Third Meeting, January 10th, 1896.

Dr Peddie in the Chair.

Symmedians of a Triangle and their concomitant Circles

By J. S. Mackay, M.A., LL.D.

NOTATION

A B C	= vertices of the fundamental triangle
A′ B′ C′	= mid points of BC CA AB
D E F	= points of contact of sides with incircle
	= other triads of points defined as they occur
D_1 E_1 F_1	= points of contact of sides with first excircle
	And so on
G	= centroid of ABC
I	= incentre of ABC
I_1 I_2 I_3	= 1st 2nd 3rd excentres of ABC
J, J_1 J_2 J_3	= quartet of points defined in the text
K	= insymmedian point of ABC
K_1 K_2 K_3	= 1st 2nd 3rd exsymmedian points of ABC
L M N	= projections of K on the sides of ABC
L_1 M_1 N_1	= " " K_1 " " "
	And so on
O	= circumcentre of ABC
R S T	= feet of the insymmedians
R′ S′ T′	= " " exsymmedians
X Y Z	= " " perpendiculars from A B C

https://doi.org/10.1017/S0013091500031758 Published online by Cambridge University Press

INTRODUCTORY

DEFINITION. The isogonals * of the medians of a triangle are called the *symmedians* †

If the internal medians be taken, their isogonals are called the *internal symmedians* ‡ or the *insymmedians*; if the external medians be taken, their isogonals are called the *external symmedians*, or the *exsymmedians*

The word symmedians, used without qualification or prefix, may, as in the title of this paper, be regarded as including both insymmedians and exsymmedians (*cyclists* include both *bicyclists* and *tricyclists*); frequently however when used by itself it denotes insymmedians, just as the word medians denotes internal medians

It is hardly necessary to say that as medians and symmedians are particular cases of isogonal lines, the theorems proved regarding the latter are applicable to the former. Medians and symmedians however have some special features of interest, which are easier to examine and recognise than the corresponding ones of the more general isogonals

DEFINITION. Two points D D′ are *isotomic* § with respect to BC when they are equidistant from the mid point of BC

It is a well-known theorem (which may be proved by the theory of transversals) that

If three concurrent straight lines AD BE CF be drawn from the vertices of ABC to meet the opposite sides in D E F, and if D′ E′ F′ be isotomic to D E F with respect to BC CA AB, then AD′ BE′ CF′ are concurrent

DEFINITION. If O and O′ be the points of concurrency of two such triads of lines, then O and O′ are called *reciprocal points* §

---

* See *Proceedings of Edinburgh Mathematical Society*, XIII. 166-178 (1895)

† This name was proposed by Mr Maurice D'Ocagne as an abbreviation of *la droite symétrique de la médiane* in the *Nouvelles Annales*, 3rd series, II 451 (1883). It has replaced the previous name *antiparallel median* proposed by Mr E. Lemoine in the *Nouvelles Annales*, 2nd series, XII 364 (1873). Mr D'Ocagne has published a monograph on the Symmedian in Mr De Longchamps's *Journal de Mathématiques Élémentaires*, 2nd series, IV. 173-175, 193-197 (1885)

‡ The names *symédiane intérieure* and *symédiane extérieure* are used by Mr Clément Thiry in *Le troisième livre de Géométrie*, p. 42 (1887)

§ Mr De Longchamps in his *Journal de Mathématiques Élémentaires*, 2nd series, V. 110 (1886).

Instead of saying that D D′ are isotomic points with respect to BC, it is sometimes said that AD AD′ are isotomic lines with respect to angle A

§ 1

Construction for an insymmedian

FIGURE 12

Let ABC be the triangle

Draw the internal median AA′ to the mid point of BC; and make $\angle$ B A R = $\angle$ CAA′

AR is the insymmedian from A

The angle C A A′ is described clockwise, and the angle B A R counter-clockwise;
consequently AA′ AR are symmetrically situated with respect to the bisector of the interior angle B A C

Hence since AA′ is situated inside triangle ABC,
AR is inside ABC

The following construction* leads to a simple proof of a useful property of the insymmedians

FIGURE 13

From AC cut off AB_1 equal to AB
and ,, AB ,, ,, AC_1 ,, ,, AC
If B_1C_1 be drawn, it will intersect BC at L the foot of the bisector of the interior angle B A C

Hence if A A′, which is obtained by joining A to the point of internal bisection of BC, be the internal median from A, the corresponding insymmedian A R is obtained by joining A to the point of internal bisection of B_1C_1

* Mr Maurice D'Ocagne in *Journal de Mathématiques Élémentaires et Spéciales*, IV. 539 (1880). This construction, which recalls Euclid's *pons asinorum*, is substantially equivalent to a more complicated one given by Const. Harkema of St Petersburg in Schlömilch's *Zeitschrift*, XVI. 168 (1871)

https://doi.org/10.1017/S0013091500031758 Published online by Cambridge University Press

(1) BC and B_1C_1 are antiparallel with respect to angle A

(2) Since the internal median AA′ bisects internally all parallels to BC, therefore the insymmedian AR bisects internally all antiparallels to BC

(3) The insymmedians of a triangle bisect the sides of its orthic triangle *

(4) *The projections of B and C on the bisector of the interior angle BAC are P and Q. If through P a parallel be drawn to AB, and through Q a parallel be drawn to AC, these parallels will intersect* † *on the insymmedian from A*

[The reader is requested to make the figure]

Let A″ be the point of intersection of the parallels, and A′ the mid point of BC

It is well known ‡ that A′P is parallel to AC, that A′Q is parallel to AB, and that

$$A'P = \tfrac{1}{2}(AC - AB) = A'Q$$

Hence the figure A′PA″Q is a rhombus,

and A″ is the image of A′ in the bisector of angle A

Now since A′ lies on the median from A

A″ must lie on the corresponding symmedian

(5) The three internal medians are concurrent at a point, called the *centroid*; hence, by a property§ of Isogonals, the three insymmedians are concurrent at a point

[Other proofs of this statement will be given later on]

Various names have been given to this point, such as minimum-point, Grebe's point, Lemoine's point, centre of antiparallel medians. The designation symmedian point, suggested by Mr Tucker ‖ is the one now most commonly in use

---

* Dr Franz Wetzig in Schlömilch's *Zeitschrift*, XII. 288 (1867)

† Mr Maurice D'Ocagne in the *Nouvelles Annales*, 3rd series, II. 464 (1883)

‡ See *Proceedings of the Edinburgh Mathematical Society*, XIII. 39 (1895)

§ *Proceedings of the Edinburgh Mathematical Society*, XIII. 172 (1895)

‖ *Educational Times*, XXXVII. 211 (1884)

https://doi.org/10.1017/S0013091500031758 Published online by Cambridge University Press

The symmedian point has three points harmonically associated with it; when it is necessary to distinguish it from them, the name insymmedian point will be used

The insymmedian point and the centroid of a triangle are *isogonally conjugate points*

(6) If XYZ be the orthic triangle of ABC the insymmedian points of the triangles AYZ XBZ XYC are situated on the medians* of ABC

(7) *The insymmedian from the vertex of the right angle in a right-angled triangle coincides with the perpendicular from that vertex to the hypotenuse*†, *and the three insymmedians intersect at the mid point of this perpendicular* ‡

The first part of this statement is easy to establish. The second part follows from the fact that the orthic triangle of the right-angled triangle reduces to the perpendicular

§ 1′

Construction for an exsymmedian

FIGURE 14

Let ABC be the triangle

Draw the external median AA_∞ parallel to BC;
and make $\angle BAR' = \angle CAA_\infty$

AR′ is the exsymmedian from A

The angle CAA_∞ is described counterclockwise, and the angle BAR′ clockwise;
consequently AA_∞ AR′ are symmetrically situated with respect to the bisector of the exterior angle BAC

Hence since AA_∞ is situated outside triangle ABC,
AR′ is outside ABC

The following construction leads to a simple proof of a useful property of the exsymmedians

* Dr Franz Wetzig in Schlömilch's *Zeitschrift*, XII. 288 (1867)

† C. Adams's *Eigenschaften des...Dreiecks*, p. 2 (1846)

‡ Mr Clément Thiry's *Le troisième livre de Géométrie*, p. 42 (1887)

FIGURE 15

From CA cut off AB_1 equal to AB
and „ BA „ „ AC_1 „ „ AC
If B_1C_1 be drawn, it will intersect BC at L′ the foot of the bisector of the exterior angle BAC

Hence if AA_∞, which is obtained by joining A to the point of external bisection of BC (that is, by drawing through A a parallel to BC) be the external median from A, the corresponding exsymmedian AR′ is obtained by joining A to the point of external bisection of B_1C_1 (that is, by drawing through A a parallel to B_1C_1)

(1′) BC and B_1C_1 are antiparallel with respect to angle A

(2′) Since the external median AA_∞ bisects externally all parallels to BC, therefore the exsymmedian AR′ bisects externally all antiparallels to BC

(3′) The exsymmedians of a triangle are parallel to the sides of its orthic triangle*

(4′) *The projections of B and C on the bisector of the exterior angle BAC are P′ and Q′. If through P′ a parallel be drawn to AB, and through Q′ a parallel be drawn to AC, these parallels will intersect on the insymmedian from A*

The proof follows from the fact that A′P′ is parallel to AC, that A′Q′ is parallel to AB, and that

$$A'P' = \tfrac{1}{2}(AC + AB) = A'Q'$$

(5′) The external medians from any two vertices and the internal median from the third vertex are concurrent at a point; hence, by a property of Isogonals, the corresponding exsymmedians and insymmedian are concurrent at a point

[Other proofs of this statement will be given later on]

* Dr Wetzig in Schlomilch's *Zeitschrift*, XII. 288 (1867)

https://doi.org/10.1017/S0013091500031758 Published online by Cambridge University Press

Three points are thus obtained, and they are sometimes called the *exsymmedian* points

The three points obtained by the intersections of the external medians of ABC are the vertices of the triangle formed by drawing through A B C parallels to BC CA AB ; that is, they are the points anticomplementary* to A B C

Hence the exsymmedian points of a triangle are *isogonally conjugate* to the anticomplementary points of the vertices of the triangle

(6′) *The tangents to the circumcircle of a triangle at the three vertices are the three exsymmedians of the triangle*†

FIGURE 14

For ∠ BAR′ = ∠ CAA$_x$

= ∠ ACB

therefore AR′ touches the circle ABC at A

(7′) When the triangle is right-angled two of the exsymmedians are parallel, or they intersect at infinity on the perpendicular drawn from the vertex of the right angle to the hypotenuse

§ 2

The distances of any point in an insymmedian from the adjacent sides are proportional to those sides‡

FIGURE 16

Let AA′ be the internal median, AR the insymmedian from A

From R draw RV RW perpendicular to AC AB ;

and „ A′ „ A′P A′Q „ „ „ „

Then. RW : RV = A′P : A′Q

= AB : AC

---

* See *Proceedings of the Edinburgh Mathematical Society*, I. 14 (1894)

† C. Adams's *Eigenschaften des... Dreiecks*, p. 5 (1846)

‡ Ivory in Leybourn's *Mathematical Repository*, new series, Vol. I. Part I. p. 26 (1804). Lhuilier in his *Élémens d'Analyse*, p. 296 (1809) proves that

RW : RV = sinC : sinB

§ 2′

The distances of any point in an exsymmedian from the adjacent sides are proportional to those sides

FIGURE 17

Let AA_∞ be the external median, AR′ the exsymmedian from A

From R′ draw R′V′ R′W′ perpendicular to AC AB;
and from A_1 any point in the external median, draw A_1P' A_1Q' perpendicular to AC AB

Then $$R'W' : R'V' = A_1P' : A_1Q'$$
$$= AB : AC$$

§ 3

*The segments into which an insymmedian from any vertex divides the opposite side are proportional to the squares of the adjacent sides**

FIGURE 16

Let AR be the insymmedian from A

Draw RV RW perpendicular to AC AB

Then $$AB : AC = RW : RV$$

therefore $$AB^2 : AC^2 = AB \cdot RW : AC \cdot RV$$
$$= ABR : ACR$$
$$= BR : CR$$

Another demonstration, by Mr Clément Thiry, will be found in *Annuaire Scientifique du Cercle des Normaliens* (published at Gand, no date given), p. 104

* Ivory in Leybourn's *Mathematical Repository*, new series, Vol. I. Part I. p. 27 (1804). Lhuilier in his *Élémens d'Analyse*, p. 296 (1809) proves that

$$BR : CR = \sin^2 C : \sin^2 B$$

§ 3′

*The segments into which an exsymmedian from any vertex divides the opposite side are proportional to the squares of the adjacent sides**

FIGURE 17

Let AR′ be the exsymmedian from A
Draw R′V′ R′W′ perpendicular to AC AB

Then $AB : AC = R'W' : R'V'$

therefore $AB^2 : AC^2 = AB \cdot R'W' : AC \cdot R'V'$

$= ABR' : ACR'$

$= BR' : CR'$

§ 4

The insymmedians of a triangle are concurrent

FIRST DEMONSTRATION

FIGURE 18

Let AR BS CT be the insymmedians

Then $BR : CR = AB^2 : AC^2$

$CS : AS = BC^2 : BA^2$

$AT : BT = CA^2 : CB^2$

therefore $$\frac{BR}{CR} \cdot \frac{CS}{AS} \cdot \frac{AT}{BT} = -1$$

since of the ratios BR : CR CS : AS AT : BT all are negative; therefore AR BS CT are concurrent

* C. Adams's *Eigenschaften des ... Dreiecks*, pp. 3-4 (1846). Pappus in his *Mathematical Collection*, VII. 119 gives the following theorem as a lemma for one of the propositions in Apollonius's *Loci Plani*:

