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UNITARY REPRESENTATIONS CORRESPONDING TO MEASURES WITH BOUNDED SUPPORT IN INFINITE DIMENSIONS

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Introduction

Let E be a real Hausdorff locally convex space with topological dual E', topologised by the strong topology. Let (x, x') denote the bilinear mapping defining the duality between E and E' ($x \in E$, $x' \in E'$). By a unitary representation of E' we mean an operator-valued function $U(x') = U_{x'}$ defined on E', whose values are unitary operators in a separable Hilbert space H such that

$$U_{x'_1 + x'_2} = U_{x'_1} \circ U_{x'_2}, \qquad x'_1, x'_2 \in E'.$$

U is called cyclic if there exists a vector $h \in H$ such that $\{U_{x'} h : x' \in E'\}$ is total. Without loss of generality we may suppose that $\|h\| = (h, h)_H^{1/2} = 1$ (by $(h_1, h_2)_H$ we denote the inner product on H, $h_1, h_2 \in H$). The vector h is called a cyclic vector for the representation U. Let $\mathscr{L}(H)$ denote the space of operators on H with the norm topology. We call U strongly continuous if the mapping $x' \in E' \mapsto U_{x'} v \in H$ is continuous for each $v \in H$. Let $\mathbb{R}$ be the field of real numbers and let n be a positive integer. Then, if $E' \cong \mathbb{R}^n$, the following result is obtained from Bochner's theorem.

Theorem. *Let $U : x' \in \mathbb{R}^n \mapsto U_{x'} \in \mathscr{L}(H)$ be a strongly continuous cyclic unitary representation with cyclic vector h. Then,*

(i) *There exists a Radon probability μ on $\mathbb{R}^n$ such that*

$$(U_{x'} h, h)_H = \int_{\mathbb{R}^n} \exp(i(x, x'))\, d\mu(x), \qquad x' \in \mathbb{R}^n.$$

(ii) *There exists an isometry between H and $L^2(\mu)$ which transforms $U_{x'}$ into the operator of multiplication by* $\exp(i(x, x'))$, $x' \in E'$.

It is natural to ask if this theorem is true in general for infinite dimensional E. The answer is positive if, for instance, E is quasicomplete and E' is nuclear, as is well known (see **[5]**, p. 365, Th. 5; **[10]**, p. 236, Cor. 2 and p. 233, examples). When E is a separable Hilbert space the theorem is not true, since the unitary representation that may be associated to the Gaussian probability is strongly continuous, for instance. There exists actually a bijection between the strongly continuous cyclic unitary representations and the cylindrical probabilities on E which are scalarly concentrated on the balls of E (see

[**10**], p. 187, Prop. 2; p. 192, Def. 1; p. 193, Th. 1). In this paper we characterise the representations corresponding to Radon probabilities with bounded support. We use a theorem of Bochner type for the Fourier transforms of such measures. Finally, similar results are proved for semi-reflexive dual nuclear spaces.

By a Radon measure with bounded support on a locally convex space E we mean a Radon measure on E, concentrated on some bounded subset of E ([**4**], p. 116). Such measures on Hilbert spaces have been considered in [**7**] in relation to the Navier–Stokes equation. In [**2**], the authors study some special problems for measures with compact support on the real line.

