6
Rectifiability of Measures

6.1 Some Basic Facts and Examples

Badger [45] has a nice survey covering parts of this topic. Recall that we have defined a measure $\mu \in \mathcal{M}(\mathbb{R}^n)$ to be m-rectifiable if there are Lipschitz maps $f_i\colon \mathbb{R}^m \to \mathbb{R}^n$ such that

$$\mu\left(\mathbb{R}^n \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R}^m)\right) = 0,$$

without requiring absolute continuity with respect to $\mathcal{H}^m$. In many works, the additional condition of absolute continuity with respect to $\mathcal{H}^m$ is added. This would avoid, for example, the fractal measures μ_s below being rectifiable.

Studying sets E with $0 < \mathcal{H}^m(E) < \infty$ is, in many respects, the same as studying measures μ with almost everywhere positive and finite upper m-density because μ and $\mathcal{H}^m \llcorner \{x\colon 0 < \Theta^{*m}(\mu, x) < \infty\}$ are mutually absolutely continuous by Theorem 1.3. Also $\mu \ll \mathcal{H}^m$ if and only if $\Theta^{*m}(\mu, x) < \infty$ for μ almost all $x \in \mathbb{R}^n$. If the upper density is infinite, we can have completely different rectifiable measures, in particular many lower-dimensional measures are such. One can show fairly easily that all AD-s-regular measures with $0 < s < m$ are m-rectifiable, see [312, Theorem 4.1]. For example, take $0 < s < 1$ and let μ_s be the product with itself of a standard $s/2$-dimensional Cantor measure in $\mathbb{R}$. Then μ_s is 1-rectifiable, but all attempts to approximate with lines clearly must fail. This example also shows that for general measures, using Lipschitz maps is quite a different thing than using C^1 maps. The measure μ_s is doubling in the sense that $\mu(B(x, 2r)) \lesssim \mu(B(x, r)$ for $x \in \operatorname{spt}\mu$ and $r > 0$. A more dramatic example was given by Garnett, Killip and Schul in [220]. They constructed a 1-rectifiable doubling measure μ whose support is the whole space $\mathbb{R}^n$. Then μ is purely unrectifiable with respect to Lipschitz graphs. In these examples, the lower density is infinite. One of the few general things one can say about rectifiable measures is that the lower density is positive, see [49, Lemma 2.7].

Theorem 6.1 *If $\mu \in \mathcal{M}(\mathbb{R}^n)$ is m-rectifiable, then $\Theta^m_*(\mu, x) > 0$ for μ almost all $x \in \mathbb{R}^n$.*

There has been a lot of recent interest in finding criteria for the rectifiability of measures in terms of variants of Jones-type square functions. Recall the definition of $\beta_E(x, r)$ from (3.1). As already mentioned in Section 4.2, $\beta_E(x, r) \to 0$ does not imply rectifiability even for AD-1-regular sets. For measures with finite upper density bilateral approximation together with positive lower density implies rectifiability by Theorem 4.9, but not without positive lower density because of the example of Preiss in [382, 5.9] of a purely unrectifiable measure with flat tangent measures. But multiscale β sums and integrals in the spirit of Theorems 3.16 and 3.17 have led to interesting results for measures.

6.2 Square Functions in General Dimensions

Azzam and Tolsa proved their Theorem 4.20 for more general measures. In analogy to (4.9), we define

$$\beta^{m,2}_\mu(x, r)^2 = \inf_{V \text{ affine } m\text{-plane}} r^{-m} \int_{B(x,r)} \left(\frac{d(y, V)}{r}\right)^2 d\mu y. \tag{6.1}$$

Theorem 6.2 *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$. If $0 < \Theta^{*m}(\mu, x) < \infty$ for μ almost all $x \in \mathbb{R}^n$, then μ is m-rectifiable if and only if*

$$\int_0^1 \beta^{m,2}_\mu(x, r)^2 r^{-1} dr < \infty \tag{6.2}$$

for μ almost all $x \in \mathbb{R}^n$.

Observe that (6.2) alone does not imply rectifiability; it is satisfied by the Lebesgue measure.

Edelen, Naber and Valtorta improved in [184, Theorem 2.19] the sufficient condition for rectifiability:

Theorem 6.3 *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$. If $\Theta^{*m}(\mu, x) > 0$ and*

$$\int_0^1 \beta^{m,2}_\mu(x, r)^2 r^{-1} dr < \infty$$

for μ almost all $x \in \mathbb{R}^n$, then $\mu(\mathbb{R}^n \setminus E) = 0$ for some m-rectifiable set E. If also $\Theta^m_*(\mu, x) < \infty$ for μ almost all $x \in \mathbb{R}^n$, then $\mu \ll \mathcal{H}^m$, so μ is m-rectifiable.