If $AB^2 : AC^2 = BR' : CR'$
then $BR' \cdot CR' = AR'^2$

Second Demonstration

Figure 19

Let BK CK the insymmedians from B C cut each other at K : to prove that K lies on the insymmedian from A

Through K draw

EF′ antiparallel to BC with respect to A						
FD′	,,	,, CA	,,	,,	,,	B
DE′	,,	,, AB	,,	,,	,,	C

Because FD′ is antiparallel to CA
therefore BK bisects FD′
Similarly CK bisects DE′

Now $\angle D'DK = \angle A = \angle DD'K$;
therefore $KD = KD'$,
therefore $KD = KD' = KE' = KF$

Again $\angle E'EK = \angle B = \angle EE'K$;
therefore $KE = KE'$
Similarly $KF = KF'$;
therefore $KE = KF'$;
therefore K is on the insymmedian from A

Third Demonstration*

Figure 20

On the sides of ABC let squares X Y Z be described either all outwardly to the triangle or all inwardly. Produce the sides of the squares Y Z opposite to AC and AB to meet in A′; the sides of the squares Z X opposite to BA and BC to meet in B′; the sides of the squares X Y opposite to CB and CA to meet in C′

* E. W. Grebe in Grunert's *Archiv*, IX. 258 (1847)

Let BB′ and CC′ meet at K

Then BB′ is the locus of points whose distances from AB and BC are in the ratio $c : a$;
CC′ is the locus of points whose distances from AC and BC are in the ratio $b : a$;
therefore the ratio of the distances of K from AB and AC is $c : b$
that is, K lies on AA′

The eight varieties of position which the squares may occupy relatively to the sides of the triangle may be thus enumerated :

(1) X outwardly	Y outwardly	Z outwardly
(2) X inwardly	Y inwardly	Z inwardly
(3) X inwardly	Y outwardly	Z outwardly
(4) X outwardly	Y inwardly	Z inwardly
(5) X outwardly	Y inwardly	Z outwardly
(6) X inwardly	Y outwardly	Z inwardly
(7) X inwardly	Y inwardly	Z outwardly
(8) X outwardly	Y outwardly	Z inwardly

If the construction indicated in the enunciation of the third demonstration be carried out on these eight figures

(1)	and	(2)	will	give	the	insymmedian	point	K
(3)	,,	(4)	,,	,,	,,	first exsymmedian	,,	K_1
(5)	,,	(6)	,,	,,	,,	second ,,	,,	K_2
(7)	,,	(8)	,,	,,	,,	third ,,	,,	K_3

Now that the existence of the insymmedian point is established, it may be well to give that property of the point which was the first to be discovered.

*The sum of the squares of the distances of the insymmedian point from the sides is a minimum**

* "Yanto" in Leybourn's *Mathematical Repository*, old series, III. 71 (1803). See *Proceedings of the Edinburgh Mathematical Society*, XI. 92-102 (1893)

In the identity

$$(x^2+y^2+z^2)(a^2+b^2+c^2)-(ax+by+cz)^2$$
$$=(bz-cy)^2+(cx-az)^2+(ay-bx)^2$$

let a b c denote the sides of the triangle,

x y z the distances of any point from the sides

Then the left side of the identity is a minimum when the right sides is a minimum

But $a^2+b^2+c^2$ is fixed, and so is $ax+by+cz$, since it is equal to 2Δ; therefore $x^2+y^2+z^2$ is a minimum when the right side is 0

Now the right side is the sum of three squares, and can only be 0 when each of the squares is 0 ;

therefore $$bz-cy=cx-az=ay-bx=0$$

therefore $$\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$$

Hence the point which has the sum of the squares of its distances from the sides a minimum is that point whose distances from the sides are proportional to the sides

[The proof here given is virtually that of Mr Lemoine, in his paper communicated to the Lyons meeting (1873) of the *Association Française pour l'avancement des Sciences*. Another demonstration by Professor Neuberg will be found in Rouché et de Comberousse's *Traité de Géométrie*, First Part, p. 455 (1891)]

§ 4′

The insymmedian from any vertex of a triangle and the exsymmedians from the two other vertices are concurrent *

First Demonstration

Figure 21

Let AR be the insymmedian from A, and BS′ CT′ the exsymmedians from B C

* C. Adams's *Eigenschaften des...Dreiecks*, pp. 3-4 (1846)

https://doi.org/10.1017/S0013091500031758 Published online by Cambridge University Press

Then $BR : CR = AB^2 : AC^2$

$CS' : AS' = BC^2 : BA^2$

$AT' : BT' = CA^2 : CB^2$

therefore $$\frac{BR}{CR} \cdot \frac{CS'}{AS'} \cdot \frac{AT'}{BT'} = -1$$

since of the ratios $BR : CR$, $CS' : AS'$, $AT' : BT'$ two are positive and one negative ;
therefore AR BS′ CT′ are concurrent

Hence also AR′ BS CT′ ; AR′ BS′ CT are concurrent

The points of concurrency of

AR BS′ CT′ ; AR′ BS CT′ ; AR′ BS′ CT

will be called the 1st 2nd 3rd exsymmedian points, and will be denoted by K_1 K_2 K_3

Second Demonstration*

Figure 22

About ABC circumscribe a circle ; draw BK_1 CK_1 tangents to it at B C

Then BK_1 CK_1 are the exsymmedians from B C : to prove AK_1 to be the insymmedian from A

Through K_1 draw DE antiparallel to BC, and let AB AC meet it at D E

Then $\angle BDK_1 = \angle ACB = \angle DBK_1$;

therefore $BK_1 = DK_1$

Similarly $CK_1 = EK_1$;

therefore $DK_1 = EK_1$;

therefore AK_1 is the insymmedian from A

From this mode of demonstration it is clear that if K_1 be taken as centre and K_1B or K_1C as radius and a circle be described, that circle will cut AB AC at the extremities of a diameter

* Professor J. Neuberg in *Mathesis*, I. 173 (1881)

(1) *The insymmedians of a triangle pass through the poles of the sides of the triangle with respect to the circumcircle*

For K_1 is the pole of BC with respect to the circumcircle

(2) The six internal and external symmedians of a triangle meet three and three in four points which are collinear in pairs with the vertices

FIGURE 25

(3) If triangle ABC be acute-angled, the points

A	B	C	will be situated on the lines
K_2K_3	K_3K_1	K_1K_2 ;	

and the circle ABC will be the incircle of triangle $K_1K_2K_3$

If, however, triangle ABC be obtuse-angled, suppose at C, then the point A will be situated on K_2K_3 produced

B	,,	,,	,,	,,	K_3K_1	,,
C	,,	,,	,,	,,	K_1K_2 ;	

and the circle ABC will be an excircle of triangle $K_1K_2K_3$

FIGURE 26

(4) Hence the relation in which triangle ABC stands to $K_1K_2K_3$ will, if ABC be acute-angled, be that in which triangle DEF stands to ABC ; or, if ABC be obtuse-angled, it will be that in which one of the triangles $D_1E_1F_1$ $D_2E_2F_2$ $D_3E_3F_3$ stands to ABC

(5) If DEF be considered as the fundamental triangle, then A B C are the first, second, and third exsymmedian points, and the concurrent triad AD BE CF meet at the insymmedian point of DEF

If $D_1E_1F_1$ be considered as the fundamental triangle, then A B C are the first, second, and third exsymmedian points, and the concurrent triad AD_1 BE_1 CF_1 meet at the insymmedian point of $D_1E_1F_1$

Similarly for triangles $D_2E_2F_2$ $D_3E_3F_3$

(6) The points of concurrency* of the triads

---

* The concurrency may be established by the theory of transversals

https://doi.org/10.1017/S0013091500031758 Published online by Cambridge University Press

FIGURE 27

$$\left.\begin{array}{lll} AD & BE & CF \\ AD_1 & BE_1 & CF_1 \\ AD_2 & BE_2 & CF_2 \\ AD_3 & BE_3 & CF_3 \end{array}\right\} \text{ are } \left\{\begin{array}{l} \Gamma \\ \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \end{array}\right.$$

Γ *being called* the Gergonne point of* ABC, *and* Γ_1 Γ_2 Γ_3 *the* associated Gergonne points

Hence the Gergonne point and its associates are the insymmedian points of the four DEF triangles

(7) With respect to BC

D and D_1 are isotomic points, so are D_2 and D_3; and a similar relation holds for the E points with respect to CA, and for the F points with respect to AB. Hence the triads

$$\left.\begin{array}{lll} AD_1 & BE_2 & CF_3 \\ AD & BE_3 & CF_2 \\ AD_3 & BE & CF_1 \\ AD_2 & BE_1 & CF \end{array}\right\} \text{ which are concurrent† at } \left\{\begin{array}{l} J \\ J_1 \\ J_2 \\ J_3 \end{array}\right.$$

furnish the four pairs of reciprocal points,

Γ	Γ_1	Γ_2	Γ_3	(Gergonne points)
J	J_1	J_2	J_3	(Nagel points)

(8) Since AD passes through J_1
BE " " J_2
CF " " J_3

therefore Γ is situated on each of the straight lines AJ_1 BJ_2 CJ_3; in other words, the triangles ABC $J_1J_2J_3$ are homologous and have Γ for centre of homology

* By Professor J. Neuberg. J. D. Gergonne (1771-1859) was editor of the *Annales de Mathématiques* from 1810 to 1831

† Many of the properties of the J points were given by C. H. Nagel in his *Untersuchungen über die wichtigsten zum Dreiecke gehörigen Kreise* (1836). This pamphlet I have never been able to procure. Since 1836 some of these properties have been rediscovered several times

https://doi.org/10.1017/S0013091500031758 Published online by Cambridge University Press

Since AD_1 passes through Γ_1

BE_2 „ „ Γ_2

CF_3 „ „ Γ_3;

therefore J is situated on each of the straight lines $A\Gamma_1$ $B\Gamma_2$ $C\Gamma_3$; in other words, the triangles ABC $\Gamma_1\Gamma_2\Gamma_3$ are homologous and have J for centre of homology

Similarly ABC is homologous with

$$J\,J_3J_2 \qquad J_3J\,J_1 \qquad J_2J_1J$$

the centres of homology being respectively

$$\Gamma_1 \qquad \Gamma_2 \qquad \Gamma_3;$$

and ABC is homologous with

$$\Gamma\,\Gamma_3\Gamma_2 \qquad \Gamma_3\Gamma\,\Gamma_1 \qquad \Gamma_2\Gamma_1\Gamma$$

the centres of homology being respectively

$$J_1 \qquad J_2 \qquad J_3$$

(9) *If* $\Gamma' \qquad \Gamma_1' \qquad \Gamma_2' \qquad \Gamma_3'$ *be the points of concurrency of lines drawn from A′ B′ C′, the mid points of the sides, parallel to the triads of angular transversals which determine the points*

$$\Gamma \qquad \Gamma_1 \qquad \Gamma_2 \qquad \Gamma_3$$

then $\Gamma\Gamma' \qquad \Gamma_1\Gamma_1' \qquad \Gamma_2\Gamma_2' \qquad \Gamma_3\Gamma_3'$

are concurrent at the centroid of ABC

The points Γ' Γ_1' ... as belonging to triangle A′B′C′ correspond to the points Γ Γ_1 ... as belonging to triangle ABC; hence as ABC A B′C′ are similar and oppositely situated and have G for their homothetic centre, $\Gamma\Gamma'$ $\Gamma_1\Gamma_1'$ ... pass through G

(10) The Γ' points are complementary to the Γ points, and the tetrad

$$I\Gamma' \qquad I_1\Gamma_1' \qquad I_2\Gamma_2' \qquad I_3\Gamma_3'$$

are concurrent at the insymmedian point* of ABC

* William Godward in the *Lady's and Gentleman's Diary* for 1867, p. 63. He contrasts this point, in reference to one of its properties, with the centroid of ABC, and recognises it as the point determined by Mr Stephen Watson in 1865. See § 8 (2) of this paper.

(11) The J points are anticomplementary to the I points, and the tetrad*

$$IJ \quad I_1J_1 \quad I_2J_2 \quad I_3J_3$$

are concurrent at G the centroid of ABC

(12) J_1 J_2 J_3 J form an orthic tetrastigm *

FIGURE 26

(13) AI BI CI intersect EF FD DE

at the feet of the medians of triangle DEF;

AD BE CF intersect EF FD DE

at the feet of the internal symmedians

AI_1 BI_1 CI_1 intersect E_1F_1 F_1D_1 D_1E_1

at the feet of the medians of triangle $D_1E_1F_1$;

AD_1 BE_1 CF_1 intersect E_1F_1 F_1D_1 D_1E_1

at the feet of the internal symmedians

Similarly for triangles $D_2E_2F_2$ $D_3E_3F_3$

(14) *The external symmedians of any triangle are also the external symmedians of three other associated triangles*

FIGURE 26

Let DEF be the triangle

Circumscribe a circle about DEF, and draw tangents to it at D E F. Let these tangents intersect at A B C. Then $D_1E_1F_1$ $D_2E_2F_2$ $D_3E_3F_3$ are the three triangles associated with DEF

To determine their vertices it is not necessary to find I_1 I_2 I_3 and to draw perpendiculars to BC CA AB

Make $CD_1 = BD$ $CE_1 = CD_1$ and $BF_1 = BD_1$ and the triangle $D_1E_1F_1$ is determined

Similarly for $D_2E_2F_2$ $D_3E_3F_3$

* William Godward in *Mathematical Questions from the Educational Times* II. 87, 88 (1865)

(15) *I_1A' I_2B' I_3C' are concurrent* at the insymmedian point of $I_1I_2I_3$*

FIGURE 26

For BC is antiparallel to I_2I_3 with respect to $\angle I_2I_1I_3$ and A′ is its mid point;

therefore I_1A' is the insymmedian of $I_1I_2I_3$ from I_1

Similarly I_2B' ,, ,, ,, ,, ,, ,, I_2

and I_3C' ,, ,, ,, ,, ,, ,, I_3

(16) IA' I_3B' I_2C' ; I_3A' IB' I_1C ; I_2A' I_1B' IC' are concurrent †

respectively at the insymmedian points of the triangles

II_3I_2 I_3II_1 I_2I_1I

§ 5

The internal and external symmedians from any vertex are conjugate harmonic rays with respect to the sides of the triangle which meet at that vertex ‡

FIGURE 25

For $BR : CR = AB^2 : AC^2$

$= BR' : CR'$

therefore B R C R′ form a harmonic range

and AR AR′ are conjugate harmonic rays with respect to AB AC

---

* Geometricus (probably Mr William Godward) in *Mathematical Questions from the Educational Times*, III. 29-31 (1865). The method of proof is not his.

