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AN EXTREMAL PROBLEM FOR HARMONIC FUNCTIONS IN THE BALL

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ABSTRACT. In this note we obtain a sharp estimate for a radial derivative of bounded harmonic functions in the ball.

The celebrated Schwarz-Pick Lemma for analytic functions in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ states that for $f : \mathbb{D} \longrightarrow \mathbb{D}$, analytic

$$(1) \qquad |f'(z)| \leq \frac{1}{1-|z|^2}, \quad z \in \mathbb{D} \text{ fixed}$$

and the equality holds if and only if f is a Möbius transformation which sends z into the origin (cf. [G, Lemma 1.2]).

In this note we indicate an elementary argument that allows one to obtain estimates similar to (1) for magnitudes of derivatives of bounded harmonic functions in the unit ball in $\mathbb{R}^n$. Since in that case the right-hand side is not nearly as pretty as in (1), we restrict ourselves to the case of a radial derivative for $n = 3$.

Let $B = \{x \in \mathbb{R}^3 : \sum_1^3 x_i^2 < 1\}$ be the unit ball, $S^2 = \partial B$.

THEOREM. *For u harmonic in B, $\|u\| \leq 1$ and $x^0 \in B$-fixed we have*

$$(2) \qquad \left|\frac{\partial u}{\partial |x|}\Big|_{x^0}\right| \leq \frac{(9-|x^0|^2)^2}{3\sqrt{3}(1-|x^0|^2)\big[(|x^0|^2+3)^{3/2}+3\sqrt{3}(1-|x^0|^2)\big]}.$$

(2) is sharp and equality holds if and only if $u = \pm u_0$, where u_0 equals $+1$ on a "spherical cap" $0 \leq \theta \leq \theta_0 = \arccos \frac{5|x^0|-|x^0|^3}{|x^0|^2+3}$, and -1 on the rest of the sphere. (θ is the latitude with respect to the axis passing through x^0 and the origin.)

NOTE. For $|x^0| \longrightarrow 1$ the left-hand side in (2) tends to $8/3\sqrt{3}(1-|x^0|^2)^{-1}$. This provides a sharp asymptotic estimate on the growth of the normal derivative of u near S^2.

PROOF. Choose our coordinate system so that the x_3-axis passes through x^0 and switch to spherical coordinates $x_1 = r\sin\theta\cos\varphi$, $x_2 = r\sin\theta\sin\varphi$, $x_3 = r\cos\theta$, $0 \leq r \leq 1$, $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$, so $\frac{\partial u}{\partial |x|}\Big|_{x^0} = \frac{\partial u}{\partial r}\Big|_{(r_0,0,0)}$. Writing down the Poisson integral representation for u (see [K, Ch. VIII, § 4]), we have

$$(3) \qquad u(x) = \frac{1}{4\pi}\int_{S^2} \frac{1-|x|^3}{|x-y|^3} u(y)\, d\sigma(y),$$

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where $d\sigma$ is Lebesgue measure on S^2. Whence, in spherical coordinates,

$$u(r,0,0) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{1-r^2}{(1+r^2-2r\cos\theta)^{3/2}}u(\theta,\varphi)\sin\theta\,d\theta\,d\varphi. \tag{4}$$

Differentiating (4) with respect to r we obtain after some algebraic manipulations

$$\begin{aligned}\frac{\partial u(r,0,0)}{\partial r}\Big|_{r=r_0} &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}u(\theta,\varphi)\frac{r_0^3+r_0^2\cos\theta-5r_0+3\cos\theta}{(1+r_0^2-2r_0\cos\theta)^{5/2}}\sin\theta\,d\theta\,d\varphi\\ &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}u(\theta,\varphi)f(\theta)\sin\theta\,d\theta\,d\varphi.\end{aligned} \tag{5}$$

Obviously, the maximum M in (5) is attained if and only if $u(\theta,\varphi) = \operatorname{sign} f(\theta) := \begin{cases}+1, & f(\theta)\geq 0,\\ -1, & f(\theta)\leq 0\end{cases}$. It is easy to see that $f(\theta)$ changes sign on $[0,\pi]$ only once, at θ_0 : $\cos\theta_0 = \frac{5r_0-r_0^3}{r_0^2+3}$ (≤ 1, as $0\leq r_0\leq 1$). Setting $\cos\theta = t$ we obtain

