ON INTEGRAL FUNCTIONS HAVING PRESCRIBED ASYMPTOTIC GROWTH. II

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1. One of the authors published in 1965 a paper with identical title **(1)**, in which the following result was proved:

THEOREM A. *Let* $\phi(r)$ *be increasing and convex in* $\log r$, *with*

$$\phi(r) \neq O(\log r) \qquad (r \to \infty).$$

Then there is an integral function $f(z)$ *such that as* $r \to \infty$

(i) $\log M(r,f) \sim \phi(r)$,

(ii) $T(r,f) \sim \phi(r)$.

In the present paper various improvements of this result will be discussed. In § 2 we shall show that by a suitable modification of the original construction one can make sure that in addition to (i) and (ii) also

(iii) $N(r, 1/(f-c)) \sim \phi(r) \qquad (r \to \infty)$

is satisfied for every finite constant c. This improves a result of Edrei and Fuchs **(2)**. In § 3 we use a different construction to prove that (i) can be replaced by

$$|\log M(r,f) - \phi(r)| < \tfrac{1}{2}\log r + \log 3. \tag{1.1}$$

In § 4 we show, by means of an example, that (1.1) is essentially the best possible. Finally, in § 5 we prove that if $\phi(r)$ satisfies an additional condition, then the right-hand side of (1.1) can be replaced by a constant.

2. In this section we shall prove the following theorem.

THEOREM 1. *Let* $\phi(r)$ *be any real function of* r *which is increasing and convex in* $\log r$ *and such that* $\phi(r) \neq O(\log r)$ $(r \to \infty)$. *Then there is an integral function* $f(z)$ *satisfying* (i) *and* (ii), *i.e.*

$$\begin{aligned} \log M(r,f) &\sim \phi(r) \qquad (r \to \infty), \\ T(r,f) &\sim \phi(r) \qquad (r \to \infty), \end{aligned} \tag{2.1}$$

and also (iii), *i.e.*

$$N(r, 1/(f-c)) \sim T(r,f) \qquad (r \to \infty) \tag{2.2}$$

for any finite constant c.

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In **(1)** it was shown that there is an integral function $g(z)$ satisfying (2.1). The series for this function has non-negative coefficients and for large r the series is dominated by at most three terms relative to the maximum term. This means that

$$g(z) = \sum_{0}^{\infty} b_n z^n,$$

where $b_n \geqslant 0$ $(n \geqslant 0)$ and to each $r \geqslant 0$ there corresponds three integers n_1, n_2, n_3, such that

$$g(z) = b_{n_1} z^{n_1} + n_{n_2} z^{n_2} + n_{n_3} z^{n_3} + o(\mu(r, g)) \qquad (r \to \infty, \ |z| = r).$$

For some values of r it may be possible to absorb one or two of the displayed terms in $o(\mu(r, g))$. In the context of **(1)** the question did not arise as to whether or not each non-zero term of the series of $g(z)$ became in turn the maximum term. However, for our present purposes it is an advantage for $g(z)$ to have this property. We can in fact assume that it does have this property without loss of generality, since if it did not, then non-zero terms which do not become maximum terms for any value of r could be dropped from the series and the resulting function would still satisfy (2.1). That this is so is clear from the proof of **(1**, Theorem A**)**.

We shall arrive at a function $f(z)$ satisfying both (2.1) and (2.2) by dropping certain terms from the series of $g(z)$. In what follows we shall relate the relevant asymptotic behaviour of this $f(z)$ to $\log \mu(r, g)$. It is obvious, from the nature of the series of $g(z)$, that $\log M(r, g) \sim \log \mu(r, g)$ $(r \to \infty)$ and consequently this will not affect the validity of our results.

Let

$$g(z) = \sum_{1}^{\infty} A_{\lambda_n} r^{\lambda_n}$$

satisfy (2.1) and suppose that $A_{\lambda_n} > 0$ $(n \geqslant 1)$ and that $A_{\lambda_n} r^{\lambda_n}$ is the maximum term of $g(z)$ for $r_n \leqslant r < r_{n+1}$ $(n \geqslant 1)$. Since each term of the series of $g(z)$ is in turn the maximum term, it follows easily that when r satisfies $r_n \leqslant r \leqslant r_{n+1}$ the sequence $\{A_{\lambda_\nu} r^{\lambda_\nu}\}_{\nu \geqslant 1}$ is non-decreasing for $\nu \leqslant n$ and non-increasing for $\nu \geqslant n$. For each $r \geqslant 0$ we let J_r be a set of three integers which includes the suffices n of the ranks λ_n of the dominant terms of $g(z)$ for $|z| = r$. In particular we define, for each $r \geqslant 0$, J_r to consist of three integers such that if $n \in J_r$ and $\nu \notin J_r$, then $A_{\lambda_n} r^{\lambda_n} \geqslant A_{\lambda_\nu} r^{\lambda_\nu}$. Since $\sum_{1}^{\infty} A_{\lambda_n} r^{\lambda_n}$ is dominated by three of its terms at most relative to the maximum term, as $r \to \infty$ it follows that

$$\max_{r_n \leqslant r \leqslant r_{n+1}} \frac{\sum_{1}^{\infty} A_{\lambda_\nu} r^{\lambda_\nu} - \sum_{\nu \in J_r} A_{\lambda_\nu} r^{\lambda_\nu}}{\mu(r, g)} = \delta_n \to 0 \qquad (n \to \infty).$$

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If λ_n is the central index of $g(z)$ for $|z| = r$, then J_r will consist of three of the integers $n-2$, $n-1$, n, $n+1$, $n+2$.

