) \\ &= \alpha(H)\alpha(E) = \alpha(K)\alpha(H) = \alpha(K)\alpha(E) \\ &= XY = (\text{Inn}(K))X = (\text{Inn}(K))Y. \end{aligned}$$

Take  $R := (H \cap K)C_H(K) \trianglelefteq H$ . Then,  $N := \alpha(R) \trianglelefteq \alpha(H) = X$  and  $M := \pi(R) \trianglelefteq \pi(H) = Y$ . It follows that  $N = \alpha(H \cap K) \leq \alpha(K) = \text{Inn}(K)$ . On the other hand,  $M = \pi(C_H(K))$  and if  $(b, \lambda_b) \in C_H(K)$ , then for every  $k \in K$ ,

$$(b, \lambda_b)(k, 1)(b, \lambda_b)^{-1} = (b + \lambda_b(k) - b, 1) = (k, 1),$$

that is,  $\lambda_b$  coincides with the inner automorphism of  $K$  induced by  $-b$ . Thus,  $M \leq \text{Inn}(K)$ . Moreover, we see that

$$\begin{aligned} Y/M &\cong (H/\text{Ker } \pi|_H)/(R/\text{Ker } \pi|_H) \cong H/R \\ &\cong (H/\text{Ker } \alpha|_H)/(R/\text{Ker } \alpha|_H) \cong X/N; \end{aligned}$$

here the isomorphism  $\gamma: Y/M \rightarrow X/N$  is given by  $\gamma(\lambda_b M) = \alpha_b \lambda_b N$ , where  $\alpha_b$  is the inner automorphism of  $K$  induced by  $b$ . Given  $a \in \gamma(\lambda_b M)$ , we have that  $a\lambda_b^{-1} \in \alpha_b N \subseteq \text{Inn}(K)$ . Furthermore, given  $x \in \text{Inn}(K)$ , we have that  $x = \alpha_b$  for some  $b \in B$  and so  $\gamma(\lambda_b M) = \alpha_b \lambda_b N = x\lambda_b N$  with  $(\alpha_b \lambda_b)\lambda_b^{-1} = x$ .

Since  $\text{Ker } \pi|_H \cap \text{Ker } \alpha|_H = (H \cap K) \cap C_H(K) = 1$ , we have that  $|R| = |H \cap K||C_H(K)|$  and  $|M| = |R/(H \cap K)| = |C_H(K)|$ ,  $|N| = |R/C_H(K)| = |H \cap K|$ . As  $|X| = |K|/|C_H(K)|$  and  $|Y| = |K|/|H \cap K|$ , the claim about the order follows.

Item (e) follows by the fact that  $H$  is isomorphic to the multiplicative group  $(C, \cdot)$  of  $B$ , so that  $T$  and  $V$  are respectively isomorphic to  $\text{Ker } \alpha|_H$  and  $\text{Ker } \pi|_H$ .

Now, suppose that  $\text{Aut}(K)$  possesses subgroups  $X$  and  $Y$  satisfying conditions (a)–(d). Let

$$W = \{(x, y) \mid x \in X, y \in Y, \gamma(yM) = xN\}$$

be a subdirect product of  $X$  and  $Y$  with amalgamated factor group  $Y/M \cong X/N$  (see [6, Chapter A, Definition 19.2]). By [6, Chapter A, Proposition 19.1], and the hypothesis, we have that  $|W| = |K|$ . Since  $Z(K)$  is trivial, the map  $\zeta: K \rightarrow \text{Inn}(K)$ , where  $\zeta(k)$  is the inner automorphism of  $K$  induced by  $k$ , is an isomorphism. By hypothesis, the map  $W \rightarrow \text{Inn}(K)$  given by  $(x, y) \mapsto xy^{-1}$  is surjective. Since  $|W| = |\text{Inn}(K)| = |K|$ , it is a bijection. We can consider  $H = \{(b, y) \mid (x, y) \in W, \zeta(b) = xy^{-1}\} \subseteq \text{Hol}(K)$ . Given  $(b, y), (b_1, y_1) \in H$ , we have that  $(b, y)(b_1, y_1) = (b + \gamma(b_1), yy_1)$ ,  $\zeta(b) = xy^{-1}$ , and  $\zeta(b_1) = x_1y_1^{-1}$  with  $(x, y), (x_1, y_1) \in B$ . Then

