A STABILITY RESULT FOR THE LINEAR DIFFERENTIAL EQUATION $x''+f(t)x=0$

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Suppose that the real-valued function $f(t)$ is positive, continuous and monotonic increasing for $t \geqq t_0$. If $x = x(t)$ is a solution of the equation

$$x''+f(t)x = 0 \qquad (' = d/dt) \tag{1}$$

for $t \geqq t_0$, it is known that the solution $x(t)$ oscillates infinitely often as $t \to \infty$, and that the successive maxima of $|x(t)|$ decrease, with increasing t. In particular $x(t)$ is bounded as $t \to \infty$.

The purpose here is to give a condition on $f(t)$ which ensures that $x(t) \to 0$ as $t \to \infty$.

THEOREM. *Suppose that $f(t)$ is positive, continuous and monotonic increasing for $t \geqq t_0$ and that $f(t)$ has continuous derivatives of orders $\leqq 3$. If, for some α, $1 < \alpha \leqq 2$, and $F = f^{-1/\alpha}$,*

$$\int_{t_0}^{\infty} |F'''| dt < \infty,$$

then every solution $x(t)$ of (1) tends to $x = 0$ as $t \to \infty$.

A particular case of the theorem, corresponding to $\alpha = 2$, has been obtained very recently by Lazer [4].

PROOF. With the solution $x = x(t)$ we define the function $y(t)$ by

$$y(t) = x^2 \left(\frac{2F^{1-\alpha}}{\alpha-1} + F'' \right) - 2xx'F' + \frac{2x'^2 F}{\alpha-1}.$$

Working out the differentiation with respect to t and reducing the result by means of $x'' = -xf = -xF^{-\alpha}$, we obtain

$$y'(t) = x^2 F''' + \frac{2(2-\alpha)}{\alpha-1} x'^2 F'.$$

Hence

$$y(t) = y(t_0) + \int_{t_0}^{t} x^2 F''' dt + \frac{2(2-\alpha)}{\alpha-1} \int_{t_0}^{t} x'^2 F' dt$$
$$\leqq y(t_0) + \int_{t_0}^{\infty} x^2 |F'''| dt = K \text{ (say)}$$

since $F' \leqq 0$.

Now
$$F^{1-\alpha} = f^{1-1/\alpha} \to \infty, \text{ as } t \to \infty,$$
and
$$|F''| = |F''(t_0) + \int_{t_0}^{t} F''' dt|$$
$$\leqq |F''(t_0)| + \int_{t_0}^{\infty} |F'''| dt;$$
so that F'' is bounded as $t \to \infty$.

Thus, given any $\varepsilon > 0$, we can choose $T > t_0$ such that at $t = T$,

(i)
$$\frac{K}{\varepsilon} < \frac{2F^{1-\alpha}}{\alpha-1} + F''$$
and

(ii)
$$x'(T) = 0.$$
Then
$$x^2(T) \frac{K}{\varepsilon} < y(T) \leqq K$$
or
$$x^2(T) < \varepsilon.$$
It follows now that
$$x^2(t) < \varepsilon \text{ whenever } t > T$$
and so
$$\lim_{t\to\infty} x(t) = 0,$$
as required.

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