We have, for $n \geqslant 2$,

$$\frac{A_{\lambda_{n-1}} r^{\lambda_{n-1}}}{A_{\lambda_n} r^{\lambda_n}} \begin{cases} = 1 & (r = r_n), \\ < 1 & (r > r_n), \end{cases}$$

and

$$\frac{A_{\lambda_{n+1}} r^{\lambda_{n+1}}}{A_{\lambda_n} r^{\lambda_n}} \begin{cases} = 1 & (r = r_{n+1}), \\ < 1 & (r < r_{n+1}). \end{cases}$$

Hence for $n \geqslant 2$ there is a (unique) value ρ_n satisfying $r_n < \rho_n < r_{n+1}$ such that

$$A_{\lambda_{n-1}} \rho_n^{\lambda_{n-1}} = A_{\lambda_{n+1}} \rho_n^{\lambda_{n+1}} = K_n, \text{ say.}$$

Observe that in $r_n \leqslant r < \rho_n$ we have $A_{\lambda_{n-1}} r^{\lambda_{n-1}} > A_{\lambda_{n+1}} r^{\lambda_{n+1}}$ and in $\rho_n < r \leqslant r_{n+1}$ we have $A_{\lambda_{n+1}} r^{\lambda_{n+1}} > A_{\lambda_{n-1}} r^{\lambda_{n-1}}$. Hence, from the monotonic nature of $\{A_{\lambda_\nu} r^{\lambda_\nu}\}_{\nu \geqslant 1}$, it follows that for $r_n \leqslant r \leqslant \rho_n$ the set J_r contains $n-1$ and n and for $\rho_n \leqslant r \leqslant r_{n+1}$ the set J_r contains n and $n+1$. Therefore, for $n \geqslant 2$ and r satisfying $\rho_n \leqslant r \leqslant \rho_{n+1}$ the set J_r contains n and $n+1$.

Let $A_{\lambda_n} r_n^{\lambda_n} = \eta_n$ $(n \geqslant 1)$ and define two complementary sets I_1 and I_2 of the integers $n \geqslant 2$ as follows: $n \in I_1$ if

$$K_n \geqslant [\sqrt{\delta_n} + 1/(\log \eta_n)]\, A_{\lambda_n} \rho_n^{\lambda_n}$$

and $n \in I_2$ otherwise. We now define a subsequence $\{\mu_\nu\}$ of $\{\lambda_\nu\}$ recursively in the following manner. Take $\mu_1 = \lambda_1$ and suppose that $\mu_1, \mu_2, \ldots, \mu_\nu$ have been specified. Suppose $\mu_\nu = \lambda_n$. Define $\mu_{\nu+1} = \lambda_{n+1}$ if $n+1 \in I_2$ and $\mu_{\nu+1} = \lambda_{n+2}$ if $n+1 \in I_1$. It is clear that the subsequence μ_ν does not omit two consecutive λ_ν's. We shall show that the function

$$f(z) = \sum_{\nu=1}^{\infty} A_{\mu_\nu} z^{\mu_\nu}$$

satisfies both (2.1) and (2.2) of Theorem 1.

LEMMA 1. *$f(z)$ satisfies* (2.1).

Consider first of all those intervals $[r_n, r_{n+1}]$ such that λ_n occurs in the subsequence $\{\mu_\nu\}$. Clearly the proof of (**1**, Theorem A) is applicable to $f(z)$ in such intervals and so

$$\left.\begin{matrix} T(r,f) \\ \log M(r,f) \end{matrix}\right\} \sim \log \mu(r,f) = \log \mu(r,g) \qquad (r_n \leqslant r \leqslant r_{n+1},\ r \to \infty).$$

Consider now those intervals $[r_n, r_{n+1}]$ such that λ_n does not occur in the subsequence $\{\mu_\nu\}$. As we have already pointed out, $\{\mu_\nu\}$ cannot omit two consecutive λ_ν's and hence in this case λ_{n-1}, λ_{n+1} both occur in $\{\mu_\nu\}$. From the construction of $\{\mu_\nu\}$ it follows that $n \in I_1$ and so

$$\frac{A_{\lambda_{n-1}} \rho_n^{\lambda_{n-1}}}{A_{\lambda_n} \rho_n^{\lambda_n}} = \frac{A_{\lambda_{n+1}} \rho_n^{\lambda_{n+1}}}{A_{\lambda_n} \rho_n^{\lambda_n}} \geqslant \sqrt{\delta_n} + \frac{1}{\log \eta_n}. \tag{2.3}$$

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Hence

$$\frac{A_{\lambda_{n-1}} r^{\lambda_{n-1}}}{A_{\lambda_n} r^{\lambda_n}} \geqslant \sqrt{\delta_n} + \frac{1}{\log \eta_n} \qquad (r_n \leqslant r \leqslant \rho_n)$$

and

$$\frac{A_{\lambda_{n+1}} r^{\lambda_{n+1}}}{A_{\lambda_n} r^{\lambda_n}} \geqslant \sqrt{\delta_n} + \frac{1}{\log \eta_n} \qquad (\rho_n \leqslant r \leqslant r_{n+1}).$$

From the monotonic nature of $\{A_{\lambda_\nu} r^{\lambda_\nu}\}_{\nu \geqslant 1}$ and the definition of J_r we see that J_{ρ_n} consists of $\lambda_{n-1}, \lambda_n, \lambda_{n+1}$. Therefore, by the definition of δ_n,

$$\left(\sum_{\nu \leqslant n-2} + \sum_{\nu \geqslant n+2}\right) A_{\lambda_\nu} \rho_n^{\lambda_\nu} \leqslant \delta_n A_{\lambda_n} \rho_n^{\lambda_n}. \tag{2.4}$$

The series for $f(z)$ contains $A_{\lambda_{n-1}} z^{\lambda_{n-1}}$ and $A_{\lambda_{n+1}} z^{\lambda_{n+1}}$ and all its terms, with the exception perhaps of $A_{\lambda_{n-2}} z^{\lambda_{n-2}}$ and $A_{\lambda_{n+2}} z^{\lambda_{n+2}}$, are contained in

$$\left(\sum_{\nu < n-2} + \sum_{\nu > n+2}\right) A_{\lambda_\nu} z^{\lambda_\nu}.$$