Mr W. J. Miller adds in a note that I_1A' divides I_2I_3 into parts which have to one another the duplicate ratio of the adjacent sides of the triangle $I_1I_2I_3$ and similarly for I_2B' I_3C'; and that the point of concurrency is such that the sum of the squares of the perpendiculars drawn therefrom on the sides of the triangle $I_1I_2I_3$ is a minimum, and these perpendiculars are moreover proportional to the sides on which they fall.

† Professor Johann Döttl in his *Neue merkwürdige Punkte des Dreiecks*, p. 14 (no date) states the concurrency, but does not specify what the points are.

‡ C. Adams's *Eigenschaften des...Dreiecks*, p. 5 (1846)

Hence also for C S A S′ and A T B T′

(1) The following triads of points are collinear :

R′ S T ; R S′ T ; R S T′ ; R′ S′ T′

(2) *The following are harmonic ranges* *

$A\ K\ R\ K_1$; $B\ K\ S\ K_2$; $C\ K\ T\ K_3$
$A\ K_2\ R'\ K_3$; $B\ K_1\ S'\ K_3$; $C\ K_1\ T'\ K_2$

For B R C R′ is a harmonic range ;
therefore A.BRCR′ is a harmonic pencil ;
and its rays are cut by the transversals $BKSK_2$ and B K_1 S′ K_3 ;
therefore B K S K_2 B K_1 S′ K_3 are harmonic ranges

(3) *If D E F be the points in which AK BK CK cut the circumcircle of ABC, then the following are harmonic ranges*

$A\ R\ D\ K_1$; $B\ S\ E\ K_2$; $C\ T\ F\ K_3$

FIGURE 25

For K_1B K_1C are tangents to the circle ABC, and K_1DRA is a secant through K_1 ;
therefore this secant is cut harmonically † by the chord of contact BC and the circumference

(4) *R′ is the pole of AK, with respect to the circumcircle* ‡

Since AR′ is the tangent at A
therefore AR′ is the polar of A
Now BR′ „ „ „ „ K_1 ;
therefore R′ is the pole of AK_1

Similarly for S′ and T′

(5) *R′D S′E T′F are tangents to the circumcircle*

For AD is the polar of R′ with respect to the circumcircle, that is, AD is the chord of contact of the tangents from R′

* The first of these is mentioned by Dr Franz Wetzig in Schlömilch's *Zeitschrift*, XII. 289 (1867)

† This is one of Apollonius's theorems. See his *Conics*, Book III., Prop. 37-40

‡ C. Adams's *Eigenschaften des...Dreiecks*, pp. 3-4 (1846)

https://doi.org/10.1017/S0013091500031758 Published online by Cambridge University Press

(6) *The straight line R′S′T′ is the polar of K with respect to the circumcircle*

For AK_1 BK_2 CK_3 pass through K;
therefore their respective poles R′ S′ T′ will lie on the polar of K

(7) R′S′T′ is perpendicular to OK, and its distance from O is equal to R^2/OK, where R denotes the radius of the circumcircle

R′S′T′ is sometimes called *Lemoine's line*

(8) *R′S′T′ is the trilinear polar* of K, or it is the line harmonically associated with the point K*

For	ST TR RS meet BC CA AB
at	R′ S′ T′;
therefore	R′S′T′ is the trilinear polar of K

(9) The three triangles ABC RST $K_1K_2K_3$ taken in pairs will have the same axis of homology, namely the trilinear polar of K

(10) *The following triads of points are collinear*

R′ E F; *S′ F D*; *T′ D E*

For BCEF is an encyclic quadrilateral,
and BE CF intersect at K;
therefore EF intersects BC on the polar of K
Now the polar of K intersects BC at R′;
therefore EF passes through R′

(11) *If BF CE intersect at D′*
CD AF „ „ E′
AE BD „ „ F′
then A D D′; B E E′; C F F′ are collinear;
and so are D′ E′ F′

Since BE CF intersect at K
therefore BF CE intersect at a point on the polar of K
Similarly for CD AF and for AE BD;
therefore D′ E′ F′ are collinear

* Mr J. J. A. Mathieu in *Nouvelles Annales*, 2nd series, IV. 404 (1865)

https://doi.org/10.1017/S0013091500031758 Published online by Cambridge University Press

Again BCEF is an encyclic quadrilateral, and

BE CF intersect at K
BC EF „ „ R′
BF CE „ „ D′;

therefore triangle KR′D′ is self-conjugate with respect to the circumcircle;
therefore KD′ is the polar of R′
But AK „ „ „ „ R′;
therefore A D D′ are collinear

(12) The following triads of lines are concurrent:

	AK BF CE;	BK CD AF;	CK AE BD
at	D′ ;	E′ ;	F′

and D′ E′ F′ are situated on R′S′T′

(13) *The straight lines which join the mid point of each side of a triangle to the mid point of the corresponding perpendicular of the triangle are concurrent at the insymmedian point* *

FIGURE 23

Let K K_1 be the insymmedian and first exsymmedian points of ABC;
let A′ be the mid point of BC and let A′K meet the perpendicular AX at P_1
Join $A'K_1$

Then $A'K_1$ is parallel to AX
Now since A K R K_1 is a harmonic range
therefore $A'.AKRK_1$ is a harmonic pencil;
therefore AX which is parallel to the ray $A'K_1$ is bisected by the conjugate ray A′K

§ 6

If L M N be the projections of K on the sides, then K is the centroid † *of triangle LMN*

* Dr Franz Wetzig in Schlömilch's *Zeitschrift*, XII. 289 (1867)
† E. W. Grebe in Grunert's *Archiv*, IX. 253 (1847)

FIGURE 28

Through L draw a parallel to MK, meeting NK produced in K′ Join K′M

Then triangle KLK′ has its sides respectively perpendicular to BC CA AB ;

therefore KL : LK′ = BC : CA

But KL : KM = BC : CA

therefore LK′ = KM ;

therefore KLK′M is a parallelogram ;

therefore KK bisects LM,

that is, KN is a median of LMN

Similarly KL KM are medians ;

therefore K is the centroid of LMN

Another demonstration by Professor Neuberg will be found in *Mathesis*, I. 173 (1881)

(1) *The sides of LMN are proportional to the medians of ABC, and the angles of LMN are equal to the angles which the medians of ABC make with each other* *

Since KL KM KN are two-thirds of the respective medians of LMN, and are proportional to BC CA AB ;
therefore the medians of LMN are proportional to BC CA AB ;
therefore the sides of LMN are proportional to the medians of ABC

See *Proceedings of the Edinburgh Mathematical Society*, I. 26 (1894)

The second part of the theorem follows from (37) on p. 25 of the preceding reference, and from the fact that the angles

CGB′ AGC′ BGA′

are respectively equal to the angles

CBG + GCB ACG + GAC BAG + GBA

Or it may be proved as follows :

Since G is the point isogonally conjugate to K, therefore AG BG CG are respectively perpendicular to MN NL LM

See *Proceedings of the Edinburgh Mathematical Society*, XIII. 178 (1895)

* Dr Franz Wetzig in Schlömilch's *Zeitschrift*, XII. 297 (1867)

(2) If LMN be considered as the fundamental triangle, K its centroid, and if at the vertices L M N perpendiculars be drawn to the medians KL KM KN a new triangle ABC is formed having K for its insymmedian point

(3) *The sum of the squares of the sides of the triangle LMN inscribed in ABC is less than the sum of the squares of the sides of any other inscribed triangle* *

The proof of this statement depends on the following lemma :

Given two fixed points M N and a fixed straight line BC ; that point L on BC for which $NL^2 + LM^2$ is a minimum is the projection on BC of the mid point of MN

(4) *If through every two vertices and the centroid of a triangle circles be described, the triangle formed by joining their centres will have for centroid and insymmedian point the circumcentre and the centroid of the fundamental triangle* †

FIGURE 39

Let O_1 O_2 O_3 be the centres of the three circles, and let the circles O_2 O_3 cut BC in the points E F

Join AG cutting BC in its mid point A′, and draw O_2P O_3Q perpendicular to BC

Then $\quad A'B \cdot A'F = A'A \cdot A'G = A'C \cdot A'E$

therefore $\quad A'F = A'E$

and $\quad A'Q = A'P$

Now O_1A' bisects BC perpendicularly ;
therefore O_1A' passes through the circumcentre of ABC and bisects O_2O_3
therefore O_1A' is a median of triangle $O_1O_2O_3$

Similarly for O_2B' and O_3C' ;
therefore O the circumcentre of ABC is the centroid of $O_1O_2O_3$

---

* Mr Emile Lemoine in the *Journal de Mathématiques Élémentaires*, 2nd series, III. 52-3 (1884)

† This theorem and the proof of it have been taken from Professor W. Fuhrmann's *Synthetische Beweise planimetrischer Sätze*, pp. 101-2 (1890)

Again if through A B C perpendiculars be drawn to the medians AG BG CG these perpendiculars will form a triangle UVW whose vertices will be situated on the circumferences* of O_1 O_2 O_3 and which will be similar to the triangle $O_1O_2O_3$
Also the triangles UVW $O_1O_2O_3$ have G for their centre of similitude

Now triangle UVW has G for its insymmedian point ;
therefore G is also the insymmedian point of triangle $O_1O_2O_3$

(5) *If* L_1 M_1 N_1 L_2 M_2 N_2 L_3 M_3 N_3

be the projections on BC CA AB *of*

K_1 K_2 K_3

then $K_1N_1L_1M_1$ $K_2L_2M_2N_2$ $K_3M_3N_3L_3$

are parallelograms †

FIGURE 40

The points K_1 L_1 B N_1 are concyclic

therefore $\angle L_1K_1N_1 = \angle ABC$

The points K_1 M_1 C L_1 are concyclic

therefore
$$\begin{aligned}\angle K_1L_1M_1 &= \angle K_1CM_1\\ &= \angle K_2CA\\ &= \angle ABC\end{aligned}$$

therefore K_1N_1 is parallel to L_1M_1

Similarly K_1M_1 „ „ „ L_1N_1

The point K_1 is the first of the three points harmonically associated with the centroid of the triangle $L_1M_1N_1$; the point K_2 is the second of the three points harmonically associated with the triangle $L_2M_2N_2$; and the point K_3 is the third of the three points harmonically associated with the triangle $L_3M_3N_3$

* For proof of some of the statements made here, see *Proceedings of the Edinburgh Mathematical Society*, I. 36-7 (1894)

† Dr Franz Wetzig in Schlömilch's *Zeitschrift*, XII. 298 (1867)

§ 7

If AK BK CK be produced to meet the circumcircle in D E F the triangle DEF has the same insymmedians as ABC

First Demonstration

Figure 29

From K draw KL KM KN perpendicular to BC CA AB and join MN NL LM

Since the points B L K N are concyclic

therefore $\angle KLN = \angle KBN$
$= \angle EBA$
$= \angle EDA$

Since the points C L K M are concyclic

therefore $\angle KLM = \angle KCM$
$= \angle FCA$
$= \angle FDA$

Hence $\angle MLN = \angle EDF$

Similarly $\angle LMN = \angle DEF$ or $\angle MNL = \angle EFD$

and triangles LMN DEF are directly similar

But since $\angle KLN = \angle KDE$

and $\angle KLM = \angle KDF$

therefore the point K in triangle LMN corresponds to its isogonally conjugate point in triangle DEF

Now K is the centroid of triangle LMN ;

therefore K is the insymmedian point of triangle DEF

Second Demonstration

Figure 25

Let AR′ BS′ CT′ be the exsymmedians

Since AK is the polar of R′, and BC EF both pass through R′ not only will the tangents to the circumcircle at B C meet on the polar of R′ but also the tangents at E F

But the tangents at E F meet on the insymmedian of DEF from D; therefore the insymmedian AD is common to triangles ABC DEF

Similarly for the insymmedians BE CF

The cosymmedian triangles ABC DEF are homologous, the insymmedian point K being their centre of homology, and R'S'T' their axis of homology

(1) *If two triangles be cosymmedian the sides of the one are proportional to the medians of the other* *

For triangle DEF is similar to triangle LMN

Or thus:

Let G be the centroid of ABC

Join GB GC

Then $\angle EDF = \angle EDA + \angle ADF$

$= \angle KBA + \angle KCA$

$= \angle GBC + \angle GCB$

since the points G K are isogonally conjugate

Similarly $\angle DEF = \angle GCA + \angle GAC$

and $\angle EFD = \angle GAB + \angle GAB$

A reference to the *Proceedings of the Edinburgh Mathematical Society*, I. 25 (1894) will show that this proves the theorem.