1. Hilbert space case

In this section we suppose that E is a real separable Hilbert space. Let μ be a Radon probability on E for the weak topology, concentrated on a closed ball centered at origin $\Omega \subset E$. We consider the Hilbert space $L^2(\mu)$, topologised by the usual norm. If $x' \in E'$, the operator of multiplication by $\exp(i(.,x'))$ is a unitary operator on $L^2(\mu)$ whose adjoint is the multiplication by $\exp(-i(.,x'))$. We denote it by $M_{x'}$. We consider the representation $x' \in E' \mapsto M_{x'} \in \mathscr{L}(L^2(\mu))$. If x' converges to x'_0 in E', $\exp(i(.,x'))$ converges to $\exp(i(.,x'_0))$ uniformly on the ball Ω; therefore this representation is continuous (not only strongly continuous!). Moreover, the set of linear combinations of the functions $\exp(i(.,x'))$, $x \in E'$, is dense for the topology of uniform convergence on Ω in the space of complex weakly continuous functions on Ω ([**4**], p. 45; p. 105). In turn, the last space is dense in $L^2(\mu)$. It follows that the vector $f_0 \equiv 1$ is cyclic for the representation M. Let $\mathbb{C}$ be the field of complex numbers. Let $E'_{\mathbb{C}}$ denote the complexified space from E' with the product topology. We may extend that representation to $E'_{\mathbb{C}}$ by

$$M_{z'}(f) = \exp(i(.,z'))f(.), \qquad f \in L^2(\mu),$$

where

$$(x,z') = (x,x') + i(x,y'), \qquad x \in E,$$

if $z' = x' + iy' \in E'_{\mathbb{C}}$.

The fact that μ is concentrated on Ω implies $M_{z'} \in \mathscr{L}(L^2(\mu))$ (not unitary!). The mapping $z' \in E'_{\mathbb{C}} \mapsto M_{z'} \in \mathscr{L}(L^2(\mu))$ is continuous for the same reason as above. It is also a G-holomorphic function. Indeed, if $z'_1, z'_2 \in E'_{\mathbb{C}}$ and $g(\lambda) = M'_{z'_1 + \lambda z'_2}$, $\lambda \in \mathbb{C}$, then $I = \int_\gamma g(\lambda)\, d\lambda \in \mathscr{L}(L^2(\mu))$ for each closed path $\gamma \subset \mathbb{C}$, If $f_1, f_2 \in L^2(\mu)$,

$$I(f_1)(f_2) = \int_\gamma \int_\Omega \exp(i(x,z'_1)) \exp(i(x,z'_2))\lambda) f_1(x) \overline{f_2(x)}\, d\mu(x)\, d\lambda$$

$$= \int_\Omega \int_\gamma \exp(i(x,z'_1)) f_1(x) \overline{f_2(x)} \exp(i(x,z'_2)\lambda)\, d\lambda\, d\mu(x) = 0,$$

whence g is an entire function (vectorial Morera's theorem). Finally, $\|M_{z'}\|$

$=\sup\{\|M_{z'}(f)\|_2:\|f\|_2\leqq 1\}=\exp(r\|\operatorname{Im} z'\|)$, where r is the radius of Ω, $\operatorname{Im} z'=y'$, if $z'=x'+iy'$, $x', y'\in E'$. In short, we have verified that the representation $x'\in E'\mapsto M_{x'}\in\mathscr{L}(L^2(\mu))$ admits an entire extension to $E'_{\mathbb{C}}$ such that $\|M_{z'}\|\leqq\exp(r\|\operatorname{Im} z'\|)$, $z'\in E'_{\mathbb{C}}$, for certain $r\geqq 0$. Now, we proceed to prove the reciprocal assertion.

The following lemma is a consequence of Prokhorov's theorem and the arguments of [**10**], p. 189, Prop. 3, for cylindrical probabilities in Hilbert spaces. Let $\{P_n\}_{n=1}^{\infty}$ be the sequence of projections associated to a fixed orthonormal basis in E.