Consider the interval $[r_n, \rho_n]$. For $r_n \leqslant r \leqslant \rho_n$ the sum of the terms of $f(r)$ with at most three exceptions is not more than

$$\left(\sum_{\nu < n-2} + \sum_{\nu \geqslant n+2}\right) A_{\lambda_\nu} r^{\lambda_\nu}.$$

For this sum we find that for $r_n \leqslant r \leqslant \rho_n$,

$$\begin{aligned} \frac{\sum_{\nu < n-2} + \sum_{\nu \geqslant n+2} A_{\lambda_\nu} r^{\lambda_\nu}}{\mu(r,f)} &\leqslant \frac{\sum_{\nu < n-2} A_{\lambda_\nu} r^{\lambda_\nu}}{A_{\lambda_{n-1}} r^{\lambda_{n-1}}} + \frac{\sum_{\nu \geqslant n+2} A_{\lambda_\nu} r^{\lambda_\nu}}{A_{\lambda_{n-1}} r^{\lambda_{n-1}}} \\ &\leqslant \frac{\sum_{\nu < n-2} A_{\lambda_\nu} r_n^{\lambda_\nu}}{A_{\lambda_{n-1}} r_n^{\lambda_{n-1}}} + \frac{\sum_{\nu \geqslant n+2} A_{\lambda_\nu} \rho_n^{\lambda_\nu}}{A_{\lambda_{n-1}} \rho_n^{\lambda_{n-1}}} \\ &\leqslant \delta_n + \sqrt{\delta_n}, \end{aligned} \tag{2.5}$$

where we have used the fact that $A_{\lambda_{n-1}} r_n^{\lambda_{n-1}} = \mu(r_n, g)$ and that J_{r_n} cannot contain any $\nu < n - 2$ in the first estimate, and (2.3) and (2.4) in the second. Consideration of the interval $[\rho_n, r_{n+1}]$ in a similar fashion shows that the series for $f(z)$ with at most three terms omitted when compared with $\mu(r,f)$ satisfies the bound given in (2.5).

Hence the series of $f(z)$ relative to its maximum term is dominated by at most three of its terms. Therefore, as in **(1)**,

$$\left.\begin{matrix} \log M(r,f) \\ T(r,f) \end{matrix}\right\} \sim \log \mu(r,f) \qquad (r \to \infty).$$

To complete Lemma 1 it only remains to show that

$$\log \mu(r,f) \sim \log \mu(r, g) \qquad (r \to \infty). \tag{2.6}$$

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We consider $r \to \infty$ with $r_n \leqslant r \leqslant r_{n+1}$. If λ_n occurs in the subsequence $\{\mu_\nu\}$, then the result is obvious. If λ_n does not occur in the subsequence μ_ν, then $\lambda_{n-1}, \lambda_{n+1}$ do occur and (2.3) is satisfied. In this case

$$(2.7) \qquad A_{\lambda_{n-1}} r^{\lambda_{n-1}} \geqslant \mu(r,f)/(\log \eta_n) \qquad (r_n \leqslant r \leqslant \rho_n)$$

and

$$(2.8) \qquad A_{\lambda_{n+1}} r^{\lambda_{n+1}} \geqslant \mu(r,f)/(\log \eta_n) \qquad (\rho_n \leqslant r \leqslant r_{n+1}).$$

Since $\log \log \eta_n = o(\log \mu(r,f))$ $(r_n \leqslant r \leqslant r_{n+1}, r \to \infty)$, we see that (2.7) and (2.8) give (2.7) as $r \to \infty$ through values under consideration.

This completes the proof of Lemma 1.

LEMMA 2. *The series of $f(z)$ relative to the maximum term is dominated by at most two terms.*

We consider first of all $r \to \infty$ through values $r_n \leqslant r \leqslant r_{n+1}$, where λ_n does not occur in the subsequence $\{\mu_\nu\}$. We deal separately with $r_n \leqslant r \leqslant \rho_n$ and $\rho_n \leqslant r \leqslant r_{n+1}$. From the proof of Lemma 1 it follows that what we have to show is that if λ_{n-2} occurs in $\{\mu_\nu\}$, then

$$(2.9) \qquad A_{\lambda_{n-2}} r^{\lambda_{n-2}}/\mu(r,f) \to 0 \qquad (r_n \leqslant r \leqslant \rho_n, \ r \to \infty),$$

and if λ_{n+2} occurs in $\{\mu_\nu\}$, then

$$(2.10) \qquad A_{\lambda_{n+2}} r^{\lambda_{n+2}}/\mu(r,f) \to 0 \qquad (\rho_n \leqslant r \leqslant r_{n+1}, \ r \to \infty).$$

If λ_{n-2} is in $\{\mu_\nu\}$, then, since λ_{n-1} is also in $\{\mu_\nu\}$, we see, from the construction of $\{\mu_\nu\}$, that

$$\frac{A_{\lambda_{n-2}} \rho_{n-1}{}^{\lambda_{n-2}}}{A_{\lambda_{n-1}} \rho_{n-1}{}^{\lambda_{n-1}}} < \sqrt{\delta_{n-1}} + \frac{1}{\log \eta_{n-1}}$$

so that, as $\rho_{n-1} < r_n$,

$$\frac{A_{\lambda_{n-2}} r^{\lambda_{n-2}}}{A_{\lambda_{n-1}} r^{\lambda_{n-1}}} < \sqrt{\delta_{n-1}} + \frac{1}{\log \eta_{n-1}} \qquad (r_n \leqslant r \leqslant \rho_n).$$

From this inequality, (2.9) follows. In a similar manner if λ_{n+2} occurs in $\{\mu_\nu\}$, then

$$\frac{A_{\lambda_{n+2}} \rho_{n+1}{}^{\lambda_{n+2}}}{A_{\lambda_{n+1}} \rho_{n+1}{}^{\lambda_{n+1}}} < \sqrt{\delta_{n+1}} + \frac{1}{\log \eta_{n+1}}$$

and so, as $\rho_{n+1} > r_{n+1}$,

$$\frac{A_{\lambda_{n+2}} r^{\lambda_{n+2}}}{A_{\lambda_{n+1}} r^{\lambda_{n+1}}} < \sqrt{\delta_{n+1}} + \frac{1}{\log \eta_{n+1}} \qquad (\rho_n \leqslant r \leqslant r_{n+1}).$$

From this inequality (2.10) follows.