(2) *The ratio of the area of ABC to that of its cosymmedian triangle DEF is* †

$$(-a^2+2b^2+2c^2)(2a^2-b^2+2c^2)(2a^2+2b^2-c^2) : 27a^2b^2c^2$$

Let $\triangle'$ be the triangle whose sides are the medians of ABC and which is similar to DEF;

and let R' be the radius of its circumcircle

Then $\triangle' = \frac{3}{4}ABC = \frac{3\triangle}{4}$;

and $DEF : \triangle' = R^2 : R'^2$

---

* Dr John Casey in *Proceedings of the Royal Irish Academy*, 2nd series, IV. 546 (1886)

† Rev. T. C. Simmons in Milne's *Companion to the Weekly Problem Papers*, p. 150 (1888)

https://doi.org/10.1017/S0013091500031758 Published online by Cambridge University Press

Hence $$\mathrm{DEF} = \frac{3\triangle \mathrm{R}^2}{4\mathrm{R}'^2}$$

$$= \frac{3\triangle}{4} \cdot \frac{a^2b^2c^2}{16\triangle^2} \cdot \left(\frac{3\triangle}{m_1 m_2 m_3}\right)^2$$

$$= \frac{27\triangle a^2b^2c^2}{64 m_1{}^2 m_2{}^2 m_3{}^2}$$

The values of m_1 m_2 m_3 are given in the *Proceedings of the Edinburgh Mathematical Society*, I. 29 (1894)

(3) *If BD CD be joined, DR DR′ are an insymmedian and an exsymmedian of triangle* * *DCB*

FIGURE 24

Draw AA_1 parallel to BC to meet the circumcircle at A_1 and let A_1K_1 meet the circle at D_1

Then triangle ACK_1 is congruent to A_1BK_1

therefore $$\angle CAD = \angle BA_1D_1$$

therefore DD_1 is parallel to BC

Now since BC is the polar of K_1 and AA_1 DD_1 are parallel, therefore AD_1 A_1D intersect on BC at its mid point A′

Again $$\angle CDR = \angle CDA$$
$$= \angle BDA_1$$
$$= \angle BDA'$$

therefore DR is isogonal to the median DA′
But DR′ is a tangent to the circumcircle at D ;
therefore DR′ is an insymmedian

(4) Hence BR BK_1 are an insymmedian and an exsymmedian of triangle BDA ;
CR CK_1 an insymmedian and an exsymmedian of triangle CAD

(5) $$AR'^2 + BK_1{}^2 = K_1R'^2$$

Let O be the centre of the circumcircle ABC

* C. Adams in his *Eigenschaften des...Dreiecks*, pp. 4-5 (1846) gives (3)—(7)

https://doi.org/10.1017/S0013091500031758 Published online by Cambridge University Press

Then $AR'^2 = OR'^2 - OA^2$

$= OA'^2 + A'R'^2 - OA^2$

$BK_1^2 = A'B^2 + A'K_1^2$

$= OB^2 - OA'^2 + A'K_1^2$

therefore $AR'^2 + BK_1^2 = A'R'^2 + A'K_1^2$

$= K_1R'^2$

(6) *OR is perpendicular to K_1R'*

For R′ is the pole of AK_1

and K_1 „ „ „ „ BC

therefore K_1R' is the polar of R

therefore OR is perpendicular to K_1R'

(7) *AR' is a mean proportional between $A'R'$ and RR'*

Since B R C R′ form a harmonic range,
and A′ is the mid point of BC

therefore $BR' : A'R' = RR' : CR'$

therefore $A'R' \cdot RR' = BR' \cdot CR'$

$= AR'^2$

(8) $AB \cdot CD = AC \cdot BD = \frac{1}{2} AD \cdot BC$

FIGURE 24

For $AB^2 : AC^2 = BR : CR$

$= BD^2 : CD^2$

therefore $AB : AC = BD : CD$

The last property follows from Ptolemy's theorem
that $AB \cdot CD + AC \cdot BD = AD \cdot BC$

(9) The distances of R from the four sides of the quadrilateral ABDC are proportional to those sides.

This follows from § 2

DEFINITION. The four points A B D C form a harmonic system of points on the circle; and hence ABDC is called a *harmonic quadrilateral.*

This name was suggested to Mr Tucker by Professor Neuberg in 1885

The first systematic study of harmonic quadrilaterals was made by Mr Tucker. In his article "Some properties of a quadrilateral in a circle, the rectangles under whose opposite sides are equal," read to the London Mathematical Society on 12th February 1885, he states that in his attempt to extend the properties of the Brocard points and circle to the quadrilateral he "was brought to a stand at the outset by the fact that the equality of angles does not involve the similarity of the figures for figures of a higher order than the triangle. Limiting the figures, however, by the restriction that they shall be circumscriptible" he arrived at a large number of beautiful results all of which cannot unfortunately be given here.

Starting with the encyclic quadrilateral ABCD whose diagonals intersect at E, and investigating the condition that a point P can be found such that

$$\angle PAB = \angle PBC = \angle PCD = \angle PDA$$

*he finds, by analytical considerations, that a condition for the exist*ence of such a point is that the rectangles under the opposite sides of the quadrilateral must be equal. He then shows that if there be one Brocard point P for the quadrilateral there will be a second P′; that the lines

$$PA \quad PB \quad PC \quad PD; \qquad P'A \quad P'B \quad P'C \quad P'D$$

intersect again in four points which, with P P′ lie on the circumference of a circle with diameter OE, where O is the centre of the circle ABCD.

Next, if through E parallels be drawn to the sides of the quadrilateral, these parallels will meet the sides in eight points which lie on a circle concentric with the previous one.

Lastly he shows that the symmedian points (ρ_1 ρ_2) of ABD BCD lie on AC; the symmedian points (σ_1 σ_2) of ABC ADC lie on BD; the lines $O\rho_1$ $O\rho_2$ $O\sigma_1$ $O\sigma_2$ are the diameters of the Brocard circles of the triangles ABD BCD ABC ACD respectively; the centres of the four Brocard circles lie two and two on straight lines, parallel to AC BD; the circles themselves intersect two and two on the diagonals AC BD at their mid points, that is, where the Brocard circle of the quadrilateral meets the diagonals.

Mr Tucker's researches were taken up by Messrs Neuberg and

https://doi.org/10.1017/S0013091500031758 Published online by Cambridge University Press

Tarry, Dr Casey, and the Rev. T. C. Simmons, and there now exists a tolerably extensive theory of harmonic polygons. The reader who wishes to pursue this subject may consult

Mr R. Tucker's memoir which appeared in *Mathematical Questions from the Educational Times*, Vol. XLIV. pp. 125–135 (1886)

Professor Neuberg *Sur le Quadrilatère Harmonique* in *Mathesis*, V. 202–204, 217–221, 241–248, 265–269 (1885)

Dr John Casey's memoir (read 26th January 1886) "On the harmonic hexagon of a triangle" in the *Proceedings of the Royal Irish Academy*, 2nd series, Vol. IV. pp. 545–556

A memoir by Messrs Gaston Tarry and J. Neuberg *Sur les Polygones et les Polyèdres Harmoniques* read at the Nancy meeting (1886) of the *Association Française pour l'avancement des sciences.* See the Report of this meeting, second part, pp. 12–26

A memoir by the Rev. T. C. Simmons (read 7th April 1887) "A new method for the investigation of Harmonic Polygons" in the *Proceedings of the London Mathematical Society*, Vol. XVIII. pp. 289–304

Dr Casey's *Sequel to the First Six Books of the Elements of Euclid*, 6th edition, pp. 220–238 (1892)

§ 8

The Cosine or second Lemoine Circle

If through the insymmedian point of a triangle, antiparallels be drawn to the three sides, the six points in which they meet the sides are concyclic

Figure 19

Let K be the insymmedian point of ABC
and through K let there be drawn EF′ FD′ DE′
respectively antiparallel to BC CA AB
the D points being on BC, the E's on CA, the F's on AB

Then EF′ FD′ DE′ are each bisected at K

Now $\angle KDD' = \angle A = \angle KD'D$

therefore $KD = KD'$ and $DE' = FD'$

Hence also $DE' = EF'$

therefore K is equidistant from the six points D D′ E E′ F F′

https://doi.org/10.1017/S0013091500031758 Published online by Cambridge University Press

[This theorem was first given by Mr Lemoine at the Lyons meeting (1873) of the *Association Française pour l'avancement des sciences*, and the circle determined by it has hence been called one of Lemoine's circles (the second).

The existence of the circle however, and the six points through which it passes were discovered by Mr Stephen Watson of Haydonbridge in 1865, and its diameter expressed in terms of the sides of the triangle. See *Lady's and Gentleman's Diary* for 1865, p. 89, and for 1866, p. 55

In the same publication Mr Thomas Milbourn in 1867 announced a neat relation connecting the diameter of this circle with the diameter of the circumcircle, and here, as far as the *Diary* is concerned, the inquiry seemed to have stopped]

(1) The figures DD′E′F EE′F′D FF′D′E are rectangles *

It may be interesting to give the way in which these three rectangles made their first appearance.

(2) *Three rectangles may be inscribed in any triangle so that they may have each a side coincident in direction with the respective sides of the triangle, and their diagonals all intersecting in the same point, and one circle may be circumscribed about all the three rectangles* †

FIGURE 30

Let ABC be the triangle

Draw AX perpendicular to BC;

and produce CB to Q making BQ equal to CX

About ACQ circumscribe a circle cutting AB at P

Join PC; and draw BE parallel to PC and meeting AC at E

From E draw ED′ parallel to AB and EF perpendicular to AB, and let these lines meet BC AB at D′ F

About D′EF circumscribe a circle cutting BC CA AB again in D E′ F′. The six points D D′ E E′ F F′ are the vertices of the required rectangles

Draw CZ perpendicular to AB,

and let ED′ meet CP at R

The similar triangles ABX CBZ give

$$AX : CZ = AB : CB$$
$$= BQ : BP$$
$$= CX : BP;$$

therefore $$AX : CX = CZ : BP$$

* Mr Lemoine at the Lille meeting (1874) of the *Association Française pour l'avancement des sciences*

† Mr Stephen Watson in the *Lady's and Gentleman's Diary* for 1865, p. 89, and for 1866, p. 55

https://doi.org/10.1017/S0013091500031758 Published online by Cambridge University Press

But $EF : CZ = AE : AC$

$= AB : AP$

$= ED' : ER$

$= ED' : BP$;

therefore $EF : ED' = CZ : BP$;

therefore $AX : CX = EF : ED'$;

therefore the right-angled triangles AXC FED′ are similar

Hence $\angle XAC = \angle EFD' = \angle D'E'E$;

therefore D′E′ is parallel to AX

Now $\angle FE'D' = \angle FED' =$ a right angle;

therefore FE′ is parallel to BC,
and DD′E′F is one of the rectangles

Again because $\angle EFF'$ is right,
therefore EF′ is a diameter;
therefore $\angle F'D'E$ is right, as well as $\angle F'DE$ $\angle F'E'E$;
therefore EE′F′D FF′D′E are the other rectangles

(3) *To find the diameter* of the circle DEF*

FIGURE 30

$$AB^2 = BC^2 + CA^2 - 2BC \cdot CX$$

$$= BC^2 + CA^2 - 2BC \cdot BQ$$

$$= BC^2 + CA^2 - 2AB \cdot BP;$$

therefore $2AB \cdot BP = BC^2 + CA^2 - AB^2$

Add $2AB^2$ to both sides;

then $2AB \cdot AP = BC^2 + CA^2 + AB^2$

therefore $$AP = \frac{a^2 + b^2 + c^2}{2c}$$

But $EF : CZ = AB : AP$

therefore $$AP = \frac{AB \cdot CZ}{EF} = \frac{2\Delta}{EF}$$

* Mr Stephen Watson in the *Lady's and Gentleman's Diary* for 1866, p. 55

https://doi.org/10.1017/S0013091500031758 Published online by Cambridge University Press

Hence $\qquad \dfrac{2\triangle}{\text{EF}} = \dfrac{a^2+b^2+c^2}{2c}$

and $\qquad \text{EF} = \dfrac{4c\triangle}{a^2+b^2+c^2}$

Lastly $\qquad \text{D}'\text{F} : \text{EF} = \text{AC} : \text{AX}$

therefore $\qquad \text{D}'\text{F} : \dfrac{4c\triangle}{a^2+b^2+c^2} = b : \dfrac{2\triangle}{a}$

therefore $\qquad \text{D}'\text{F} = \dfrac{2abc}{a^2+b^2+c^2}$ = the diameter

The following is another proof

FIGURE 19

Triangles AEF′ ABC are similar
and AK is a median of AEF′;

therefore $\qquad \text{EF}' : \text{AK} = \text{BC} : m_1$

therefore $\qquad \text{EF}' = \dfrac{\text{AK} \cdot \text{BC}}{m_1}$

$$= \frac{2abc}{a^2+b^2+c^2}$$

For the value of AK, namely, $\dfrac{2abm_1}{a^2+b^2+c^2}$

see Formulæ connected with the Symmedians, at the end of this paper.

