EXTREME OPERATORS ON H_∞

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(Received 5 May, 1975)

Let A_1 and A_2 be sup-norm algebras, each containing the constant functions. Let $P(A_1, A_2)$ denote the set of bounded linear operators from A_1 to A_2 which carry 1 into 1 and have norm 1. Several authors have considered the problem of describing the extreme points of $P(A_1, A_2)$. In the case where A_1 is the algebra of continuous complex functions on some compact Hausdorff space, and A_2 is the algebra of complex scalars, Arens and Kelley proved that the extreme operators in $P(A_1, A_2)$ are exactly the multiplicative ones (see [**1**]). It was shown by Phelps in [**6**] that if A_1 is self-adjoint, then every extreme point of $P(A_1, A_1)$ is multiplicative. In [**4**], Lindenstrauss, Phelps, and Ryff exhibited non-multiplicative extreme points of $P(A, A)$ and $P(H_\infty, H_\infty)$, where A and H_∞ are, respectively, the disk algebra, and the algebra of bounded analytic functions on the open unit disk D. The extreme multiplicative operators in $P(A, A)$ were described in [**6**]. Rochberg proved in [**8**] that, if T is a member of $P(A, A)$ which carries the identity on D into an extreme point of the unit ball of A, then T is multiplicative and is an extreme point of $P(A, A)$. Rochberg's paper [**9**] is a study of certain extremal subsets of $P(A, A)$, namely, those of the form $K(F, G) = \{T \in P(A, A) : TF = G\}$, where F and G are inner functions in A. We proved in [**5**] that, if F is non-constant, then $K(F, G)$ contains an extreme point of $P(A, A)$.

In this note we show that the set of extreme elements of $P(H_\infty, H_\infty)$ contains " many " non-multiplicative members. In fact, we show that if F is a member of the unit ball of H_∞ which has a continuous extension to $\bar{D}$, and if G is an extreme point of the unit ball of H_∞ such that $G(D) \subseteq F(D) \backslash F(\partial D)$, then there is an extreme point T of $P(H_\infty, H_\infty)$ such that $TF = G$. Unless G is of the form $G = F \circ h$ for some $h \in H_\infty$, the operator T cannot be multiplicative. We also apply our methods to showing that neither $P(H_\infty, H_\infty)$ nor $P(A, A)$ is the weak operator-closed convex hull of its multiplicative elements.

Let $P = P(H_\infty, H_\infty)$ and let U denote the unit ball of H_∞. For $f, g \in U$, let $K(f, g)$ denote the set $\{T \in P : Tf = g\}$. If g is an extreme point of U, then $K(f, g)$ is an *extreme subset* of P, i.e. $cT + (1-c)S \in K(f, g)$, where $c \in (0, 1)$ and $S, T \in P$, implies $S, T \in K(f, g)$. We will use B to denote the space of bounded linear operators from H_∞ to itself. The weakest topology on B for which the linear functionals of the form $T \to Th(z)$ are continuous will be indicated by τ. By a result due to Kadison [**3**], the unit ball of B is τ-compact. It follows that P and also sets of the form $K(f, g)$ are τ-compact.

LEMMA 1. *If g is an extreme point of U and if $K(f, g)$ is non-empty, then there is an extreme point T of P such that $Tf = g$.*

Proof. The lemma follows from the Krein–Milman theorem and the fact that $K(f, g)$ is an extreme subset of P.

LEMMA 2. *Let g be a function in H_∞. Let f be a member of U having a continuous extension to $\bar{D}$. If $\overline{g(D)} \subseteq f(D) \backslash f(\partial D)$, then $K(f, g)$ is non-empty.*

https://doi.org/10.1017/S0017089500002901 Published online by Cambridge University Press

Proof. Note that the lemma holds if f is a constant. For the remainder of the proof, we will assume that f is not constant. Let E be a closed disk of radius t centred at 0, where t is less than 1 but chosen large enough so that $\overline{g(D)} \subseteq f(E)\backslash f(\partial E)$. Since $\overline{g(D)}$ is contained in one of the connected components of $f(E)\backslash f(\partial E)$, it follows that the integer

$$N = (2\pi i)^{-1} \int_{\partial E} f'(\xi)(f(\xi) - g(w))^{-1} d\xi$$

is independent of $w \in D$, and not equal to zero. Define the operator T on H_∞ by

$$Tk(w) = (2\pi i N)^{-1} \int_{\partial E} k(\xi) f'(\xi)(f(\xi) - g(w))^{-1} d\xi.$$

Since $Tk(w) = N^{-1} \sum_{j=1}^{N} k(z_j)$, where the z_j are the roots of $f(z) = g(w)$ which lie in E, it follows that T has norm less than 1. (Let n_j denote the order of z_j as a zero of $h(z) = f(z) - g(w)$. In the sum above each z_j is counted n_j times.) It is clear that $T1 = 1$ and $Tf = g$. Thus $T \in K(f, g)$.