We consider next $r \to \infty$ through values $r_n \leqslant r \leqslant r_{n+1}$, where λ_n occurs in the subsequence $\{\mu_\nu\}$. Again we deal separately with the cases $r_n \leqslant r \leqslant \rho_n$

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and $\rho_n \leqslant r \leqslant r_{n+1}$. Consider $r_n \leqslant r \leqslant \rho_n$. Suppose at first that λ_{n-1} does not occur in $\{\mu_\nu\}$. As was pointed out earlier in $r_n \leqslant r \leqslant \rho_n$, two of the three largest terms of $g(r)$ are $A_{\lambda_{n-1}} r^{\lambda_{n-1}}$ and $A_{\lambda_n} r^{\lambda_n}$. In the present case, when we are assuming that λ_n occurs in $\{\mu_\nu\}$ but λ_{n-1} does not, it follows that as $r \to \infty$ $(r_n \leqslant r \leqslant \rho_n)$ there can be at most only one other term of $f(r)$ comparable with $\mu(r,f)$. Thus in this case the series for $f(r)$ is dominated by at most two terms.

Suppose now that $r_n \leqslant r \leqslant \rho_n$ and both λ_{n-1} and λ_n occur in $\{\mu_\nu\}$. In this case we have

$$\frac{A_{\lambda_{n+1}} \rho_n^{\lambda_{n+1}}}{A_{\lambda_n} \rho_n^{\lambda_n}} < \sqrt{\delta_n} + \frac{1}{\log \eta_n}$$

and so

$$A_{\lambda_{n+1}} r^{\lambda_{n+1}} = o(\mu(r,f)) \qquad (r \to \infty,\ r_n \leqslant r \leqslant \rho_n). \tag{2.11}$$

From (2.4),

$$\sum_{\nu \geqslant n+2} A_{\lambda_\nu} r^{\lambda_\nu} = o(\mu(r,f)) \qquad (r \to \infty,\ r_n \leqslant r \leqslant \rho_n). \tag{2.12}$$

From (2.11) and (2.12) it follows that the only possible term of $f(r)$ other than $A_{\lambda_{n-1}} r^{\lambda_{n-1}}$ which is comparable with $\mu(r,f)$ as $r \to \infty$ $(r_n \leqslant r \leqslant \rho_n)$ in the present case is $A_{\lambda_{n-2}} r^{\lambda_{n-2}}$, if this in fact does occur in the series of $f(z)$. If it does not, then we have the required result at once. If it does, then (2.9) shows that $A_{\lambda_{n-2}} r^{\lambda_{n-2}}$ is small relative to $\mu(r,f)$ as $r \to \infty$ under our present assumptions. Note that (2.9) is valid provided λ_{n-2}, λ_{n-1} occur in $\{\mu_\nu\}$. Hence we obtain the required result in this case also.

Similar considerations give the result as $r \to \infty$ with $\rho_n \leqslant r \leqslant r_{n+1}$ and λ_n occurring in $\{\mu_\nu\}$.

Hence the proof of Lemma 2 is complete, since we have shown it to be true in all possible cases.

LEMMA 3. *Let $h(z) = \sum b_n z^{\nu_n}$ be an integral function such that each term is in turn the maximum term and relative to its maximum term the series for $h(z)$ is dominated by at most two terms. Then for any finite c,*

$$N(r, 1/(h-c)) \sim T(r,h) \qquad (r \to \infty).$$

Let $|b_n| r^{\nu_n}$ be the maximum term for $r_n \leqslant r < r_{n+1}$. Define σ'_n, σ''_n by

$$\frac{|b_{n-1}| {\sigma_n'}^{\nu_{n-1}}}{|b_n| {\sigma_n'}^{\nu_n}} = \frac{1}{2}, \qquad \frac{|b_{n+1}| {\sigma_n''}^{\nu_{n+1}}}{|b_n| {\sigma_n''}^{\nu_n}} = \frac{1}{2}. \tag{2.13}$$

Since the series is dominated by at most two terms when n is large, we have $\sigma'_n < \sigma''_n$. In $\sigma'_n \leqslant r \leqslant \sigma''_n$ when n is large, the only other terms apart from $|b_n| r^n$ that qualify as dominant terms are $|b_{s-1}| r^{\nu_{n-1}}$ and $|b_{n+1}| r^{\nu_{n+1}}$. Since there is only one such other term, it follows from (2.13) that

$$|h(z)| \geqslant \mu(r,h)[1 - \tfrac{1}{2} + o(1)] \qquad (|z| = r,\ \sigma'_n \leqslant r \leqslant \sigma''_n).$$

When r is large enough as above, i.e. $\sigma'_n \leqslant r \leqslant \sigma''_n$, then

$$(2.14)\qquad \frac{1}{2\pi}\int_0^{2\pi} \log|h(re^{i\theta})|\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \log^+|h(re^{i\theta})|\, d\theta = T(r, h).$$

Now

$$(2.15)\qquad N(r, 1/(h-c)) = \int_0^r \frac{n(t, 1/(h-c)) - n(0, 1/(h-c))}{t}\, dt + n(0, 1/(h-c)) \log r$$

and so, by Jensen's theorem,

$$N(r, 1/(h-c)) = \frac{1}{2\pi}\int_0^{2\pi} \log|h(re^{i\theta}) - c|\, d\theta + O(1).$$

Clearly $h(z) - c$ satisfies the same conditions as $h(z)$ when $|z| = r$ is large and hence, from (2.14),

$$(2.16)\qquad N(r, 1/(h-c) = T(r, h-c) + O(1) = T(r, h) + O(1)$$

provided $\sigma'_n \leqslant r \leqslant \sigma''_n$.