$$\begin{aligned}M &= \frac{1}{2}\int_0^{\pi}|f(\cos\theta)|\sin\theta\,d\theta = \frac{1}{2}\int_{-1}^{1}|f(t)|\,dt\\ &= \frac{1}{2}\Big[\int_{-1}^{t_0}f(t)\,dt-\int_{t_0}^{1}f(t)\,dt\Big] = F(t_0)-\frac{1}{2}\big(F(1)+F(-1)\big),\end{aligned} \tag{6}$$

where $F(t) = \int f(t)\,dt$, $t_0 = \cos\theta_0 = \frac{5r_0-r_0^3}{r_0^2+3}$. After elementary but tedious calculations one finds

$$\begin{aligned}F(t) :&= \int\frac{(5r_0-r_0^3)-(r_0^2+3)t}{(1+r_0^2-2r_0t)^{5/2}}\\ &= \frac{2(r_0^2+3)}{3(2r_0)^{5/2}}\left[\frac{1+r_0^2}{r_0}+t_0-3t\right]\left[\frac{1+r_0^2}{r_0}-t\right]^{-3/2}.\end{aligned} \tag{7}$$

Substituting (7) into (6) and carefully following all the "nice" cancellations that come along we obtain (2).

COROLLARY. *For u as above*

$$\|\operatorname{grad} u|_{x=0}\| \leq \frac{3}{2}. \tag{8}$$

The equality holds if and only if u equals $+1$ *on a hemisphere and* -1 *on the remaining hemisphere.*

REMARK. This corresponds very well to the physical intuition: the largest electrostatic force at the origin occurs as we keep the potential equal to $+1$ on one hemisphere and -1 on the other hemisphere.

Although (8) follows immediately from (2) by letting $x_0 = 0$, we would like to give an independent (short) proof. From (3) it follows that for any $j = 1,2,3$

$$\begin{aligned}\frac{\partial u}{\partial x_j}\Big|_{x=0} &= \frac{1}{4\pi}\int_{S^2}\frac{-2x_j|x-y|^3+3\frac{x_i-y_j}{|x-y|^5}}{|x-y|^6}u(y)\,d\sigma(y)\Big|_{x=0}\\ &= -\frac{3}{4\pi}\int_{S^2}y_ju(y)\,d\sigma(y).\end{aligned}$$

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Thus, for $j = 1, 2, 3$

$$\max |\partial_j u(0)| = \frac{3}{4\pi}\|y_j\|_{L^1(\sigma)} = \frac{3}{4\pi}\cdot 2\pi = \frac{3}{2}.$$

REMARKS. (i) For $n = 2$, a similar argument yields the following analogue of (8) for $u : \|u\| \leq 1$, harmonic in $\mathbb{D}$

$$\|\operatorname{grad} u|_{z=0}\| \leq \frac{2}{2\pi}\|\operatorname{Re} z\|_{L^1(\mathbb{T})} = \frac{4}{\pi} \tag{9}$$

($\mathbb{T} = \partial\mathbb{D} = \{z : |z| = 1\}$). From this, arguing as above or following Pick's proof of the invariant form of Schwarz' Lemma one easily obtains for such u:

$$\left|\frac{\partial u}{\partial |z|}\Big|_{z=z^0}\right| \leq \frac{4}{\pi(1-|z^0|^2)} \tag{10}$$

and equality only holds for $\pm u_0$, where

$$u_0 : u_0|_{\mathbb{T}} = \begin{cases} +1, & |\theta - \arg z^0| \leq \arccos \frac{2|z^0|}{1+|z^0|^2} \\ -1, & \text{elsewhere.} \end{cases}$$

Moreover, since Möbius automorphisms of the disk preserve harmonic functions, we can see at once that (10) holds if one replaces $\|\frac{\partial u}{\partial |z|}\|$ by $\|\operatorname{grad} u\|$ with extremal functions being those of (9) composed with an appropriate Möbius transformation.

Unfortunately, this is no longer true in $\mathbb{R}^n$, $n \geq 3$, since Möbius automorphisms of the ball preserve harmonicity only up to a non-constant scalar factor. Thus, the problem of finding $\max\{\|\operatorname{grad} u|_{X^0 \in B}\| : \Delta u = 0, \|u\|_\infty \leq 1\}$ transfers into a much more complicated extremal problem at the origin as $n \geq 3$.

(ii) It is not hard to see that in $\mathbb{R}^n$ the constant in the right-hand side of (8) behaves as $\sqrt{n}$, for $n \longrightarrow \infty$.

(iii) Professor A. Weitsman pointed out that an easy proof of the Corollary can also be obtained by applying the standard symmetrization technique, as e.g. in [B].

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