(4) *If d denote the diameter of circle DEF,*
and $\quad D \quad$,, ,, ,, ,, ,, *ABC,*
*then** $\quad \dfrac{1}{d^2} + \dfrac{1}{D^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2}$

For $\qquad \dfrac{1}{d} = \dfrac{a^2+b^2+c^2}{2abc}. \qquad \dfrac{1}{\text{D}} = \dfrac{4\triangle}{2abc};$

therefore $\qquad \dfrac{1}{d^2} + \dfrac{1}{\text{D}^2} = \dfrac{(a^2+b^2+c^2)^2 + (4\triangle)^2}{4a^2b^2c^2}$

* Mr Thomas Millbourn in the *Lady's and Gentleman's Diary* for 1867, p. 71, and for 1868, p. 75

$$= \frac{(a^2+b^2+c^2)^2+4c^2a^2 - (a^2-b^2+c^2)^2}{4a^2b^2c^2}$$

$$= \frac{4c^2a^2+2(c^2+a^2)2b^2}{4a^2b^2c^2}$$

$$= \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$

(5) *The centre of the circle DEF is the insymmedian point of the triangle ABC*

FIGURE 30

Because $\angle EFD' = \angle XAC$

therefore their complements are equal

that is $\angle D'FB = \angle ACX$;

therefore D'F is antiparallel to CA with respect to B

Hence E'D „ „ „ AB „ „ „ C

and F'E „ „ „ BC „ „ „ A

But these antiparallels are all bisected at the centre of the circle DEF;
therefore the centre of the circle is the insymmedian point K

(6) *The intercepts DD' EE' FF' made by the circle DEF on the sides of ABC are proportional to the cosines of the angles of ABC*

For triangle DD'E' is right-angled;

therefore $DD' = DE'\cos D'DE'$

$= DE'\cos A$

Similarly $EE' = EF'\cos B$

$FF' = FD'\cos C$

and DE' EF' FD' are all equal

Hence the name *cosine circle*, given to it by Mr Tucker

(7) *The triangles EFD F'D'E' are directly similar to ABC, and congruent to each other*

https://doi.org/10.1017/S0013091500031758 Published online by Cambridge University Press

FIGURE 19

For $\angle DEF = \angle DD'F$

$= \angle A$

and $\angle EFD = \angle EE'D$

$= \angle B$

therefore EFD is similar to ABC

In like manner for F'D'E'

Now since EFD F'D'E' are similar to each other and are inscribed in the same circle they are congruent

(8) The angles which	F D	D E	E F
make with	A B	B C	C A
are equal to the angles which	F'D'	D'E'	E'F'
make with	B C	C A	A B

§ 9

THE TRIPLICATE RATIO OR FIRST LEMOINE CIRCLE

If through the insymmedian point of a triangle parallels be drawn to the three sides, the six points in which they meet the sides will be concyclic

FIRST DEMONSTRATION *

FIGURE 31

Let K be the insymmedian point of ABC,
and through K let there be drawn EF' FD' DE'
respectively parallel to BC CA AB,
the D points being on BC, the E's on CA, the F's on AB

Then AFKE' is a parallelogram;
therefore AK bisects FE';
therefore F E' is antiparallel to BC ;
therefore F E' „ „ „ EF';

* This mode of proof is due to Mr R. F. Davis. See Fourteenth General Report (1888) of the Association for the Improvement of Geometrical Teaching, p. 39.

therefore the points E E′ F F′ are concyclic
Hence " " F F′ D D′ " "
and " " D D′ E E′ " "

Now if these three circles be not one and the same circle, their radical axes, which are DD′ EE′ FF′ or BC CA AB, must meet in a point, the radical centre
But BC CA AB do not meet in a point ;
therefore these three circles are one and the same,
that is, the six points D D′ E E′ F F′ are concyclic

Second Demonstration

Figure 32

Let O be the circumcentre of ABC, and let OA OK be joined.

Because AFKE′ is a parallelogram
therefore AK bisects FE′ at U ;
therefore FE′ is antiparallel to BC
therefore OA is perpendicular to FE′
If therefore through U a perpendicular be drawn to FE′
it will be parallel to OA, and will pass through O′,
the mid point of OK

Hence also the perpendiculars to DF′ and ED′ through their mid points V and W will pass through O′ ;
that is, O′ the mid point of OK is the centre of a circle which passes through D D′ E E′ F F′

This theorem also was first given by Mr Lemoine at the Lyons meeting (1873) of the *Association Française*, and the circle determined by it has been called one of Lemoine's circles (the first). In his famous article "Sur quelques propriétés d'un point remarquable d'un triangle" (famous for having given the impulse to a long series of researches, Mr Lemoine's own being not the least prominent among them all) he states that the centre of the circle is the mid point of the line joining the centre of antiparallel medians (or as it is now called, the symmedian point) to the circumcentre ; and that the intercepts made by the circle on the sides of the triangle are proportional to the cubes of the sides to which they belong.

Ten years later Mr Tucker, unaware of Mr Lemoine's researches, rediscovered the circle with many of its leading properties, and gave to it the name of *triplicate ratio* circle. See his papers "The Triplicate-Ratio Circle" in the *Quarterly Journal of Mathematics*, XIX. 342-348 (1883) and in the Appendix to the *Proceedings of the London Mathematical Society*, XIV. 316-321 (1883) and "A Group of Circles" in the *Quarterly Journal*, XX. 57-59 (1884).

The parallels drawn through K, the symmedian point, to the sides of ABC are often called Lemoine's parallels, and the hexagon they determine DD'EE'FF' Lemoine's hexagon.

(1) *If E'F F'D D'E be produced to meet and form a triangle, then the incircle of this triangle will have O' for its centre, and its radius will be half the radius of the circumcircle of ABC*

For $O'U = \frac{1}{2}OA \quad O'V = \frac{1}{2}OB \quad O'W = \frac{1}{2}OC$;

therefore $O'U = O'V = O'W$;

and O'U is perpendicular to E'F, O'V to F'D, O'W to D'E

(2) The figures DD'EF' EE'FD' FF'DE'
are symmetrical trapeziums;

therefore $E'F = F'D = D'E$

(3) *Triangles FDE E'F'D' are directly similar to ABC and congruent to each other*

For $\angle FDE = \angle FF'E = \angle B$

and $\angle DEF = \angle DD'F = \angle C$

therefore FDE is similar to ABC

In like manner for E'F'D'

Now since FDE E'F'D' are similar to each other and are inscribed in the same circle, therefore they are congruent.

It is not difficult to show that if K be any point in the plane of ABC and through it parallels be drawn to the sides, as in the figure, the triangles DEF D'E'F' are equal in area.

See Vuibert's *Journal de Mathématiques Élémentaires*, VIII. 12 (1883)

(4) *The following three triangles are directly similar to ABC :*

KDD' E'KE FF'K

For EF' FD' DE' are parallel to the sides

(5) *The following six triangles are inversely similar to ABC :*

AE'F KFE' DBF' F'KD D'EC ED'K

For E'F F'D D'E are antiparallel to the sides

(6) *The triangles* cut off from ABC by E'F F'D D'E are together equal to triangle DEF or D'E'F'*

---

* Properties (6)—(9) are due to Mr Tucker. See *Quarterly Journal*, XIX. 344, 346 (1883)

https://doi.org/10.1017/S0013091500031758 Published online by Cambridge University Press

For $AE'F = \frac{1}{2}AE'KF = EKF$

$$DBF' = \frac{1}{2}BF'KD = FKD$$

$$D'EC = \frac{1}{2}CD'KE = DKE$$

therefore $AE'F + DBF' + D'EC = DEF = D'E'F'$

(7) *The following six angles are equal :*

$$DFB \quad EDC \quad FEA \qquad D'E'C \quad F'D'B \quad E'F'A$$

For the arcs E'F F'D D'E are equal

(8) If each of these angles be denoted by ω

$$\angle AFE = \angle AE'F' = B + C - \omega$$

$$\angle BDF = \angle BF'D' = C + A - \omega$$

$$\angle CED = \angle CD'E' = A + B - \omega$$

(9) The following points are concyclic :

B C E' F C A F' D A B D' E

(10) *The radical axis of the circumcircle and the triplicate ratio circle is the Pascal line of Lemoine's hexagon*

Let FE' meet BC at X

Since the points B F E' C are concyclic,

therefore $XB \cdot XC = XF \cdot XE'$

therefore X has equal potencies with respect to the circumcircle and the triplicate ratio circle ;
therefore X is a point on their radical axis

Hence if DF' meet CA at Y, and ED' meet AB at Z,
Y and Z are points on the radical axis ;
therefore the radical axis is the straight line XYZ

(11) The radical axis is the polar of K with respect to the triplicate ratio circle

(12) The diagonals of Lemoine's hexagon

E'F' DE F'D' EF D'E' FD

intersect on the polar of K with respect to the triplicate ratio circle

FIGURE 33

(13) *If the chords*

$EF\ E'F' \qquad FD\ F'D' \qquad DE\ D'E'$

intersect in $\qquad p \qquad\qquad q \qquad\qquad r$

the triangles ABC pqr *are homologous* *

Let Bq Cr meet at T, and, for the moment, denote the distances of T from BC CA AB by α β γ

Then $\alpha : \gamma$ = perp. on BC from q : perp. on AB from q

$= \quad DD' \quad : \quad FF'$

from the similarity of triangles DD'q F'Fq

$= \quad a^3 \quad : \quad c^3$

Similarly $\alpha : \beta = \quad a^3 \quad : \quad b^3$;

therefore $\beta : \gamma = \quad b^3 \quad : \quad c^3$;

therefore the point p lies on AT

(14) *The intersections of the antiparallel chords with Lemoine's parallels, that is, of*

$E'F\ \ EF'' \qquad F''D\ \ FD' \qquad D'E\ \ DE'$

namely $\qquad P \qquad\qquad Q \qquad\qquad R$

are collinear *

The quadrilateral EE'FF' is inscribed in the circle DEF, and

EE' FF' $\qquad$ EF E'F' $\qquad$ E'F EF'

meet in $\qquad$ A $\qquad$ p $\qquad$ P

therefore triangle ApP is self-conjugate with respect to circle DEF ;

therefore P is the pole of Ap with respect to DEF

Similarly Q ,, ,, ,, Aq ,, ,, ,; ,,

and R ,, ,, ,, Ar ,, ,, ,, ,,

Now Ap Bq Cr are concurrent at T

therefore P Q R are collinear on the polar of T with respect to the circle DEF

* Dr John Casey. See his *Sequel to Euclid*, 6th ed., p. 190 (1892)

https://doi.org/10.1017/S0013091500031758 Published online by Cambridge University Press

(15) *If the intersections of*

	DE F′D′	*EF D′E′*	*FD E′F′*
be	*l*	*m*	*n*

the triangles ABC lmn are similar and oppositely situated

Since the arcs E′F F′D D′E are equal

therefore ∠ E′mF = ∠ E′nF = 2ω ;

therefore the points E′ m n F are concyclic ;

therefore ∠ E′nm = ∠ E′Fm = ∠ E′F′E ;

therefore mn is parallel to EF′ and to BC

Similarly for the other sides

(16) *The triangles pqr lmn are homologous and K is their centre of homology*

For E′F′D′FED is a Pascal hexagram ;

therefore the intersections of

	E′F′ FE	F′D′ ED	D′F DE′
namely	p	l	K

are collinear

Similarly q m K ; r n K are collinear

§10

Tucker's Circles *

If triangles ABC $A_1B_1C_1$ be similar and similarly situated and have K the symmedian point for centre of similitude, the six points in which the sides of $A_1B_1C_1$ meet the sides of ABC are concyclic

Figure 34

Let the six points be D D′ E E′ F F′

Since AFA$_1$E′ is a paralellogram

therefore AK bisects E′F ;

therefore E′F is antiparallel to BC ,

therefore E′F „ „ „ EF′ ;

* See the *Quarterly Journal*, XX. 57-59 (1884)

therefore the points E E′ F F′ are concyclic

Similarly „ „ F F′ D D′ „ „

and „ „ D D′ E E′ „ „

therefore the six points D D′ E E′ F F′ are concyclic

(1) *To find the centre of the circle DD′EE′FF′* *

Let O O_1 be the circumcentres of ABC $A_1B_1C_1$

Then O O_1 K are collinear

and OA is parallel to O_1A_1

Now since E′F is antiparallel to BC

therefore OA is perpendicular to E′F

Hence if E′F meet AA_1 at U,

a line through U parallel to AO will bisect E′F perpendicularly, and also bisect OO_1

Similarly the perpendicular bisectors of F′D and D′E will bisect OO_1 ;

therefore the centre of the circle is the mid point of OO_1

(2) *Triangles FDE E′F′D′ are directly similar to ABC and congruent to each other*

Figure 34

Since F′E D′F E′D are respectively parallel

to BC CA AB

therefore the arcs E′F F′D D′E are equal ;

therefore $\angle$ EFD = $\angle$ D′FF′ = $\angle$ A

Similarly $\angle$ FDE = $\angle$ E′DD′ = $\angle$ B

therefore FDE is similar to ABC

In like manner for E′F′D′

Now since FDE E′F′D′ are similar to each other and are inscribed in the same circle therefore they are congruent

* The properties (1), (2), (3), (6), (7) are due to Mr Tucker. See *Quarterly Journal*, XX. 59, 57, XIX. 348, XX. 59 (1884, 3)

(3) If T be the mid point of OO_1

$$TU = \frac{1}{2}(OA + O_1A_1)$$

Similarly, if V W be the mid points of F′D D′E,

$$TV = \frac{1}{2}(OB + O_1B_1)$$

$$TW = \frac{1}{2}(OC + O_1C_1)$$

Hence if E′F F′D D′E be produced to meet and form a triangle $A_2B_2C_2$

T will be the incentre of the triangle $A_2B_2C_2$ and the radius of the incircle will be an arithmetic mean between the radii of the circumcircles of ABC and $A_1B_1C_1$

(4) The triangle $A_3B_3C_3$ formed by producing E′F F′D D′E will have its sides respectively parallel to those of $K_1K_2K_3$ formed by drawing through A B C tangents to the circumcircle ABC

FIGURE 35

(5) Triangles $A_3B_3C_3$ $K_1K_2K_3$ have K for their centre of homology

(6) When the triangle $A_1B_1C_1$ becomes the triangle ABC, the Tucker circle DD′EE′FF′ becomes the circumcircle

(7) When the triangle $A_1B_1C_1$ reduces to the point K that is when the parallels B_1C_1 C_1A_1 A_1B_1 to the sides of ABC pass through K the Tucker circle DD′EE′FF′ becomes the triplicate ratio or first Lemoine circle

(8) When the triangle A_3 B_3 C_3 reduces to the point K, that is when the antiparallels B_3C_3 C_3A_3 A_3B_3 to the sides of ABC pass through K, the Tucker circle DD′EE′FF′ becomes the cosine or second Lemoine circle

(9) *If F′D D′E meet at* A_3

D′E E′F ,, ,, B_3

E′F F′D ,, ,, C_3

then AK BK CK pass through A_3 B_3 C_3

https://doi.org/10.1017/S0013091500031758 Published online by Cambridge University Press

FIGURE 28

Let R denote the foot of the symmedian AK

Then $\quad KF' : KE = BD : CD'$

$= c^2 : b^2$

and $\quad RD : RD' = BR - BD : CR - CD'$

$= c^2 : b^2$

therefore F′D KR ED′ are concurrent

§ 11

TAYLOR'S CIRCLE

The six projections of the vertices of the orthic triangle on the sides of the fundamental triangle are concyclic

FIGURE 36

Let the projections of X on CA AB be Y_1 Z_1

" " " Y " AB BC " Z_2 X_2

" " " Z " BC CA " X_3 Y_3

Then $\quad AZ : AZ_1 = AH : AX$

$= AY : AY_1$

therefore YZ is parallel to Y_1Z_1

Now Y_3Z_2 is antiparallel to YZ

therefore Y_3Z_2 " " " Y_1Z_1

therefore Y_1 Y_3 Z_2 Z_1 are concyclic

Similarly Z_2 Z_1 X_3 X_2 " "

and X_3 X_2 Y_1 Y_3 " "

therefore the six points X_3 X_2 Y_1 Y_3 Z_2 Z_1 are concyclic

The property, that the six projections of the vertices of the orthic triangle on the sides of the fundamental triangle are concyclic, seems to have been first published in Mr Vuibert's *Journal de Mathématiques Élémentaires* in November 1877. See Vol. II. pp. 30, 43. It is proposed by Eutaris. This name, as my friend Mr Maurice D'Ocagne informs me, was assumed anagrammatically by M. Restiau, at that time a répétiteur in the Collège Chaptal, Paris.