It is known **(4**, p. 280**)** that there is a constant a such that

$$(2.17)\qquad N(r, 1/(h-a)) \sim T(r, h) \qquad (r \to \infty).$$

Consider $\sigma''_n \leqslant r \leqslant \sigma'_{n+1}$. If n is large enough, then both $h(z) - a$ and $h(z) - c$ have ν_n zeros in $|z| \leqslant \sigma''_n$ and ν_{n+1} zeros in $|z| \leqslant \sigma'_{n+1}$. Therefore, from (2.15), for $\sigma''_n \leqslant r \leqslant \sigma'_{n+1}$

$$(2.18)\qquad |N(r, 1/(h-\alpha)) - N(r, 1/(h-c))| \leqslant (\nu_{n+1} - \nu_n) \times \log(\sigma'_{n+1}/\sigma''_n) + O(1).$$

Now

$$\frac{|b_{n+1}|\sigma''^{\,\nu_n+1}_n}{|b_n|\sigma''^{\,\nu_n}_n} = \frac{1}{2} = \frac{|b_n|\sigma'^{\,\nu_n}_{n+1}}{|b_{n+1}|\sigma'^{\,\nu_n+1}_{n+1}}$$

or

$$(2.19)\qquad (\nu_{n+1} - \nu_n) \log(\sigma'_{n+1}/\sigma''_n) = \log 4.$$

From (2.16), (2.17), (2.18), and (2.19) we obtain Lemma 3.

From Lemmas 1, 2, and 3 we arrive at the result of Theorem 1.

3. In this section we prove the following theorem:

THEOREM 2. *Let $\phi(r)$ be an increasing and logarithmically convex function defined and positive for $r \geqslant 1$, subject to the condition that for every $n > 0$,*

$$(3.1)\qquad \phi(r)/r^n \to \infty \qquad (r \to \infty).$$

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Then there exists an entire function $f(z)$ with positive coefficients such that for $r \geqslant 9/5$,

$$1/(3\sqrt{r}) < \phi(r)/M(r,f) < 3\sqrt{r}. \tag{3.2}$$

Proof. $\phi^*(r) = \frac{1}{3}\sqrt{r}\,\phi(r)$ satisfies the same conditions as $\phi(r)$. Therefore we can represent $\log \phi^*(r)$ by

$$\log \phi^*(r) = \log \phi^*(1) + \int_1^r \frac{\psi(\rho)}{\rho}\,d\rho,$$

where $\psi(\rho)$ is a positive increasing function and, by condition (3.1),

$$\lim_{\rho\to\infty} \psi(\rho) = \infty.$$

We write $n_0 = [\psi(1)]$, $r_{n_0} = \rho_0 = 1$. For $n > n_0$ we define the sequence $\{r_n\}$ by

$$\psi(r_n - 0) \leqslant n \leqslant \psi(r_n + 0).$$

$\{r_n\}$ is an increasing sequence. We shall now define the sequence n_m (and the sequence $\rho_m = r_{n_m}$) by recursion.

First we introduce the notation

$$\left.\begin{aligned} h(\mu,\nu) &= \int_{r_\mu}^{r_\nu} \frac{\nu - \psi(\rho)}{\rho}\,d\rho \\ d(\mu,\nu) &= \int_{r_\mu}^{r_\nu} \frac{\psi(\rho) - \mu}{\rho}\,d\rho \end{aligned}\right\} \quad (\nu \geqslant \mu).$$

Suppose now that n_m is already defined. For a given positive number $c > 1$ we define l_m and k_m such that

$$\begin{aligned} h(n_m, j_m) &\leqslant \log c < h(n_m, j_m + 1), \\ d(n_m, k_m) &\leqslant \log c < d(n_m, k_m + 1), \end{aligned} \tag{3.3}$$

and we define n_{m+1} by

$$n_{m+1} = \max\{j_m, k_m, n_m + 1\}.$$

We note that $n_{m+1} > n_m$ and that $\rho_{m+2} > \rho_m$. We define the positive numbers c_m by

$$\log c_m + n_m \log \rho_m = \log \phi^*(\rho_m). \tag{3.4}$$

We shall prove that

$$f(z) = \sum_{m=0}^{\infty} c_m z^{n_m}$$

is an entire function which has the desired property. We now write

$$l_m(r) = \log c_m + n_m \log r.$$

For $r < \rho_m$,

$$\log \phi^*(r) - l_m(r) = \int_r^{\rho_m} \frac{n_m - \psi(\rho)}{\rho}\,d\rho \geqslant 0$$

is a decreasing function of r, while for $r > \rho_m$,

$$\log \phi^*(r) - l_m(r) = \int_{\rho_m}^{r} \frac{\psi(\rho) - n_m}{\rho} d\rho \geqslant 0$$

is an increasing function of r. Hence (taking (3.4) into account),

$$l_m(r) \leqslant \log \phi^*(r) \tag{3.5}$$

with equality for $r = \rho_m$. Hence, for every n and m

$$l_m(\rho_m) \geqslant l_n(\rho_m).$$

Hence the central-index and maximal-term of $f(z)$ are given by

$$\nu(\rho_m, f) = n_m,$$
$$\log \mu(\rho_m, f) = \max_n l_n(\rho_m) = l_m(\rho_m).$$

For every m, there exists a σ_m such that

$$\rho_m \leqslant \sigma_m \leqslant \rho_{m+1}$$

and

$$l_m(\sigma_m) = l_{m+1}(\sigma_m).$$

Then, clearly, for $\sigma_{m-1} \leqslant r < \sigma_m$,

$$\begin{aligned} \nu(r, f) &= n_m, \\ \log \mu(r, f) &= \max_n l_n(r) = l_m(r). \end{aligned} \tag{3.6}$$

We shall now prove a few lemmas.

LEMMA 4.

$$0 \leqslant \log \phi^*(r) - \log \mu(r, f) \leqslant \max\{\log r, \log c\}.$$

The first inequality is an immediate consequence of (3.5) and (3.6). To prove the second inequality, we assume that $\rho_m \leqslant r < \rho_{m+1}$.