Paper [184] contains much more related material with quantitative estimates. As in the proof of Theorem 4.20, several delicate stopping time arguments are key tools in the proofs. Tolsa gave in [419] a different proof for the second statement based on [42].

Badger and Naples [47] characterized measures that live on countably many Lipschitz graphs in terms of Jones-type sums where the cubes are restricted to lie in cones. Dabrowski characterized rectifiability and big pieces of Lipschitz graphs in terms of conical square functions, see [125, 127].

But what if we don't make any density assumptions? Edelen, Naber and Valtorta proved in [184, Theorem 2.17] the following Reifenberg-type theorem, which is a very special case of their results:

Theorem 6.4 *Let* $\mu \in \mathcal{M}(\mathbb{R}^n)$ *with* $\operatorname{spt}\mu \subset B(0,1)$ *and* $\eta > 0$. *Suppose that*

$$\int_0^2 \int \beta_\mu^{m,2}(x,r)^2 \, d\mu x \, r^{-1} dr \le \eta^2.$$

Then $\mu = \mu_1 + \mu_2$ *where* $\mu_1(B(0,1) \setminus E) = 0$ *for some m-rectifiable set E,* $\mu_2(\mathbb{R}^n) \le C(n)\eta$ *and* $\Theta^m(\mu_2, x) = 0$ *for* μ_2 *almost all* $x \in \mathbb{R}^n$.

Here μ_2 could, for example, be $\mathcal{L}^n \llcorner A$ for some A with $\mathcal{L}^n(A) > 0$. For Hilbert and Banach space versions, see [185]. The proof is based on corona-type decompositions. Naber gives a proof also in [354]. He formulates it in terms of neck decompositions. Roughly this means that most of $B(0,1)$ is covered with balls B whose neck regions have small measure. The neck region of B is a complement in B of two sets. For x in the first, there is a good approximation by planes at all scales. That part is rectifiable. The second set is a union of balls $B(x, r_x)$ in which there is a good approximation by planes at the scales bigger than r_x. Of course this is very vague, and the interested reader should consult the references above and in the next sentence. Variants of the neck decomposition have been used in several places, see [93, 262, 358].

6.3 Square Functions and One-Dimensional Measures

Badger and Schul introduced new Jones-type square functions to study the one-dimensional rectifiability of measures. Define a variant of the quadratic β for $\mu \in \mathcal{M}(\mathbb{R}^n)$ and for cubes Q by

$$\beta_\mu^1(Q)^2 = \inf_{L \text{ a line}} \mu(Q)^{-1} \int_Q \left(\frac{d(y,L)}{d(Q)} \right)^2 d\mu y. \tag{6.3}$$

All the cubes Q in this section will be dyadic cubes of side length at most 1. Then

$$J_\mu(x) = \sum_Q \beta^1_\mu(3Q)^2 \chi_Q(x), x \in \mathbb{R}^n$$

is essentially the original Jones function for measures. In [49], Badger and Schul proved that $J_\mu(x) < \infty$ for μ almost all $x \in \mathbb{R}^n$ if μ is 1-rectifiable and $\mu \ll \mathcal{H}^1$. The absolute continuity is needed: for the rectifiable measures $\mu_s, 0 < s < 1$ (recall the beginning of this chapter), we have $\beta^1_{\mu_s}(3Q) \sim 1$ for the squares Q (with $d(Q) \leq 1$) meeting $\operatorname{spt}\mu_s$, so $J_\mu = \infty$ on $\operatorname{spt}\mu_s$. The next definition avoids this situation:

$$\tilde{J}_\mu(x) = \sum_Q \beta^1_\mu(3Q)^2 \frac{d(Q)}{\mu(Q)} \chi_Q(x), x \in \mathbb{R}^n. \tag{6.4}$$

Theorem 6.5 *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$. If μ is 1-rectifiable, then $\tilde{J}_\mu(x) < \infty$ for μ almost all $x \in \mathbb{R}^n$. The converse holds if* $\limsup_{r\to 0} \frac{\mu(B(x,2r))}{\mu(B(x,r))} < \infty$ *for μ almost all $x \in \mathbb{R}^n$.*

The proof of the first part, in [49], is based on Jones's travelling salesman theorem. The proof of the second part is in [51].