The same property, along with three others, is given in Catalan's *Théorèmes et Problèmes*, 6th ed., pp. 132-4 (1879). It occurs also in a question proposed by Professor Neuberg in *Mathesis*, I. 14 (1881), and in a paper by Mr H. M. Taylor in the *Messenger of Mathematics*, XI. 177-9 (1882). A proof by Mr C. M. Jessop, somewhat shorter than that given by Mr Taylor, occurs in the *Messenger*, XII. 36 (1883) and in the same volume (pp. 181-2) Mr Tucker examines whether any other positions of X Y Z on the sides would, with a similar construction, give a six-point circle, and he shows that no other circle is possible under the circumstances.

See also *L'Intermédiaire des Mathématiciens*, II. 166 (1895).

The projections of

X on BY CZ are Y_2 Z_0
Y „ CZ AX „ Z_3 X_0
Z „ AX BY „ X_1 Y_0

With regard to the notation it may be remarked that the
X points lie on BC and on the perpendicular to it from A
Y „ „ „ CA „ „ „ „ „ „ „ B
Z „ „ „ AB „ „ „ „ „ „ „ C

Let a notation, similar to that which prevails with regard to the sides, the semiperimeter, the radii of the incircle and the excircles of triangle ABC, be adopted for triangle XYZ; that is, let

$$YZ = x \qquad ZX = y \qquad XY = z$$

$$\sigma = \tfrac{1}{2}(x+y+z) \quad \sigma_1 = \tfrac{1}{2}(-x+y+z) \quad \sigma_2 = \tfrac{1}{2}(x-y+z) \quad \sigma_3 = \tfrac{1}{2}(x+y-z)$$

and let ρ ρ_1 ρ_2 ρ_3 be the radii of the incircle and the excircles

If reference be made to the *Proceedings of the Edinburgh Mathematical Society*, XIII. 39-40 (1895), it will be found that various properties are proved with respect to triangle $I_1I_2I_3$ and its orthic triangle ABC. These properties may be transferred to triangle ABC and its orthic triangle XYZ. The transference will be facilitated by writing down in successive lines the points which correspond. They are

$$\begin{matrix} I & I_1 & I_2 & I_3 & A & B & C & A_1 & A_2 & A_3 & A_4 \\ H & A & B & C & X & Y & Z & Y_2 & Y_1 & Z_0 & Z_1 \end{matrix}$$

Hence, from Wilkinson's theorem and corollary, the three following statements relative to Fig 25 may at once be inferred

(1) Z_1 Y_2 Z_0 Y_1 X_2 Z_3 X_0 Z_2 Y_3 X_1 Y_0 X_3
are three tetrads of collinear points

https://doi.org/10.1017/S0013091500031758 Published online by Cambridge University Press

(2) $Y_1Z_1 \quad Z_2X_2 \quad X_3Y_3$

or $Y_2Z_0 \quad Z_3X_0 \quad X_1Y_0$

intersect two by two at the mid points of the sides of XYZ

(3) *

$$Y_1Z_1 = Z_2X_2 = X_3Y_3 = \sigma$$
$$Y_2Z_0 = Z_3X_2 = X_3Y_0 = \sigma_1$$
$$Y_1Z_0 = Z_3X_0 = X_1Y_3 = \sigma_2$$
$$Y_2Z_1 = Z_2X_0 = X_1Y_0 = \sigma_3$$

(4) If X′ Y′ Z′ be the mid points of
YZ ZX XY
the sides of triangle X′Y′Z′ intersect the sides of ABC in six concyclic points

(5) *Triangles ABC X′Y′Z′ are homologous, and the symmedian point K is the centre of homology*

For YZ is antiparallel to BC,
and X′ is the mid point of YZ;
therefore AX′ is the symmedian from A
Similarly BY′ CZ′ are the symmedians from B C

(6) †

$$R\cdot Y_1Z_1 = ABC \qquad R\cdot Y_2Z_0 = HCB$$
$$R\cdot Y_1Z_0 = CHA \qquad R\cdot Y_2Z_1 = BAH$$

FIGURE 37

Join O the circumcentre to A B C

Then OA OB OC are respectively perpendicular
to YZ ZX XY

therefore 2AZOY = OA·YZ
2BXOZ = OB·ZX
2CYOX = OC·XY

therefore $2\,\triangle = R\,(YZ + ZX + XY)$

* The property that Y_1Z_1 is equal to the semiperimeter of XYZ occurs in Lhuilier's *Élémens d'Analyse*, p. 231 (1809)

† The first of these equalities is given by Feuerbach, *Eigenschaften des ... Dreiecks*, §19, or Section VI., Theorem 3 (1822). The other three are given by C. Hellwig in Grunert's *Archiv*, XIX., 27 (1852). The proof is that of Messrs W. E. Heal and P. F. Mange in Artemas Martin's *Mathematical Visitor*, II. 42 (1883)

(7) *The following triangles are isosceles* :

$X'ZZ_2$ $X'Z_2Y$ $X'ZY_3$ $X'Y_3Y$

For triangles YZZ_2 YZY_3 are right-angled and X' is the mid point of their hypotenuse

Similarly there are four isosceles triangles with vertex Y'
and ,, ,, ,, ,, ,, Z'

(8) Y_1Z_1 is antiparallel to BC with respect to A
Z_2X_2 ,, ,, ,, CA ,, ,, ,, B
X_3Y_3 ,, ,, ,, AB ,, ,, ,, C

(9) Y_3Z_2 is parallel to BC
Z_1X_3 ,, ,, ,, CA
X_2Y_1 ,, ,, ,, AB

(10) *Z_1X_3 and Y_1X_2 intersect on the symmedian from A*

Let A_1 be their point of intersection

Then $AZ_1A_1Y_1$ is a parallelogram,
and AA_1 bisects Y_1Z_1
But Y_1Z_1 is antiparallel to BC with respect to A;
therefore AA_1 is the symmedian from A

Similarly Y_1X_2 Z_2Y_3 intersect on the symmedian from B;
and Z_2Y_3 Z_1X_3 ,, ,, ,, ,, ,, C

(11) *Triangles $Y_1Z_2X_3$ $Z_1X_2Y_3$ are directly similar to ABC and congruent to each other*

FIGURE 36

For $\angle X_3Y_1Z_2 = \angle X_3Y_3Z_2$
Now X_3Y_3 is parallel to XY
and Y_3Z_2 ,, ,, ,, BC
therefore $\angle X_3Y_3Z_2 = \angle YXC$
$= \angle A$
Similarly $\angle Y_1Z_2X_3 = \angle B$
therefore $Y_1Z_2X_3$ is similar to ABC

In like manner for $Z_1X_2Y_3$

Now since $Y_1Z_2X_3$ $Z_1X_2Y_3$ are similar to each other and are inscribed in the same circle, therefore they are congruent

(12) Since XYZ is the orthic triangle not only of ABC, but also of HCB CHA BAH, if the projections of X Y Z be taken on the sides of the last three triangles, three other circles are obtained

These circles are

$$X_2X_3Y_2Y_0Z_3Z_0 \qquad X_1X_0Y_3Y_1Z_0Z_3 \qquad X_0X_1Y_0Y_2Z_1Z_2$$

If they be denoted by T_1 T_2 T_3 and the circle
$X_3X_2Y_1Y_3Z_2Z_1$ „ „ T, then

T T_1			BC
T T_2			CA
T T_3	have for radical axis		AB
T_2 T_3			AX
T_3 T_1			BY
T_1 T_2			CZ

(13) *The centres of the circles T T_1 T_2 T_3 are the incentre and the excentres of the triangle X′Y′Z′*

For Y_1Z_1 Z_2X_2 X_3Y_3 are equal chords in circle T;
therefore the centre of T is equidistant from them
But these chords form by their intersection the triangle X′Y′Z′;
therefore the centre of T must be the incentre of X′Y′Z′

Hence T T_1 T_2 T_3 form an orthic tetrastigm

(14) The centres of T T_1 T_2 T_3 are the four points of concurrency of the triads of perpendiculars from X′ Y′ Z′ on the sides of ABC HCB CHA BAH

See *Proceedings of the Edinburgh Mathematical Society*, I. 66 (1894)

(15) *The circle T belongs to the group of Tucker circles* *

Figure 36

For triangle $Z_1X_2Y_3$ is similar to ABC;
and it is inscribed in ABC
Hence its circumcircle T belongs to the group of Tucker circles

* Dr Kiehl of Bromberg. See his *Zur Theorie der Transversalen*, pp. 7-8 (1881). See also *Proceedings of the London Mathematical Society*, XV. 281 (1884)

(16) *The circle T cuts orthogonally the three excircles of the orthic triangle XYZ, and each of the circles T_1 T_2 T_3 cuts orthogonally* the incircle and two of the excircles of XYZ*

FIGURE 36

Let p_1 p_2 p_3 denote the perpendiculars from
A B C on YZ ZX XY;
these perpendiculars are the radii of the three excircles of XYZ

Since triangles AYZ ABC are similar,

therefore $p_1^2 : AX^2 = AZ^2 : AC^2$

therefore $p_1^2 : AC \cdot AY_1 = AC \cdot AY_3 : AC^2$

therefore $p_1^2 = AY_1 \cdot AY_3$ the potency of the point A with respect to the circle T

Hence the circle with centre A and radius p_1 cuts the circle T orthogonally

Similarly for the other statements

(17) *The squares of the radii of the circles* †

	T	T_1	T_2	T_3
are	$\frac{1}{4}(\rho^2+\sigma^2)$	$\frac{1}{4}(\rho_1^2+\sigma_1^2)$	$\frac{1}{4}(\rho_2^2+\sigma_2^2)$	$\frac{1}{4}(\rho_3^2+\sigma_3^2)$

FIGURE 36

The triangle X′Y′Z′ is similar to XYZ, the ratio of similitude being 1 : 2 therefore the radii of the incircle and excircles of X′Y′Z′

are $\frac{1}{2}\rho$ $\frac{1}{2}\rho_1$ $\frac{1}{2}\rho_2$ $\frac{1}{2}\rho_3$

Now if T be the incentre of X′Y′Z′, the perpendicular from T to Y_1Z_1 will bisect Y_1Z_1, and will be equal to $\frac{1}{2}\rho$.

Hence if t denote the radius of circle T,

$$t^2 = TY_1^2 = (\tfrac{1}{2}\rho)^2 + (\tfrac{1}{2}\sigma)^2 = \tfrac{1}{4}(\rho^2+\sigma^2)$$

Similarly if T_1 be the first excentre of X′Y′Z′, and t_1 denote the radius of circle T_1,

$$t_1^2 = T_1Y_2^2 = (\tfrac{1}{2}\rho_1)^2 + (\tfrac{1}{2}\sigma_1)^2 = \tfrac{1}{4}(\rho_1^2+\sigma_1^2)$$

* This corollary and the mode of proof have been taken from Dr John Casey's *Sequel to Euclid*, 6th ed. p. 195 (1892)

† Dr John Casey's *Sequel to Euclid*, 5th ed. p. 195 (1892)

https://doi.org/10.1017/S0013091500031758 Published online by Cambridge University Press

(18) *The sum of the squares of the radii of the circles* T T_1 T_2 T_3 *is equal to the square of the radius of the circumcircle of ABC*

In reference to triangle ABC, the following property may be proved to be true

$$16R^2 = r^2 + r_1^2 + r_2^2 + r_3^2 + a^2 + b^2 + c^2$$

This becomes in reference to triangle X′Y′Z′

$$16(\tfrac{1}{4}R)^2 = (\tfrac{1}{2}\rho)^2 + (\tfrac{1}{2}\rho_1)^2 + (\tfrac{1}{2}\rho_2)^2 + (\tfrac{1}{2}\rho_3)^2 + (\tfrac{1}{2}x)^2 + (\tfrac{1}{2}y)^2 + (\tfrac{1}{2}z)^2$$

$$\text{or} \quad R^2 = \tfrac{1}{4}(\rho^2 + \rho_1^2 + \rho_2^2 + \rho_3^2) + \tfrac{1}{4}(x^2 + y^2 + z^2)$$

$$= \tfrac{1}{4}(\rho^2 + \rho_1^2 + \rho_2^2 + \rho_3^2) + \tfrac{1}{4}(\sigma^2 + \sigma_1^2 + \sigma_2^2 + \sigma_3^2)$$

In connection with the Taylor circles it may be interesting to compare the properties given in the *Proceedings of the Edinburgh Mathematical Society*, Vol. I. pp. 88-96 (1894). These properties were worked out before the Taylor circle had attracted much attention.