(i) Suppose first that $n_{m+1} = n_m + 1$. For $\rho_m \leqslant r \leqslant \sigma_m$, we find that

$$\begin{aligned} \log \phi^*(r) - \log \mu(r) &= \log \phi^*(r) - l_m(r) = \int_{\rho_m}^{r} \frac{\psi(\rho) - n_m}{\rho} d\rho \\ &\leqslant \int_{\rho_m}^{r} \frac{n_{m+1} - n_m}{\rho} d\rho = \log r - \log \rho_m \leqslant \log r. \end{aligned} \tag{3.7}$$

Making use of (3.7), for $\sigma_m \leqslant r < \rho_{m+1}$, we find that

$$\begin{aligned} \log \phi^*(r) - \log \mu(r) &= \log \phi^*(r) - l_{m+1}(r) \\ &= \int_{r}^{\sigma_{m+1}} \frac{n_{m+1} - \psi(\rho)}{\rho} d\rho \leqslant \int_{\sigma_m}^{\rho_{m+1}} \frac{n_{m+1} - \psi(\rho)}{\rho} d\rho \\ &= \log \phi^*(\sigma_m) - \log \mu(\sigma_m) \leqslant \log \sigma_m \leqslant \log r. \end{aligned}$$

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(ii) Suppose now that $n_{m+1} = j_m$. Then in view of (3.3) and (3.4) we have that

$$(3.8)\quad \begin{aligned} \log \phi^*(r) - \log \mu(r) &\leqslant \log \phi^*(r) - l_{m+1}(r) \\ &= \log \phi^*(r) - \{n_{m+1} \log r + \log c_{m+1}\} \\ &= \log \phi^*(r) - \{n_{m+1} \log \rho_{m+1} + \log c_{m+1}\} \\ &\qquad + n_{m+1}(\log \rho_{m+1} - \log r) \\ &= \log \phi^*(r) - \log \phi(\rho_{m+1}) + n_{m+1}(\log \rho_{m+1} - \log r) \\ &= \int_r^{\rho_{m+1}} \frac{n_{m+1} - \psi(\rho)}{\rho}\, d\rho \leqslant \int_{\rho_m}^{\rho_{m+1}} \frac{n_{m+1} - \psi(\rho)}{\rho}\, d\rho \\ &= h(n_m, n_{m+1}) = h(n_m, j_m) \leqslant \log c. \end{aligned}$$

In the same way one proves that (3.8) holds also in the case $n_{m+1} = k_m$. This completes the proof of the lemma.

COROLLARY. *For* $r \geqslant c$,

$$(3.9)\qquad 0 \leqslant \log \phi^*(r) - \log \mu(r, f) \leqslant \log r.$$

LEMMA 5. *With the notations* $H(m, p) = h(n_m, n_p)$, $D(m, p) = d(n_m, n_p)$ · $(m < p)$, *we have the following inequalities:*

$$(3.10)\quad \begin{aligned} &\text{(i) } D(m, s) \geqslant D(m, p) + D(p, s) \\ &\text{(ii) } H(m, s) \geqslant H(m, p) + H(p, s) \end{aligned} \Big\} \quad \textit{for } m < p < s,$$
$$\begin{aligned} &\text{(iii) } D(m, m+2) \geqslant \log c, \\ &\text{(iv) } H(m, m+2) \geqslant \log c, \\ &\text{(v) } D(m-2k-1, m) \geqslant D(m-2k, m) \geqslant k \log c, \\ &\text{(vi) } H(m, m+2k+1) \geqslant D(m, m+2k) \geqslant k \log c. \end{aligned}$$

Proof of (i).

$$\begin{aligned} D(m, s) &= \int_{\rho_m}^{\rho_s} \frac{\psi(\rho) - n_m}{\rho}\, d\rho \\ &= \int_{\rho_m}^{\rho_p} \frac{\psi(\rho) - n_m}{\rho}\, d\rho + \int_{\rho_p}^{\rho_s} \frac{\psi(\rho) - n_p}{\rho}\, d\rho + (n_p - n_m)(\log \rho_s - \log \rho_p) \\ &\geqslant D(m, p) + D(p, s). \end{aligned}$$

Proof of (iii). From $n_{m+2} \geqslant n_{m+1} + 1 \geqslant k_m + 1$ and $\rho_{m+2} = r_{n_{m+2}} \geqslant r_{k_m+1}$ it follows that

$$\begin{aligned} D(m, m+2) &= \int_{\rho_m}^{\rho_{m+2}} \frac{\psi(\rho) - n_m}{\rho}\, d\rho \geqslant \int_{r_{n_m}}^{r_{k_m+1}} \frac{\psi(\rho) - n_m}{\rho}\, d\rho \\ &= d(n_m, k_m + 1) > \log c \qquad \text{(by (3.3))}. \end{aligned}$$

(ii) and (iv) are proved in a similar way. (v) and (vi) are immediate consequences of (i)–(iv).

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LEMMA 6.

$$\frac{c_{m-2k-1}\,\rho_m^{\,n_{m-2k-1}}}{c_m\,\rho_m^{\,n_m}} \leqslant \frac{c_{m-2k}\,\rho_m^{\,n_{m-2k}}}{c_m\,\rho_m^{\,n_m}} \leqslant c^{-k},$$

(3.11)

$$\frac{c_{m+2k+1}\,\rho_m^{\,n_{m+2k+1}}}{c_m\,\rho_m^{\,n_m}} \leqslant \frac{c_{m+2k}\,\rho_m^{\,n_{m+2k}}}{c_m\,\rho_m^{\,n_m}} \leqslant c^{-k}.$$

In fact,

$$\begin{aligned}\log\frac{c_j\,\rho_m^{\,n_j}}{c_m\,\rho_m^{\,n_m}} &= \log c_j - \log c_m + n_j\log\rho_m - n_m\log\rho_m\\ &= n_j(\log\rho_m - \log\rho_j) + (\log c_j + n_j\log\rho_j) - (\log c_m + n_m\log\rho_m)\\ &= n_j(\log\rho_m - \log\rho_j) + \log\phi^*(\rho_j) - \log\phi^*(\rho_m)\\ &= \int_{\rho_m}^{\rho_j}\frac{\psi(\rho)-n_j}{\rho}\,d\rho = \begin{cases} -D(j,m) & \text{if } j<m,\\ -H(m,j) & \text{if } j>m,\end{cases}\end{aligned}$$

and thus (3.11) follows immediately from (3.10).