$$\zeta(b + \gamma(b_1)) = \zeta(b)\zeta(\gamma(b_1)) = \zeta(b)y\zeta(b_1)y^{-1} = xy^{-1}yx_1y_1^{-1}y^{-1} = (xx_1)(yy_1)^{-1}$$

with  $(xx_1, yy_1) = (x, y)(x_1, y_1) \in W$ . Furthermore, if  $(b, y) \in H$ , with  $\zeta(b) = xy^{-1}$ , we have that  $(b, y)^{-1} = (y^{-1}(-b), y^{-1})$  and

$$\zeta(y^{-1}(-b)) = y^{-1}\zeta(-b)y = y^{-1}\zeta(b)^{-1}y = y^{-1}yx^{-1}y = x^{-1}(y^{-1})^{-1}$$

with  $(x^{-1}, y^{-1}) = (x, y)^{-1} \in W$ . We conclude that  $H$  is a subgroup of  $\text{Hol}(K)$ . As the projection onto its first component is surjective, it turns out that  $H$  is a regular subgroup of  $\text{Hol}(K)$  by [2, Proposition 2.5] and so it is isomorphic to the multiplicative group of a brace with additive group  $K$  (see [8, Theorem 4.2]).

We finish the proof by showing that the map  $\phi: H \rightarrow W$  given by  $(b, y) \mapsto (\zeta(b)y, y)$ , where  $\zeta(b) = xy^{-1}$  and  $(x, y) \in W$ , is an isomorphism. Indeed, if  $\zeta(b) = xy^{-1}$ ,  $\zeta(b_1) = x_1y_1^{-1}$ , where  $(x, y), (x_1, y_1) \in W$ , we have that

$$\begin{aligned} \phi(b, y)\phi(b_1, y_1) &= (\zeta(b)y, y)(\zeta(b_1)y_1, y_1) = (x, y)(x_1, y_1) = (xx_1, yy_1), \\ \phi((b, y)(b_1, y_1)) &= \phi(b + \gamma(b_1), yy_1) = (\zeta(b + \gamma(b_1))yy_1, yy_1) \\ &= (\zeta(b)y\zeta(b_1)y^{-1}yy_1, yy_1) = (xy^{-1}yx_1y_1^{-1}yy_1, yy_1) \\ &= (xx_1, yy_1). \end{aligned}$$

We conclude that  $\phi$  is a group homomorphism. Assume that  $\phi(b, y) = (\zeta(b)y, y) = (1, 1)$ , with  $\zeta(b) = xy^{-1}$  and  $(x, y) \in W$ , then  $y = 1$  and so  $\zeta(b) = x = 1$ , which implies that  $b = 0$ . Consequently,  $\phi$  is injective. As  $W$  and  $H$  are finite and have the same order, we obtain that  $\phi$  is an isomorphism. Since  $C$  is isomorphic to  $H$  we have just proved that (e) holds for  $C$ . ■

### 3 A worked example

In general, we do not have that  $X \cap Y = 1$ . Let us consider  $K = \text{PSL}_2(25)$ . Its automorphism group  $A = \text{Aut}(K)$  is generated by  $\text{Inn}(K)$ , the diagonal automorphism  $d$  induced by the conjugation by the matrix

$$D = \begin{bmatrix} \zeta & 0 \\ 0 & 1 \end{bmatrix} \in \text{GL}_2(25),$$

where  $\zeta$  is a primitive 24th-root of unity of  $\text{GF}(25)$ , and the field automorphism  $f$ . The group  $A$  possesses a subgroup  $X$  generated by the inner automorphisms  $c_1$ ,  $c_2$ , and  $c_3$  induced by the matrices