(19) *If A′B′C′ be the complementary, and XYZ the orthic triangle of ABC, the Wallace lines of the points A′ B′ C′ with respect to the triangle XYZ pass through the centre of the circle T*

FIGURE 36

It is well known that the points A′ B′ C′ X Y Z are situated on the nine point circle of ABC

Since A′ is the mid point of BC

therefore A′Y = A′Z

therefore the foot of the perpendicular from A′ on YZ is X′ the mid point of YZ

Since BC bisects the exterior angle between XY and ZX the straight line joining the feet of the perpendiculars from A′ on XY and ZX will be perpendicular to BC

Hence the Wallace line A′ (XYZ) passes through X′ and is perpendicular to BC

that is, it passes through the centre of T

Similarly for the Wallace lines B′ (XYZ) and C′ (XYZ)

(20) *The Wallace lines of the points X Y Z with respect to the triangle $A'B'C'$ pass through the centre of the circle T*

[The reader is requested to make the figure]

Let the feet of the perpendiculars from X on $B'C'$ $C'A'$ $A'B'$ be L M N

Then the points A' M X N are concyclic

therefore $\angle A'MN = \angle A'XN$

$= 90° - \angle B$

$= \angle CAO$

therefore LMN is parallel to AO

But L is the mid point of AX and H_1 is situated on AO therefore LMN passes through the mid point of H_1X, that is through T

(21) *If H_1 H_2 H_3 be the orthocentres of triangles AYZ ZBX XYC the lines H_1X H_2Y H_3Z pass through the centre of circle T and are there bisected*

FIGURE 36

The orthocentre H_1 of triangle AYZ is the point of intersection of ZY_3 YZ_2 which are respectively perpendicular to CA AB

The centre T of the Taylor circle is the point of intersection of $Y'T$ $Z'T$ which are respectively perpendicular to CA AB

Hence since $XZ = 2XY'$ $XY = 2XZ'$

the quadrilaterals H_1ZXY $TY'XZ'$ are homothetic;

therefore H_1X passes through T and is there bisected

(22) Triangle $H_1H_2H_3$ is congruent and oppositely situated to triangle XYZ and T is their homothetic centre

(23) The centre T is situated on the straight line which joins O the circumcentre of ABC to the orthocentre of XYZ

For O is the orthocentre of $H_1H_2H_3$

https://doi.org/10.1017/S0013091500031758 Published online by Cambridge University Press

Not only is XYZ the orthic triangle of ABC and

triangles			similar to
AYZ	XBZ	XYC	ABC

but XYZ is the orthic triangle of HCB CHA BAH and

triangles			similar to
HYZ	XCZ	XYB	HCB
CYZ	XHZ	XYA	CHA
BYZ	XAZ	XYH	BAH

Let the orthocentres of the second, third, and fourth triads of triangles be denoted by

$$H_1' \quad H_2' \quad H_3' \qquad H_1'' \quad H_2'' \quad H_3'' \qquad H_1''' \quad H_2''' \quad H_3'''$$

The following results (among several others) will be found to be established in the *Proceedings of the Edinburgh Mathematical Society*, I. 83–87 (1894). They are quoted here, without proof, to save the reader the trouble or the expense of hunting out the reference

(24) The homothetic centre of the triangles

XYZ	$H_1'\ H_2'\ H_3'$	is	T_1
XYZ	$H_1''\ H_2''\ H_3''$	,,	T_2
XYZ	$H_1'''H_2'''H_3'''$	,,	T_3

and $T_1T_2T_3$ is similar and oppositely situated to ABC

(25) The point T is the centre of three parallelograms

$$YZH_2H_3 \qquad ZXH_3H_1 \qquad XYH_1H_2$$

and similarly T_1 T_2 T_3 are each the centre of three parallelograms

Let the incircle and the excircles of XYZ be denoted by their centres H A B C

(26) The radical axes of

	H A	H B	H C	B C	C A	A B
are	T_2T_3	T_3T_1	T_1T_2	T_1T	T_2T	T_3T

https://doi.org/10.1017/S0013091500031758 Published online by Cambridge University Press

(27) The radical centres of

	ABC	HCB	CHA	BAH
are	T	T_1	T_2	T_3

(28) X' Y' Z' are the feet of the perpendiculars of the triangle $T_1T_2T_3$

(29) The homothetic centre of the triangles

$T_1T_2T_3$	$H_1'H_1''H_1'''$	is	X
$T_1T_2T_3$	$H_2'H_2''H_2'''$	,,	Y
$T_1T_2T_3$	$H_3'H_3''H_3'''$	,,	Z

(30) The straight lines

$$HT \qquad AT_1 \qquad BT_2 \qquad CT_3$$

pass through the centroid of XYZ

(31) If G' denote this centroid

$$HG' : TG' = AG' : T_1G' = BG' : T_2G' = CG' : T_3G'$$
$$= 2 : 1$$

(32) If $HG'T$ be produced to J' so that $TJ' = HT$ then J' will be the incentre $X_1Y_1Z_1$ the triangle anticomplementary to XYZ

Similarly J_1' J_2' J_3' situated on AT_1 BT_2 CT_3 so that $T_1J_1' = AT_1$ and so on, will be the first, second, and third excircles of $X_1Y_1Z_1$

(33) The tetrads of points

$$HG'TJ' \qquad AG'T_1J_1' \qquad BG'T_2J_2' \qquad CG'T_3J_3'$$

form harmonic ranges

§ 12

Adams's Circle *

If D E F be the points of contact of the incircle with the sides of ABC, and if through the Gergonne point Γ (the point of concurrency of AD BE CF) parallels be drawn to EF FD DE, these parallels will meet the sides of ABC in six concyclic points

* C. Adams's *Die Lehre von den Transversalen*, pp. 77-80 (1843)

https://doi.org/10.1017/S0013091500031758 Published online by Cambridge University Press

FIGURE 38

Let X X′ Y Y′ Z Z′ be the six points

Join LL′ MM′ NN′

The complete quadrilateral AFΓEBC has its diagonal AΓ cut harmonically by FE BC;

therefore A U Γ D is a harmonic range;

therefore E·A U Γ D is a harmonic pencil

Now ΓMEM′ is a parallelogram;

therefore MM′ is bisected by EΓ

therefore MM′ is parallel to that ray of the harmonic pencil which is conjugate to EΓ, namely EA

In like manner NN′ is parallel to AB, and LL′ to BC

Again, since YEM′M Y′EMM′ are parallelograms,

therefore $YE = Y'E$

Similarly $Z'F = ZF$

therefore $YE : Y'E = Z'F : ZF$

Now YZ′ is parallel to EF;

therefore Y′Z is parallel to EF

In like manner Z′X is parallel to FD and X′Y to DE

Hence the two hexagons LL′MM′NN′ and XX′YY′ZZ′ are similar, and the ratio of their corresponding sides is that of 1 to 2

Lastly, since LL′ is parallel to BC

$$\angle L'L\Gamma = \angle CDE$$
$$= \angle CED$$
$$= \angle MM'\Gamma$$

therefore the points L L′ M M′ are concyclic

Similarly the points M M′ N N′ are concyclic

and the points N N′ L L′ are concyclic;

therefore all the six points are concyclic *

Hence the six points X X′ Y Y′ Z Z′ are also concyclic

* This method of proof is different from Adams's

https://doi.org/10.1017/S0013091500031758 Published online by Cambridge University Press

(1) *The centre of Adams's circle is the incentre* of ABC*

For XX′ YY′ ZZ′ are chords of Adams's circle, and they are bisected at D E F;
hence the centre of Adams's circle is found by drawing through D E F perpendiculars to XX′ YY′ ZZ′
These perpendiculars are concurrent at I the incentre of ABC

(2) *To find the centre of the circle LL′MM′NN′*

Since Γ is the homothetic centre of the two circles XX′YY′ZZ′ and LL′MM′NN′, and I is the centre of the first of these circles, therefore the centre of the second circle is situated on ΓI

If I′ denote the centre of the second circle

then $$\Gamma I : \Gamma I' = 2 : 1$$

(3) Since Γ the Gergonne point of ABC is the insymmedian point of DEF, the circle LL′MM′NN′ is the triplicate ratio or first Lemoine circle of DEF

(4) Besides the six-point circle obtained by drawing through Γ the Gergonne point of ABC parallels to the sides of triangle DEF, three other six-point circles will be obtained if through the associated Gergonne points Γ_1 Γ_2 Γ_3 parallels be drawn to the sides of the triangles $D_1E_1F_1$ $D_2E_2F_2$ $D_3E_3F_3$ respectively

The centres of these three circles are the excentres of ABC namely I_1 I_2 I_3 and the centres of the three LMN circles which correspond to them are the mid points of Γ_1I_1 Γ_2I_2 Γ_3I_3

These three LMN circles are the triplicate ratio circles of the triangles $D_1E_1F_1$ $D_2E_2F_2$ $D_3E_3F_3$

Formulae connected with the Symmedians

The sides a b c are in ascending order of magnitude

$$\left.\begin{aligned} BR &= \frac{ac^2}{b^2+c^2} \qquad CS = \frac{ba^2}{c^2+a^2} \qquad AT = \frac{cb^2}{a^2+b^2} \\ CR &= \frac{ab^2}{b^2+c^2} \qquad AS = \frac{bc^2}{c^2+a^2} \qquad BT = \frac{ca^2}{a^2+b^2} \end{aligned}\right\} \quad (1)$$

* C. Adams's *Die Lehre von den Transversalen*, p. 79 (1843)

$$\left.\begin{aligned} BR' &= \frac{ac^2}{c^2 - b^2} \qquad CS' = \frac{ba^2}{c^2 - a^2} \qquad AT' = \frac{cb^2}{b^2 - a^2} \\ CR' &= \frac{ab^2}{c^2 - b^2} \qquad AS' = \frac{bc^2}{c^2 - a^2} \qquad BT' = \frac{ca^2}{b^2 - a^2} \end{aligned}\right\} \quad (2)$$

$$RR' = \frac{2ab^2c^2}{c^4 - b^4} \qquad SS' = \frac{2a^2bc^2}{c^4 - a^4} \qquad TT' = \frac{2a^2b^2c}{b^4 - a^4} \quad (3)$$

Let the three internal medians be denoted by

$$m_1 \qquad m_2 \qquad m_3$$

Their values in terms of the sides are

$$4m_1^2 = -a^2 + 2b^2 + 2c^2$$

$$4m_2^2 = 2a^2 - b^2 + 2c^2$$

$$4m_3^2 = 2a^2 + 2b^2 - c^2$$

$$AR = \frac{2bcm_1}{b^2 + c^2} \qquad BS = \frac{2ca\,m_2}{c^2 + a^2} \qquad CT = \frac{2ab\,m_3}{a^2 + b^2} \quad (4)$$

FIGURE 12

Let AA′ AR be the internal median and symmedian from A

Then $\qquad BR \cdot CR : BA' \cdot CA' = AR^2 : AA'^2$

therefore $\qquad AR^2 = \dfrac{BR \cdot CR}{BA' \cdot CA'} \cdot AA'^2$

$$AR' = \frac{abc}{c^2 - b^2} \qquad BS' = \frac{abc}{c^2 - a^2} \qquad CT' = \frac{abc}{b^2 - a^2} \quad (5)$$

FIGURE 14

For $\qquad AR'^2 = BR' \cdot CR'$

$$(AR^2 + BR^2)b^2 + (AR^2 + CR^2)c^2 = 2b^2c^2 \quad (6)\,*$$

and so on

* Mr Clément Thiry, *Applications remarquables du Théorème de Stewart*, p. 20 (1891)

https://doi.org/10.1017/S0013091500031758 Published online by Cambridge University Press

$$\left.\begin{aligned} AK &= \frac{2bc\,m_1}{a^2+b^2+c^2} & RK &= \frac{2a^2bc\,m_1}{(b^2+c^2)(a^2+b^2+c^2)} \\ BK &= \frac{2ca\,m_2}{a^2+b^2+c^2} & SK &= \frac{2ab^2c\,m_2}{(c^2+a^2)(a^2+b^2+c^2)} \\ CK &= \frac{2ab\,m_3}{a^2+b^2+c^2} & TK &= \frac{2abc^2\,m_3}{(a^2+b^2)(a^2+b^2+c^2)} \end{aligned}\right\} \quad (7)^*$$

FIGURE 18

Since $$\frac{AS}{CS} = \frac{AKB}{CKB} \qquad \frac{AT}{BT} = \frac{AKC}{BKC}$$

therefore $$\frac{AS}{CS} + \frac{AT}{BT} = \frac{AKB + AKC}{BKC}$$

$$= \frac{AKB + AKC}{BKR + CKR}$$

Now $$\frac{AK}{RK} = \frac{AKB}{BKR} = \frac{AKC}{CKR}$$

$$= \frac{AKB + AKC}{BKR + CKR}$$

therefore $$\frac{AK}{RK} = \frac{AS}{CS} + \frac{AT}{BT}$$

$$= \frac{b^2+c^2}{a^2}$$

therefore $$\frac{AK}{AR} = \frac{b^2+c^2}{a^2+b^2+c^2}$$

$$a \cdot AK : b \cdot BK : c \cdot CK = m_1 : m_2 : m_3 \qquad (8)\dagger$$

$$a^2 \cdot AK^2 + b^2 \cdot BK^2 + c^2 \cdot CK^2 = \frac{3a^2b^2c^2}{a^2+b^2+c^2} \qquad (9)\ddagger$$