LEMMA 7. *For* $\rho_m \leqslant r \leqslant \rho_{m+1}$ *we have*

$$0 < f(r) - \{c_{m-1}\,r^{n_{m-1}} + c_m\,r^{n_m} + c_{m+1}\,r^{n_{m+1}} + c_{m+2}\,r^{n_{m+2}}\} < [4/(c-1)]\mu(r,f).$$

In fact, in view of the previous lemma,

$$\begin{aligned}0 < f(r) - \sum_{\nu=m-1}^{m+2} c_\nu\,r^{n_\nu} &= \sum_{\nu=0}^{m-2} c_\nu\,r^{n_\nu} + \sum_{\nu=m+3}^{\infty} c_\nu\,r^{n_\nu}\\ &= c_m\,r^{n_m}\sum_{\nu=1}^{m-2}\frac{c_\nu\,r^{n_\nu}}{c_m\,r^{n_m}} + c_{m+1}\,r^{n_{m+1}}\sum_{\nu=m+3}^{\infty}\frac{c_\nu\,r^{n_\nu}}{c_{m+1}\,r^{n_{m+1}}}\\ &\leqslant c_m\,r^{n_m}\sum_{\nu=1}^{m-2}\frac{c_\nu\,\rho_m^{\,n}}{c_m\,\rho_m^{\,n_m}} + c_{m+1}\,r^{n_{m+1}}\sum_{\nu=m+3}^{\infty}\frac{c_\nu\,\rho_{m+1}^{\,n}}{c_{m+1}\,\rho_{m+1}^{\,n_{m+1}}}\\ &\leqslant c_m\,r^{n_m}\sum_{k=1}^{\infty}2\cdot c^{-k} + c_{m+1}\,r^{n_{m+1}}\sum_{k=1}^{\infty}2\cdot c^{-k}\\ &= (c_m\,r^{n_m} + c_{m+1}\,r^{n_{m+1}})\frac{2}{c-1} \leqslant \frac{4\mu(r,f)}{c-1}\end{aligned}$$

as stated.

COROLLARY.

(3.12) $\mu(r,f) < f(r) < [4 + 4/(c-1)]\mu(r,f) = [4c/(c-1)]\mu(r,f).$

Now we can complete the proof of Theorem 2. From (3.12) we obtain

(3.13) $$(c-1)/4c < \mu(r)/f(r) < 1.$$

On the other hand for $r \geqslant c$ we obtain from (3.9) the inequality:

(3.14) $$1 \leqslant \phi^*(r)/\mu(r) \leqslant r.$$

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From (3.13) and (3.14) we obtain immediately that

$$(c-1)/4c < \phi^*(r)/f(r) < r,$$

$$(3.15) \qquad \sqrt{\frac{c-1}{4c}}\frac{1}{\sqrt{(r)}} < \frac{\sqrt{\frac{4c}{c-1}}\frac{1}{\sqrt{(r)}}\phi^*(r)}{f(r)} < \sqrt{\frac{4c}{c-1}}\sqrt{r}.$$

The substitution $c = 9/5$, $\phi(r) = (3/\sqrt{r})\phi^*(r)$ now gives (3.2).

4. In this section it will be convenient to make use of the following result of P. Erdös and one of the authors.

LEMMA 8 (**3**, Theorem 1). *For every entire function $f(z)$, there exists an entire function $F(z)$ with positive coefficients and with the property*

$$\tfrac{1}{6} < M(r,f)/F(r) < 3.$$

To show that (3.2) is essentially best possible we shall prove

THEOREM 3. *There exists a function $\phi_0(r)$ satisfying the conditions of Theorem 2, and having the property that for every entire function $f(z)$*

$$(4.1) \qquad \limsup_{r\to\infty} \frac{1}{\log\sqrt{r}}\left|\log\frac{\phi_0(r)}{M(r,f)}\right| \geqslant 1.$$

Proof. Let $r_0 = 0$, $r_n = 2^{n!}$ for $n \geqslant 1$, and let

$$\phi_0(r) = \frac{1}{r_1 r_2 \ldots r_n} r^{n+\frac{1}{2}} = A_n r^{n+\frac{1}{2}} \qquad \text{for } r_n \leqslant r < r_{n+1}.$$

(The function $\phi_0(r)$ defined here is of very slow growth, but by a slight modification of the construction we could obtain functions of arbitrarily fast growth which have the same property). Clearly, $\phi_0(r)$ satisfies the conditions of Theorem 2. Suppose now that for this function (4.1) is false, and that for some entire function $f(z)$, $\epsilon > 0$, and $r > R_0$, we have

$$\log[\phi_0(r)/M(r,f)] \leqslant (\tfrac{1}{2} - \epsilon)\log r.$$

Then, by Lemma 8, we could also construct an entire function with positive coefficients

$$F(z) = \sum_0^\infty a_n z^n, \text{ say,}$$

such that for $r > R_1$

$$(4.2) \qquad \log[\phi_0(r)/F(r)] \leqslant \tfrac{1}{2}(1-\epsilon)\log r.$$

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Hence, for $r_n \leqslant r < r_{n+1}$,

$$r^{-\frac{1}{2}(1-\epsilon)} \leqslant F(r)/(A_n r^{n+\frac{1}{2}}) \leqslant r^{\frac{1}{2}(1-\epsilon)},$$

$$A_n r^{-\frac{1}{2}(1-\epsilon)} \leqslant \sum_{m=0}^{\infty} a_m r^{m-n-\frac{1}{2}} \leqslant A_n r^{\frac{1}{2}(1-\epsilon)},$$