$$C_1 = \begin{bmatrix} \zeta^4 & 0 \\ 0 & \zeta^{20} \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 \\ \zeta & 1 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

respectively, and  $df$ . We have that  $c_1$  has order 3,  $\langle c_2, c_3 \rangle$  is an elementary abelian group of order 25,  $c_1$  normalises  $\langle c_2, c_3 \rangle$ ,  $(df)c_1(df)^{-1} = c_1^{-1}$ ,  $df$  has order 8, and  $df$  normalizes  $\langle c_2, c_3 \rangle$ . Then the group  $\langle df, c_1, c_2, c_3 \rangle$  has order 600.

Let  $u_1$  and  $u_2$  be the inner automorphisms induced by the conjugation by

$$U_1 = \begin{bmatrix} \zeta^3 & \zeta^{16} \\ \zeta^{13} & \zeta^{11} \end{bmatrix}, \quad U_2 = \begin{bmatrix} \zeta^5 & \zeta^5 \\ \zeta^9 & \zeta^{22} \end{bmatrix}.$$

Let  $Y = \langle u_1, dfu_2 \rangle$ . We have that  $u_1$  has order 13. Let

$$R = \begin{bmatrix} \zeta & 0 \\ 0 & \zeta \end{bmatrix} \in Z(\text{GL}_2(25)), \quad T = \begin{bmatrix} 3 & 0 \\ 4 & 2 \end{bmatrix}$$

and let  $t$  be the automorphism induced by conjugation by  $T$ . Then  $(dfu_2)^2 = dfu_2dfu_2 = d^f u_2 d^5 u_2$  is the automorphism induced by conjugation by

$$(1) \quad DU_2^{(5)}D^5U_2 = R^{15}T,$$

where  $U_2^{(5)}$  denotes the matrix whose entries are obtained from the entries of  $U_2$  by applying the Frobenius field automorphism, that is,  $(dfu_2)^2 = t$ . As  $(R^{15}T)^2 = R^3$ , we conclude that  $dfu_2$  has order 4. We can also check that  $(dfu_2)u_1(dfu_2)^{-1} = u_1^8$ . It follows that  $Y$  has order 52.

By [5],  $X$  and  $Y$  are maximal subgroups of the almost simple group  $\text{Inn}(K)\langle df \rangle$ . Observe that  $(df)^4 c_3^2 = (dfdf)^2 c_3^2 = (dd^5)^2 c_3^2 = d^{12} c_3^2$  is induced by  $D^{12}C_3^2 = R^{18}T$ , consequently,  $(df)^4 c_3^2 = t$ . This, together with Equation (1), shows that  $t \in X \cap Y$ . We note that  $\text{Inn}(K)X = \text{Inn}(K)Y = \text{Inn}(K)\langle df \rangle$ . Moreover,  $|X \cap Y|$  divides  $\gcd(|X|, |Y|) = 4$ . If  $|X \cap Y| = 4$ , then  $X \cap Y$  is contained in  $X \cap \text{Inn}(K)$ , but it is not contained in  $Y \cap \text{Inn}(K)$ . This shows that  $|X \cap Y| \leq 2$ . Hence  $|X \cap Y| = 2$ . As  $XY \subseteq \text{Inn}(K)\langle df \rangle$ ,

$$15\,600 = |\text{Inn}(K)\langle df \rangle| \geq |XY| = \frac{|X||Y|}{|X \cap Y|} = 15\,600 \cdot \frac{2}{|X \cap Y|} = 15\,600,$$

and so  $XY = \text{Inn}(K)\langle df \rangle$ .