Lemma. *Let $(\mu_n)_{n=1}^{\infty}=(\mu_{E/P_n(E)^{\perp}})_{n=1}^{\infty}$ be a cylindrical probability on E such that there exists a ball $\Omega\subset E$, centered at origin, with $\mu_n(P_n(E)\backslash P_n(\Omega))=0$, $n\in\mathbb{N}$. Then, there exists a $\sigma(E,E')$-Radon probability μ on E, concentrated on Ω, with $\mu=(\mu_n)_{n=1}^{\infty}$.*

Theorem 1.1. *Let μ be a Radon probability on E, concentrated on the closed ball $\Omega_r=\{x\in E:\|x\|\leqq r\}$, $r>0$. Let*

$$\hat{\mu}(x')=\int_{\Omega_r}\exp(i(x,x'))\,d\mu(x),\qquad x'\in E',$$

be the Fourier transform of μ. Then,

(i) *$\hat{\mu}$ is a continuous function of positive type on E' such that $\hat{\mu}(0)=1$.*

(ii) *$\hat{\mu}$ may be extended to an entire function θ on $E'_{\mathbb{C}}$ such that $|\theta(z')|\leqq\exp(r\|\operatorname{Im} z'\|)$, $z'\in E'_{\mathbb{C}}$.*

Conversely, if θ is an entire function satisfying (ii) *and whose restriction to E' satisfies* (i) *there exists a Radon probability μ on E, concentrated on Ω_r and such that $\hat{\mu}=\theta$ on E'.*

Proof. Bochner's theorem for Hilbert spaces ([**10**], p. 239, Th. 3) proves part (i). Part (ii) is proved as for the representations. Conversely, part (i) implies that there is a cylindrical probability $(\mu_n)_{n=1}^{\infty}$ on E such that if

$$\theta_n: x'_n\mapsto\theta(x'_n\circ P_n),\quad x'_n\in E'_n=P_n(E)',\qquad n\in\mathbb{N},$$

then $\mu_n(x'_n)=\theta_n(x'_n)$.
([**10**], p. 187, Prop. 2). Moreover the function

$$z'_n\in E'_{n_{\mathbb{C}}}\mapsto\theta_n(z'_n)=\theta(z'_n\circ P_n)$$

is entire for each n and it verifies

$$|\theta_n(z'_n)|\leqq\exp(r\|\operatorname{Im} z'_n\|)\quad\text{for every}\quad z'_n\in E'_{n_{\mathbb{C}}}$$

(we still denote by P_n the obvious extension of P_n on $E_{\mathbb{C}}$). Because of the Paley–Wiener theorem there is a distribution T_n with support contained in the ball of radius r in $E_n=P_n(E)\cong\mathbb{R}^n$ such that

$$\hat{T}_n(x'_n)=T_n(\exp(i(.,x'_n)))=\theta_n(x'_n)=\hat{\mu}_n(x'_n),\qquad x'_n\in\mathbb{R}^n.$$

Let $\mathscr{S}_n$ be the space of all functions rapidly decreasing at infinity in $\mathbb{R}^n$. Let $\alpha, \beta \in \mathscr{S}_n$ so that β is the Fourier transform of α, $\beta = \hat{\alpha}$. Then,

$$
\begin{aligned}
T_n(\beta) = T_n(\hat{\alpha}) &= \int_{\mathbb{R}^n} \hat{T}_n(x')\alpha(x')\,dx \\
&= \int_{\mathbb{R}^n} \hat{\mu}_n(x')\alpha(x')\,dx' \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \alpha(x')\exp(i(x,x'))\,d\mu_n(x)\,dx' \\
&= \int_{\mathbb{R}^n} \hat{\alpha}(x)\,d\mu_n(x) = \int_{\mathbb{R}^n} \beta(x)\,d\mu_n(x).
\end{aligned}
$$

Therefore, the distributions μ_n, T_n coincide on $\mathscr{S}_n$ and thus $\mu_n = T_n$ $(n \in \mathbb{N})$. It follows that μ_n has support contained in Ω_r and, according to the preceding lemma, there exists a Radon probability μ on E, concentrated on Ω_r, such that $\mu = (\mu_n)_{n=1}^{\infty}$.

