THE NUMBER OF HAMILTONIAN CIRCUITS IN LARGE, HEAVILY EDGED GRAPHS

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G is a graph on n nodes with q edges, without loops or multiple edges. We write $\alpha = q/n$ and β for the maximum degree of any node of G. We write

$$B(h, 0) = 1, \quad B(h, k) = h!/\{k!(h-k)!\}, \quad M = (n-1)!/2$$

and H for the number of Hamiltonian circuits (H.c.) in $\bar{G}$, the complement of G, or, what is the same thing, the number of those H.c. in the complete graph K_n which have no edge in common with G. Our object here is to prove the following theorem.

THEOREM 1. *If* $\alpha \to a < \infty$ *as* $n \to \infty$ *and* $\beta = o(n)$, *then*

$$H/M \to e^{-2a} \text{ as } n \to \infty. \tag{1}$$

Wright [4] proved this result for the particular case when G is a Hamiltonian circuit (when $\alpha = a = 1$) and Singmaster [3] when G is a 1-factor (when $\alpha = a = \frac{1}{2}$). Rousseau [2] found Wright's result by an improved method; our own method owes something to Rousseau's. The authors of [1] find an exact, but complicated, formula for H when G takes one of several special forms.

To prove Theorem 1, we write $J(e_1, \ldots, e_r)$ for the number of different H.c. in K_n which pass through the edges $e_1, \ldots, e_r$ belonging to G. We write

$$L_r = \sum J(e_{i_1}, \ldots, e_{i_r}),$$

where the sum is over all sets of r different edges belonging to G, and L_0 for the numder of H.c. in K_n, so that $L_0 = M$. Then, by the Exclusion–Inclusion Theorem,

$$H = \sum_{r=0}^{x-1} (-1)^r L_r + (-1)^x \theta L_x, \tag{2}$$

where x is at our choice and $0 \leqq \theta \leqq 1$. We shall take $x < n$, so that we need only consider $r < n$.

An *arc* (or more precisely, an *s-arc*) is a sequence of edges

$$P_1P_2, P_2P_3, P_3P_4, \ldots, P_sP_{s+1},$$

where the nodes $P_1, P_2, \ldots, P_{s+1}$ are all different. A set of arcs (or, as a particular case, a set of edges) is *independent* if no two of the arcs have a node in common. If the set of edges $e_1, \ldots, e_r$ consists of an independent set of arcs, R in number, we have

$$J(e_1, \ldots, e_r) = 2^{R-1}(n-r-1)! \tag{3}$$

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by the simple argument of [4] or of [2]. If the set of edges is of any other form, so that it contains a cycle or a star of 3 or more edges, then $J = 0$.

It follows that

$$L_1 = q\{(n-2)!\}. \tag{4}$$

Hence, if $a = 0$, i.e. $q = o(n)$, we have

$$L_1/M = 2q/(n-1) \to 0 \text{ as } n \to \infty$$

and, if we choose $x = 1$ in (2), then (1) follows. Henceforth, then, we may take $a > 0$ so that

$$q > C_1 n \quad (n > C_2), \tag{5}$$

where C_1, C_2 are fixed positive numbers.

We can choose $B(q, r)$ sets of r edges from G, of which Ω_r (say) are dependent. From (3) for each of the independent sets

$$J(e_1, \ldots, e_r) = 2^{r-1}(n-r-1)!,$$

while, for each of the dependent sets,

$$J(e_1, \ldots, e_r) < 2^{r-1}(n-r-1)!.$$

Hence

$$L_r = 2^{r-1}(n-r-1)!\{B(q, r) + O(\Omega_r)\}.$$

Every set of dependent edges must contain at least one 2-arc. But the number of 2-arcs in G is at most $q\beta$ (since one edge can be chosen in q ways and the second in at most $2(\beta-1)$ ways and we have then counted each 2-arc twice). The remaining $r-2$ edges in a dependent set can be chosen in at most $B(q-2, r-2)$ ways. Hence

$$\Omega_r \leqq q\beta B(q-2, r-2)$$

and so

$$L_r = 2^{r-1}(n-r-1)!B(q, r)\{1 + O(\beta r^2/q)\}.$$

Now

$$2^{r-1}B(q, r)(n-r-1)! = (2\alpha)^r MQ/r!,$$

where

$$\log Q = \sum_{s=1}^{r} \left\{\log\left(1-\frac{s-1}{q}\right) - \log\left(1-\frac{s}{n}\right)\right\}.$$

If $r = o(n)$, we have by (5)

$$\log Q = \sum_{s=1}^{r} O(s/n) = O(r^2/n).$$

Hence, if $r^2 = o(n)$, we have

$$L_r/M = \{(2\alpha)^r/r!\}\{1 + O(\beta r^2/n)\}. \tag{6}$$

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Using this in (2), we have

$$H/M = e^{-2\alpha}+O((2\alpha)^x/x!)+O(\beta e^{2\alpha}/n)$$
$$= e^{-2\alpha}+o(1),$$

if we choose x so that $x \to \infty$ as $n \to \infty$. This is Theorem 1.

Clearly the condition $\beta = o(n)$ in Theorem 1 cannot be replaced by $\beta = O(n)$, since $\beta = n-1$ implies that at least one node of $\bar{G}$ is isolated and $H = 0$. Nor can we replace $\beta = o(n)$ by $\beta \leqq bn$ for some fixed b such that $0 < b < 1$. For, take G to consist of a star and a number of isolated nodes, with

$$\beta = [bn] = q = \alpha n, \text{ so that } \alpha \to b \text{ as } n \to \infty.$$

Then

$$L_1 = q\{(n-2)!\}, \quad L_2 = q(q-1)\{(n-3)!\}/2$$

and $L_r = 0$ for $r \geqq 3$. Hence

$$H/M = (L_0 - L_1 + L_2)/M \to (1-b)^2$$

as $n \to \infty$. But $(1-b)^2 < e^{-2b}$ since $b > 0$. Hence (1) is false for this G.

We can however prove the following theorem for larger α and more restricted β, but the proof is so much more complicated that we shall present it elsewhere.

THEOREM 2. *If* A_1, A_2, ε *are any fixed positive numbers,* $A_1 < \alpha < A_2 \log n$ *and* $\beta = O(n^{1-\varepsilon})$, *then* $H \sim Me^{-2\alpha}$ *as* $n \to \infty$.

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