Let  $N = \langle c_1, c_2, c_3, (df)^2 \rangle \trianglelefteq X$ ,  $M = \langle u_1 \rangle \trianglelefteq Y$ . Then  $|N| = 150$ ,  $|M| = 13$ ,  $N \leq X \cap \text{Inn}(K)$ ,  $M \leq Y \cap \text{Inn}(K)$ ,  $Y/M \cong X/N \cong C_4$ , and  $|K| = |X||M| = |Y||N|$ . The isomorphism between  $Y/M$  and  $X/N$  is given by  $\gamma((dfu_2)^r M) = (df)^r N$  for  $0 \leq r < 4$ , and, since  $d^6 \in \text{Inn}(K)$ , it is clear that

$$\begin{aligned}(df)(dfu_2)^{-1} &= c_0 u_0^{-1} \in \text{Inn}(K), \\ (df)^2(dfu_2)^{-2} &= d^6 t^{-1} \in \text{Inn}(K), \\ (df)^3(dfu_2)^{-3} &= (df)(d^6 t^{-1} u_2^{-1})(df)^{-1} \in \text{Inn}(K).\end{aligned}$$

Let  $z \in XY \cap \text{Inn}(K)$ . Recall that  $X \cap Y = \langle t \rangle$ . Then there exist  $x \in X$ ,  $y \in Y$  with  $z = xy^{-1} = (xt)(yt)^{-1}$ . We observe that  $t = (df)^4 c_3^2 \in N$ , but  $t \notin M$  by order considerations. Given  $x \in X$ ,  $y \in Y$ , there exist  $r, s \in \{0, 1, 2, 3\}$  such that  $xN = (df)^r N$  and  $yM = (dfc_3)^s M$ . We also observe that  $x \in \text{Inn}(K)$  if, and only if,  $y \in \text{Inn}(K)$ . To prove that we can choose  $x \in X$ ,  $y \in Y$  such that  $z = xy^{-1}$  and  $y(yM) = xN$ , it is enough to prove that for such a choice we have that  $z = xy^{-1}$  and  $r = s$ . Note that if  $x \in N$ , then  $r = 0$ ; if  $x \in \text{Inn}(K) \setminus N$ , then  $r = 2$ ; and if  $x \notin \text{Inn}(K)$ , then  $r \in \{1, 3\}$ . Analogously, if  $y \in M$ , then  $s = 0$ ; if  $y \in \text{Inn}(K) \setminus M$ , then  $s = 2$ ; and if  $y \notin \text{Inn}(K)$ , then  $s \in \{1, 3\}$ . We also have that  $tM = (dfu_2)^2 M$  and that  $tN = N$ , as  $t \in \text{Inn}(K)$ ,  $t \in N$ , but  $t \notin M$ . If  $x \in N$  and  $y \in M$ , we can choose  $r = s = 0$  and  $y(yM) = xN$ . Suppose that  $x \in N$  and  $y \notin M$ . Then  $y \in \text{Inn}(K)$  and so,  $xN = N$  and  $yM = (dfu_2)^2 M$ . Consequently,  $xtN = N$ ,  $ytN = N$ , and  $y(ytN) = xtN$ . Suppose that  $x \notin N$  and  $y \in M$ . We have that  $x \in \text{Inn}(K)$  and so,  $xN = (df)^2 N$  and  $yM = M$ . It follows that  $xtN = (df)^2 N$  and  $ytM = (dfu_2)^2 M$ , that is,  $y(ytM) = xtN$ . Suppose that  $x, y \in \text{Inn}(K)$ ,  $x \notin N$ , and  $y \notin M$ . Then  $xN = (df)^2 N$ ,  $yM = (dfu_2)^2 M$ , and  $y(yM) = xN$ . Finally, suppose that  $x$  and  $y \notin M$ . Then  $xN = (df)^r N$  and  $yM = (dfu_2)^s M$ , with  $r, s \in \{1, 3\}$ . If  $r = s$ , then  $y(yM) = xN$ . If  $r \neq s$ , then  $xtN = (df)^r N$  and  $ytM = (dfu_2)^{s+2} M$ , with  $r \equiv s + 2 \pmod{4}$ . Thus  $y(ytM) = xtN$ .

It follows that  $X, Y$  satisfy all conditions of Theorem A. We can also check with GAP [7] all this information about these subgroups.

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