LEMMA 3. *Let g be a function in H_∞. Let f be a member of U having a continuous extension to $\bar{D}$. If $g(D) \subseteq f(D)\backslash f(\partial D)$, then $K(f, g)$ is non-empty.*

Proof. For each $r \in [0, 1)$, let g_r denote the function defined by: $g_r(z) = g(rz)$. Then $\overline{g_r(D)} \subseteq f(D)\backslash f(\partial D)$. It follows by Lemma 2 that an operator T_r can be chosen from $K(f, g_r)$ for each $r \in [0, 1)$. The net $\{T_r\}_{r \in [0,1)}$ has a subnet which converges in the topology τ to an operator T in P. It follows immediately from the definition of τ that T is in $K(f, g)$.

The following Theorem is an immediate consequence of Lemmas 1 and 3:

THEOREM. *Let G be an extreme element of U. Let F be a member of U having a continuous extension to $\bar{D}$. If $G(D) \subseteq F(D)\backslash F(\partial D)$, then there is an extreme element T of P such that $TF = G$.*

We observe that the hypotheses of the theorem are fulfilled if F is a finite Blaschke product.

Suppose that M is a multiplicative element in $K(f, g)$, where $f, g \in U$ and f has a continuous extension to $\bar{D}$. Since f can be approximated uniformly on D by polynomials in the identity function Z, it follows that $Mf = f \circ MZ = g$. Thus, unless g is of the form $f \circ h$, where $h \in H_\infty$, there can be no multiplicative element of P such that $Tf = g$.

Let S denote the collection of multiplicative elements in P. Let K_1 and K_2 denote, respectively, the closure in the weak operator topology and the closure in the topology τ, of the convex hull of S. Note that $K_1 \subseteq K_2$. By Milman's converse to the Krein–Milman theorem [7, p. 9], the extreme elements of K_2 lie in S. Since P contains non-multiplicative extreme points, it follows that K_2 is a proper subset of P. Thus, K_1 is a proper subset of P.

Let S_A denote the set of multiplicative operators in $P(A, A)$. Since the polynomials are dense in A, it follows that the operators in S_A are exactly those of the form

$$Mg = g \circ h \quad (\|h\| \leq 1)$$

where $h \in A$. Let $R = \{T \in S_A : TZ \text{ is not a constant of modulus } 1\}$. Note that S_A is the

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uniform closure of R. Hence, the closure of cov S_A in the weak operator topology coincides with the closure of R in the weak operator topology. Each $M \in R$ has an extension M^* in S, where M^* is defined by

$$M^* f = f \circ MZ$$

for each $f \in H$. Similarly each $V \in \text{cov}\, R$ has an extension V^* in cov S. Let T be an extreme element of $P(A, A)$ which is not multiplicative. If T is in the closure in the weak operator topology of cov S_A, then there is a net $\{V_\alpha\}$ in cov R which converges to T in the weak operator topology. Let $\{V_\beta\}$ be a subnet of $\{V_\alpha\}$ such that $\{V_\beta^*\}$ converges in the topology τ to some $T^* \in K_2$. It is easy to see that $Tg(z) = T^*g(z)$ for z in D and g in A. Thus, the set $K = \{T' \in K_2 : T' \mid A = T\}$ is non-empty. By the Krein–Milman theorem, K has an extreme element T_1'. Since K is an extreme subset of K_2, it follows that T_1' is an extreme point of K_2. By Milman's converse to the Krein–Milman theorem, T_1' must lie in S. But, if T_1' is in S, then T must be in S_A. Since we have reached a contradiction, it follows that T is not in the weak operator closure of cov S_A. Thus, $P(A, A)$ is not the weak operator closed, convex hull of its multiplicative elements.

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