Theorem 1.2. *Let $U: x' \in E' \mapsto U_{x'} \in \mathscr{L}(H)$ be a continuous cyclic unitary representation with cyclic vector h. If U admits an entire extension $\tilde{U}: E'_{\mathbb{C}} \mapsto \mathscr{L}(H)$ verifying $\|\tilde{U}_{z'}\| \leqq \exp(r\|\mathrm{Im}\, z'\|)$, $z' \in E'_{\mathbb{C}}$, for a certain $r > 0$, then*

(i) *There exists a Radon probability μ on E, concentrated on Ω_r, such that*

$$(U_{x'}h, h)_H = \int_{\Omega_r} \exp(i(x,x'))\,d\mu(x), \qquad x' \in E'.$$

(ii) *The equality of part* (i) *defines an isometric correspondence between H and $L^2(\mu)$, so that the operator corresponding by this isometry to $U_{x'}$ is the operator of multiplication by* $\exp(i(.,x'))$.

Proof. It is standard. The function $x' \in E' \mapsto \theta(x') = (U_{x'}h, h)_H$ is continuous and of positive type such that $\theta(0) = 1$. Moreover $z' \in E'_{\mathbb{C}} \mapsto (\tilde{U}_{z'}h, h)_H$ is an entire function and $|\theta(z')| \leqq \|\tilde{U}_{z'}\| \leqq \exp(r\|\mathrm{Im}\, z'\|)$, $z' \in E'_{\mathbb{C}}$. Theorem (1.1) implies part (i). The isometry of part (ii) is proved by associating to each vector $\sum_{k=1}^{n} \lambda_k U_{x'_k} h \in H$ the function $\sum_{k=1}^{n} \lambda_k \exp(i(.,x'_k)) \in L^2(\mu)$, and applying obvious arguments about density in H and $L^2(\mu)$.

2. Nuclear space case

Henceforth we suppose that E is a semi-reflexive (i.e. $E = E''$ algebraically) dual nuclear locally convex space (see [**9**]). Let $\mathscr{B}$ be the family of all subsets of E which are closed, bounded, balanced and convex. If $B \in \mathscr{B}$, let B^0 denote the polar set of B in E'. We suppose E' endowed its strong topology or topology of uniform convergence on the elements of $\mathscr{B}$. This is defined by the seminorms $q_{B^0}(x') = \sup_{x \in B} |(x,x')| = \|x'\|_B$, $x' \in E'$, where q_{B^0} is the gauge of $B^0 \subset E'$, $B \in \mathscr{B}$. Let E_B denote the linear subspace of E spanned by B and normed by the gauge of B. Let E'_{B^0} denote the quotient space $E'/q_{B^0}^{-1}(0)$ topologised by the norm defined by q_{B^0}. If $x' \in E'$, let $[x']$ denote the image through the canonical surjection $E' \to E'_{B0}$. Obviously, this surjection induces a surjection $\pi_B: E'_{\mathbb{C}} \to (E'_{B0})_{\mathbb{C}}$. If $\theta: E'_{\mathbb{C}} \to \mathbb{C}$ is an entire function (G-entire and continuous) we say that θ is factorisable if there is $B \in \mathscr{B}$ and $\eta:(E'_{B0})_{\mathbb{C}} \to \mathbb{C}$, where η is entire, such that $\theta = \eta \circ \pi_B$. By Radon probabilities we mean the ones relative to weak topology $\sigma(E, E')$.

Theorem 2.1. *Let μ be a Radon probability on E, concentrated on $B \in \mathscr{B}$. Let $\hat{\mu}(x') = \int_B \exp(i(x, x'))\, d\mu(x)$, $x' \in E'$, be its Fourier transform. Then,*

(i) *$\hat{\mu}$ is a continuous function of positive type on E' and $\hat{\mu}(0) = 1$.*

(ii) *μ may be extended to a factorisable entire function θ on $E'_{\mathbb{C}}$ such that $|\theta(z')| \leq \exp(\|\mathrm{Im}\, z'\|_B)$, $z' \in E'_{\mathbb{C}}$.*