Martikainen and Orponen showed in [310] that the second part cannot be extended to general measures. They constructed an example where $\tilde{J}_\mu$ is bounded and μ has zero lower density, so it is not rectifiable.

Consider the modified β numbers

$$\beta^{1*}_\mu(Q)^2 = \inf_{L \text{ a line}} \max_R \min\left(\frac{1}{d(3R)}, \frac{1}{\mu(3R)}\right) \int_{3R} \left(\frac{d(y,L)}{d(3R)}\right)^2 d\mu y. \tag{6.5}$$

The maximum is taken over the dyadic cubes R for which $d(Q) \leq d(R) \leq 2d(Q)$ and $3R \subset 1600\sqrt{n}Q$. Define

$$J^*_\mu(x) = \sum_Q \beta^{1*}_\mu(Q)^2 \frac{d(Q)}{\mu(Q)} \chi_Q(x), x \in \mathbb{R}^n.$$

Badger and Schul proved in [51] the following characterization:

Theorem 6.6 *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$ be such that $\Theta^m_*(\mu, x) > 0$ for μ almost all $x \in \mathbb{R}^n$. Then μ is 1-rectifiable if and only if $J^*_\mu(x) < \infty$ for μ almost all $x \in \mathbb{R}^n$.*

Combining with Theorem 6.1, we have

Theorem 6.7 *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$. The* 1*-rectifiable and purely* 1*-unrectifiable parts in the decomposition $\mu = \mu_r + \mu_u$ are given by*

$$\mu_r = \mu \llcorner \{x \colon \Theta^m_*(\mu, x) > 0 \text{ and } J^*_\mu(x) < \infty\},$$

$$\mu_u = \mu \llcorner \{x \colon \Theta^m_*(\mu, x) = 0 \text{ or } J^*_\mu(x) = \infty\}.$$

Badger, Li and Zimmerman [46] proved analogous results in Carnot groups.

In [289], Lerman used different modified Jones functions to get sufficient conditions for 1-rectifiability of measures with quantitative estimates. He did not make any a priori density or absolute continuity assumptions. Naples [362] proved extensions to Hilbert spaces.

6.4 Square Functions and Distance of Measures

Often it is more natural to approximate with Lebesgue measures on planes than with planes. Recall the metric $F_{x,r}$ and the α coefficients from (5.3) and (5.4). Azzam, Tolsa and Toro [43] used a slightly different α:

$$\tilde{\alpha}^m_\mu(x,r) = \frac{1}{r\mu(B(x,r))} \inf\{F_{x,r}(\mu, c\mathcal{H}^m \llcorner V) \colon c \geq 0, V \text{ an affine } m\text{-plane}\}.$$

They proved

Theorem 6.8 *Let* $\mu \in \mathcal{M}(\mathbb{R}^n)$ *and suppose that* $\limsup_{r\to 0} \frac{\mu(B(x,2r))}{\mu(B(x,r))} < \infty$ *for* μ *almost all* $x \in \mathbb{R}^n$*. Then* μ *is m-rectifiable and* $\mu \ll \mathcal{H}^m$ *if and only if*

$$\int_0^1 \tilde{\alpha}^m_\mu(x,r)^2 r^{-1} dr < \infty$$

for μ *almost all* $x \in \mathbb{R}^n$.

The ‘only if’ direction was proved in [417]. The difference between the β’s and α’s is something like what we had before; small β tells us locally that most of the measure lives close to a plane, but small α tells us that most of the plane also is close to the support of the measure. So in a way the β’s are smaller than the α’s but not in a precise sense.

Azzam, David and Toro [33] proved rectifiability of doubling measures under different α assumptions. They do not specify the dimension but define an $\alpha_\mu(x,r)$ minimizing first the distance to normalized m-flat measures and then minimizing over $m = 0, 1, \ldots, n$. The finiteness of $\int_0^1 \alpha_\mu(x,r)/r\, dr$ for μ almost all $x \in \mathbb{R}^n$ implies that the doubling measure μ is a sum over $m = 0, 1, \ldots, n$ of m-rectifiable measures. They also proved related quantitative results. In [34], they defined α numbers measuring self-similarity properties of a measure. If this $\alpha_\mu(x,r)$ is small, then the blow-up at the scale r is close to blow-ups at certain smaller scales, possibly after a rotation. They proved that then μ has unique flat tangent measures. From that they got rectifiability, and more, as above.