$$(4.3) \qquad \sum_{m=n+1}^{\infty} a_m r^{m-n-\frac{1}{2}} = \sum_{m=n+1}^{\infty} a_m r_{n+1}{}^{m-n-\frac{1}{2}} (r/r_{n+1})^{m-n-\frac{1}{2}}$$
$$\leqslant (r/r_{n+1})^{\frac{1}{2}} A_n r_{n+1}{}^{\frac{1}{2}(1-\epsilon)} = A_n r^{\frac{1}{2}} r_{n+1}{}^{-\epsilon\frac{1}{2}},$$

$$\sum_{m=0}^{n} a_m r^{m-n-\frac{1}{2}} \sum_{m=0}^{n} a_m r_n{}^{m-n-\frac{1}{2}} (r_n/r)^{n-m+\frac{1}{2}}$$
$$\leqslant (r_n/r)^{\frac{1}{2}} A_n r_n{}^{\frac{1}{2}(1-\epsilon)} = A_n r_n{}^{1-\frac{1}{2}\epsilon} r^{-\frac{1}{2}}.$$

Adding these two inequalities, and using the first inequality of (4.3), we find that

$$A_n r^{-\frac{1}{2}(1-\epsilon)} \leqslant \sum_{m=0}^{\infty} a_m r^{m-n-\frac{1}{2}} \leqslant A_n r^{\frac{1}{2}} r_{n+1}{}^{-\frac{1}{2}\epsilon} + A_n r_n{}^{1-\frac{1}{2}\epsilon} r^{-\frac{1}{2}},$$

$$(4.4) \qquad r^{-\frac{1}{2}(1-\epsilon)} \leqslant r^{\frac{1}{2}} r_{n+1}{}^{-\frac{1}{2}\epsilon} + r_n{}^{1-\frac{1}{2}\epsilon} r^{-\frac{1}{2}},$$
$$r^{\frac{1}{2}\epsilon} \leqslant r \cdot r_{n+1}{}^{-\frac{1}{2}\epsilon} + r_n{}^{1-\frac{1}{2}\epsilon}.$$

Let $\log r = (\log r_n)^{\frac{1}{2}} (\log r_{n+1})^{\frac{1}{2}}$. Then

$$\frac{\log r_{n+1}}{\log r} = \frac{\log r}{\log r_n} = \sqrt{n+1} > \frac{2}{\epsilon}$$

if $n > n_0(\epsilon)$. Also, for $n > n_1(\epsilon)$, $r^{\frac{1}{2}\epsilon} > r_n^{\frac{1}{2}\epsilon} > 2$. Hence, for $n > n_2(\epsilon)$,

$$2r_1{}^{-\frac{1}{2}\epsilon} < r < r_{n+1}{}^{\frac{1}{2}\epsilon} \quad \text{and} \quad 2r_n{}^{1-\frac{1}{2}\epsilon} < r_n < r^{\frac{1}{2}\epsilon}$$

so that

$$r \cdot r_{n+1}{}^{-\frac{1}{2}\epsilon} < \tfrac{1}{2} r^{\frac{1}{2}\epsilon} \quad \text{and} \quad r_n{}^{1-\frac{1}{2}\epsilon} < \tfrac{1}{2} r^{\frac{1}{2}\epsilon}$$

and finally

$$r \cdot r_{n+1}{}^{-\frac{1}{2}\epsilon} + r_n{}^{1-\frac{1}{2}\epsilon} < r^{\frac{1}{2}\epsilon},$$

which contradicts (4.4). Hence (4.1) is proved.

5. THEOREM 4. *Let $\phi(r)$ be an increasing and logarithmically convex function defined for $r \geqslant 1$. $\phi(r)$ can be written in the form*

$$\phi(r) = \phi(1) \exp \int_1^r \frac{\psi(\rho)}{\rho} \, d\rho,$$

where $\psi(\rho)$ is a positive increasing function. We also assume that for some $c > 1$ and every $r \geqslant 1$

$$(5.1) \qquad \psi(cr) - \psi(r) \geqslant 1.$$

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Then there exists an entire function $f(z)$ with positive coefficients and such that

(5.2) $$\frac{\sqrt{(c-1)}}{2c} < \frac{\phi(r)}{M(r,f)} < \frac{2c}{\sqrt{(c-1)}}.$$

Proof. $\phi^*(r) = [2/(\sqrt{c}-1)]\phi(r)$ satisfies the same conditions as $\phi(r)$, and we have

$$\log \phi^*(r) = \log \phi^*(1) + \int_1^r \frac{\psi(\rho)}{\rho}\,d\rho.$$

It is an immediate consequence of (5.1) that

$$\lim_{\rho\to\infty} \psi(\rho) = \infty.$$

We shall show that the function $f(z)$ constructed in § 3 has the desired property. It is a consequence of (5.1) that

(5.3) $$r_{n+1}/r_n \leqslant c.$$

Now we can replace Lemma 4 by

LEMMA 4*. $0 \leqslant \log \phi^*(r) - \log \mu(r,f) \leqslant \log c.$

It is only necessary to consider the case when $n_{m+1} = n_m + 1$ and

$$\rho_m \leqslant r < \sigma_m.$$

Then in view of (5.3) we have, as in (3.7),

$$\begin{aligned}\log \phi^*(r) - \log \mu(r) \leqslant \log r - \log \rho_m &\leqslant \log \rho_{m+1} - \log \rho_m \\ &= \log r_{n_m+1} - \log r_{n_m} \leqslant \log c,\end{aligned}$$

as stated. It follows immediately that

(5.4) $$1 \leqslant \phi^*(r)/\mu(r) \leqslant c.$$

From (3.13) and (5.4) we obtain

$$\frac{c-1}{4c} < \frac{\phi^*(r)}{f(r)} < c,$$

$$\frac{\sqrt{(c-1)}}{2c} < \frac{[2/\sqrt{(c-1)}]\phi^*(r)}{f(r)} < \frac{2c}{\sqrt{(c-1)}},$$

which proves (5.2).

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