Conversely, if θ is an entire function on $E'_{\mathbb{C}}$ satisfying (ii), *whose restriction to E' satisfies* (i) *there exists a Radon probability μ on E, concentrated on some element of $\mathscr{B}$ and such that $\hat{\mu} = \theta$ on E'.*

Proof. Let μ be a Radon probability. Part (i) is a consequence of [**10**], p. 193, Th. 1. It makes sense to define $\hat{\mu}(z') = \int_B \exp(i(x, z'))\, d\mu(x)$, for every $z' \in E'_{\mathbb{C}}$, and it is easy to prove that $\hat{\mu}$ is a G-holomorphic and continuous function on $E'_{\mathbb{C}}$.

Now, without loss of generality, we may suppose that E'_{B0} is a separable pre-Hilbert space since E' is nuclear. The topological dual of E'_{B0} coincides with $E''_{B00} = E_B$ (E is semireflexive) and the strong topology on E_B is the one defined by the gauge of B. Thus E_B is a separable Hilbert space whose topological dual is the completion of E'_{B0}. The continuity of the injection $E_B \to E$ implies the identity of topologies $\sigma(E_B, (E_B)')$, $\sigma(E, E')$ on B. Let v be the measure μ considered as Radon measure on the Hilbert space E_B. According to previous arguments, the function $\hat{v}:(E'_{B0})_{\mathbb{C}} \to \mathbb{C}$ in entire and, for $[x']$, $[y'] \in E'_{B0}$,

$$\begin{aligned} \hat{v}([x'] + i[y']) &= \int_B \exp(i(x, [x'] + i[y'])\, d\mu(x) \\ &= \int_B \exp(i(x, [x'])) \exp(-(x, [y']))\, d\mu(x) \\ &= \int_B \exp(i(x, x')) \exp(-(x, y'))\, d\mu(x) = \hat{\mu}(x' + iy'), \end{aligned}$$

i.e., $\hat{v} \circ \pi_B = \hat{\mu}$.

Conversely, we may suppose that E_B is a Hilbert space and θ is factorisable through E'_{B0}, i.e., there is $\eta:(E'_{B0})_{\mathbb{C}} \to \mathbb{C}$ an entire function, such that $\theta = \eta \circ \pi_B$. Evidently, the restriction of η to E'_B is a continuous function of positive type. Moreover, for every $z' \in E'_{\mathbb{C}}$,

$$|\eta(\pi_B z')| = |\theta(z')| = \exp(\|\mathrm{Im}\, z'\|_B) = \exp \|\mathrm{Im}\, \pi_B z'\|_B.$$

Therefore, the natural extension $\bar{\eta}$ of η to the completion of $(E'_{B0})_{\mathbb{C}}$ verifies (i) and (ii) of Theorem 1.1. Thus, there exists a Radon probability μ on E_B concentrated on B, such that $\hat{\mu}([x']) = \eta([x']) = \theta(x')$ for every $x' \in E'$. If we consider μ as a Radon measure on E, then $\hat{\mu}(x') = \hat{\mu}([x']) = \theta(x')$, for every $x' \in E'$, and the proof is finished.

Remark. Theorem 2.1 is also true if the word "factorisable" is dropped in condition (ii), because every entire function θ on $E'_{\mathbb{C}}$ satisfying

$$|\theta(z')| \leqq \exp(\|\operatorname{Im} z'\|_B), \qquad z' \in E'_{\mathbb{C}}$$

is factorisable. It is enough to prove that $\theta(z'_1) = \theta(z'_2)$ for $z'_1, z'_2 \in E'_{\mathbb{C}}$ with $z'_1 = z'_2$ on B. Using Cauchy's inequalities the last identity is verified by each n-linear mapping of the Taylor series of θ at the origin. Therefore, it is also verified by θ.

By means of arguments similar to those of Sections 1 and 2 and some simple additions, we obtain a bijective correspondence similar to that of Theorem 1.2.

Remark. We may state analogous results in the setting of bornological linear spaces, under suitable restrictions.

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