---

* The values of AK BK CK are given by E. W. Grebe in Grunert's *Archiv*, XI. 252 (1847)

† Dr Franz Wetzig in Schlömilch's *Zeitschrift*, XII. 293 (1867)

‡ Mr Clément Thiry, *Applications remarquables du Théorème de Stewart*, p. 38 (1891)

$$\left.\begin{aligned}
AK_1 &= \frac{2bc\,m_1}{-a^2+b^2+c^2} & RK_1 &= \frac{2a^2bc\,m_1}{(b^2+c^2)(-a^2+b^2+c^2)}\\
BK_2 &= \frac{2ca\,m_2}{a^2-b^2+c^2} & SK_2 &= \frac{2ab^2c\,m_2}{(c^2+a^2)(a^2-b^2+c^2)}\\
CK_3 &= \frac{2ab\,m_3}{a^2+b^2-c^2} & TK_3 &= \frac{2abc^2\,m_3}{(a^2+b^2)(a^2+b^2-c^2)}
\end{aligned}\right\} \quad (10)$$

$$\left.\begin{aligned}
BK_1 = CK_1 &= \frac{abc}{-a^2+b^2+c^2}\\
CK_2 = AK_2 &= \frac{abc}{a^2-b^2+c^2}\\
AK_3 = BK_3 &= \frac{abc}{a^2+b^2-c^2}
\end{aligned}\right\} \quad (11)$$

$$\left.\begin{aligned}
KK_1 &= \frac{4a^2bc\,m_1}{(a^2+b^2+c^2)(-a^2+b^2+c^2)}\\
KK_2 &= \frac{4ab^2c\,m_2}{(a^2+b^2+c^2)(a^2-b^2+c^2)}\\
KK_3 &= \frac{4abc^2\,m_3}{(a^2+b^2+c^2)(a^2+b^2-c^2)}
\end{aligned}\right\} \quad (12)$$

$$\left.\begin{aligned}
K_2K_3 &= \frac{2a^3bc}{(a^2-b^2+c^2)(a^2+b^2-c^2)}\\
K_3K_1 &= \frac{2ab^3c}{(a^2+b^2-c^2)(-a^2+b^2+c^2)}\\
K_1K_2 &= \frac{2abc^3}{(-a^2+b^2+c^2)(a^2-b^2+c^2)}
\end{aligned}\right\} \quad (13)$$

$$a^2\frac{AK_1}{KK_1} = b^2\frac{BK_2}{KK_2} = c^2\frac{CK_3}{KK_3} = \frac{a^2+b^2+c^2}{2} \qquad (14)^*$$

$$\frac{KK_1}{AK_1} : \frac{KK_2}{BK_2} : \frac{KK_3}{CK_3} = a^2 : b^2 : c^2 \qquad (15)^*$$

---

* Dr Franz Wetzig in Schlömilch's *Zeitschrift*, XII. 294 (1867)

$$\left.\begin{aligned}
BL &= \frac{a(a^2 - b^2 + 3c^2)}{2(a^2+b^2+c^2)} = \frac{a(c^2 + ca\cos B)}{a^2+b^2+c^2} \\
CL &= \frac{a(a^2 + 3b^2 - c^2)}{2(a^2+b^2+c^2)} = \frac{a(b^2 + ab\cos C)}{a^2+b^2+c^2} \\
CM &= \frac{b(3a^2 + b^2 - c^2)}{2(a^2+b^2+c^2)} = \frac{b(a^2 + ab\cos C)}{a^2+b^2+c^2} \\
AM &= \frac{b(-a^2 + b^2 + 3c^2)}{2(a^2+b^2+c^2)} = \frac{b(c^2 + bc\cos A)}{a^2+b^2+c^2} \\
AN &= \frac{c(-a^2 + 3b^2 + c^2)}{2(a^2+b^2+c^2)} = \frac{c(b^2 + bc\cos A)}{a^2+b^2+c^2} \\
BN &= \frac{c(3a^2 - b^2 + c^2)}{2(a^2+b^2+c^2)} = \frac{c(a^2 + ca\cos B)}{a^2+b^2+c^2}
\end{aligned}\right\} \quad (16)\,*$$

DISTANCES OF K FROM THE SIDES OF ABC

$$\left.\begin{aligned}
KL &= \frac{2a\Delta}{a^2+b^2+c^2} = \frac{a\sin A\sin B\sin C}{\sin^2 A + \sin^2 B + \sin^2 C} \\
KM &= \frac{2b\Delta}{a^2+b^2+c^2} = \frac{b\sin A\sin B\sin C}{\sin^2 A + \sin^2 B + \sin^2 C} \\
KN &= \frac{2c\Delta}{a^2+b^2+c^2} = \frac{c\sin A\sin B\sin C}{\sin^2 A + \sin^2 B + \sin^2 C}
\end{aligned}\right\} \quad (17)\,*$$

FIGURE 23

Draw AX perpendicular to BC

Then $$AR : KR = AX : KL$$

therefore $$KL = \frac{KR \cdot AX}{AR}$$

$$= \frac{a^2 h_1}{a^2+b^2+c^2}$$

$$= \frac{2a\Delta}{a^2+b^2+c^2}$$

* E. W. Grebe in Grunert's *Archiv*, IX, 252, 250-1 (1847)

https://doi.org/10.1017/S0013091500031758 Published online by Cambridge University Press

The following is another demonstration *

Let α β γ denote the distances of K from BC CA AB

Then
$$\frac{\alpha}{a} = \frac{\beta}{b} = \frac{\gamma}{c}$$
$$= \frac{a\alpha}{a^2} = \frac{b\beta}{b^2} = \frac{c\gamma}{c^2}$$
$$= \frac{a\alpha + b\beta + c\gamma}{a^2 + b^2 + c^2}$$
$$= \frac{2\triangle}{a^2 + b^2 + c^2}$$

$$\left.\begin{aligned} \mathrm{KL}^2 + \mathrm{KM}^2 + \mathrm{KN}^2 &= \frac{4\triangle^2}{a^2+b^2+c^2} \\ = \frac{a^2 \sin^2\mathrm{B}\sin^2\mathrm{C}}{\sin^2\mathrm{A} + \sin^2\mathrm{B} + \sin^2\mathrm{C}} &= \frac{b^2 \sin^2\mathrm{C}\sin^2\mathrm{A}}{\sin^2\mathrm{A} + \sin^2\mathrm{B} + \sin^2\mathrm{C}} \\ = \frac{c^2 \sin^2\mathrm{A}\sin^2\mathrm{B}}{\sin^2\mathrm{A} + \sin^2\mathrm{B} + \sin^2\mathrm{C}} &= \frac{\triangle}{\cot\mathrm{A} + \cot\mathrm{B} + \cot\mathrm{C}} \end{aligned}\right\} (18) \dagger$$

Distances of K_1 K_2 K_3 from the sides of ABC

$$\left.\begin{array}{llll} (\mathrm{K}_1) & \dfrac{-2a\triangle}{-a^2+b^2+c^2} & \dfrac{2b\triangle}{-a^2+b^2+c^2} & \dfrac{2c\triangle}{-a^2+b^2+c^2} \\ (\mathrm{K}_2) & \dfrac{2a\triangle}{a^2-b^2+c^2} & \dfrac{-2b\triangle}{a^2-b^2+c^2} & \dfrac{2c\triangle}{a^2-b^2+c^2} \\ (\mathrm{K}_3) & \dfrac{2a\triangle}{a^2+b^2-c^2} & \dfrac{2b\triangle}{a^2+b^2-c^2} & \dfrac{-2c\triangle}{a^2+b^2-c^2} \end{array}\right\} (19)$$

---

* Mr R. Tucker in *Quarterly Journal of Mathematics*, XIX. 342 (1883)

† The first of these values is given by "Yanto" in Leybourn's *Mathematical Repository*, old series, Vol. III. p. 71 (1803). Lhuilier in his *Élémens d'Analyse*, p. 298 (1809) gives the analogous property for the tetrahedron.

The other values are given by E. W. Grebe in Grunert's *Archiv*, IX. 251 (1847)

Grebe, *loco citato*, p. 257, gives the distances of K_3 from the sides of ABC with the following trigonometrical equivalents

$$\left.\begin{aligned}\frac{a \sin A \sin B \sin C}{\sin^2 A + \sin^2 B - \sin^2 C} &= \tfrac{1}{2} a \tan C \\ \frac{b \sin A \sin B \sin C}{\sin^2 A + \sin^2 B - \sin^2 C} &= \tfrac{1}{2} b \tan C \\ \frac{-c \sin A \sin B \sin C}{\sin^2 A + \sin^2 B - \sin^2 C} &= -\tfrac{1}{2} c \tan C\end{aligned}\right\} \quad (20)$$

If Σ Σ_1 Σ_2 Σ_3 denote the sum of the squares of the distances from the sides of ABC of K K_1 K_2 K_3

$$\Sigma_1 = \frac{4\Delta^2}{-a^2+b^2+c^2} \qquad \Sigma_2 = \frac{4\Delta^2}{a^2-b^2+c^2} \qquad \Sigma_3 = \frac{4\Delta^2}{a^2+b^2-c^2} \quad (21)\ *$$

$$\frac{1}{\Sigma} = \frac{1}{\Sigma_1} + \frac{1}{\Sigma_2} + \frac{1}{\Sigma_3} \quad (22)\ *$$

If k k_1 k_2 k_3 denote the distances from BC of K K_1 K_2 K_3

$$\frac{1}{k} + \frac{1}{k_1} = \frac{1}{k_2} + \frac{1}{k_3} = \frac{2}{h_1} \quad (23)\ *$$

This relation holds for any four harmonically associated points

$$MN^2 + NL^2 + LM^2 = \frac{12\Delta^2}{a^2+b^2+c^2} \quad (24)$$

For the left side $= 3\ (KL^2 + KM^2 + KN^2)$

$$\left.\begin{aligned}MN &= \frac{4m_1\Delta}{a^2+b^2+c^2} = \frac{m_1\Sigma}{\Delta} \\ NL &= \frac{4m_2\Delta}{a^2+b^2+c^2} = \frac{m_2\Sigma}{\Delta} \\ LM &= \frac{4m_3\Delta}{a^2+b^2+c^2} = \frac{m_3\Sigma}{\Delta}\end{aligned}\right\} \quad (25)\ \dagger$$

* Dr Franz Wetzig in Schlömilch's *Zeitschrift*, XII. 294-295 (1867)

† Both forms are given by E. W. Grebe in Grunert's *Archiv*, IX. 253 (1847)

For MN can be found by applying Ptolemy's theorem to the encyclic quadrilateral ANKM

$$\text{LMN} = \frac{12\Delta^3}{(a^2+b^2+c^2)^2} \qquad (26)\,*$$

FIGURE 28

Since K is the controid of LMN,

$$\text{LMN} = 3\text{KLK}'$$

Now KLK′ has its sides equal to KL KM KN and it is similar to ABC

therefore $$\frac{\text{KLK}'}{\text{ABC}} = \frac{\text{KL}^2+\text{KM}^2+\text{KN}^2}{a^2+b^2+c^2}$$

$$= \frac{4\Delta^2}{(a^2+b^2+c^2)^2}$$

$$\text{KBC} : \text{KCA} : \text{KAB} = a^2 : b^2 : c^2 \qquad (27)\,\dagger$$

$$\left.\begin{aligned} \text{K}_1\text{BC} : \text{K}_1\text{CA} : \text{K}_1\text{AB} &= -a^2 : b^2 : c^2 \\ \text{K}_2\text{BC} : \text{K}_2\text{CA} : \text{K}_2\text{AB} &= a^2 : -b^2 : c^2 \\ \text{K}_3\text{BC} : \text{K}_3\text{CA} : \text{K}_3\text{AB} &= a^2 : b^2 : -c^2 \end{aligned}\right\} (28)\,\ddagger$$

$$\left.\begin{aligned} \text{AA}'\cdot\text{BB}'\cdot\text{CC}' &: \text{AK}_1\cdot\text{BK}_2\cdot\text{CK}_3 \\ &= \text{AK}\cdot\text{BK}\cdot\text{CK} : \text{KK}_1\cdot\text{KK}_2\cdot\text{KK}_3 \end{aligned}\right\} (29)\,\ddagger$$

$$\left.\begin{aligned} \text{AK}_2 : \text{AK}_3 &= \text{CX} : \text{BX} \\ \text{BK}_3 : \text{BK}_1 &= \text{AY} : \text{CY} \\ \text{CK}_1 : \text{CK}_2 &= \text{BZ} : \text{AZ} \end{aligned}\right\} \qquad (30)\,\ddagger$$

* Dr Franz Wetzig in Schlömilch's *Zeitschrift*, XII. 298 (1867)

† L. C. Schulz von Strasznicki in Baumgartner and D'Ettingshausen's *Zeitschrift für Physik und Mathematik*, II. 403 (1827)

‡ Dr Franz Wetzig in Schlömilch's *Zeitschrift*, XII. 287, 293, 291 (1867)

https://doi.org/10.1017/S0013091500031758 Published online by Cambridge University Press

FIGURE 23

Draw CZ perpendicular to AB
Let the tangent at A meet K_1C produced at K_2 and draw K_2B' perpendicular to CA

From the similar triangles K_1CA' CAZ

$$K_1C : CA' = CA : AZ$$

or
$$K_1C : a = b : 2AZ$$

From the similar triangles K_2CB' CBZ

$$K_2C : CB' = CB : BZ$$

or
$$K_2C : b = a : 2BZ$$

Hence
$$K_1C \cdot AZ = \tfrac{1}{2}ab$$
$$= K_2C \cdot BZ$$

therefore
$$K_1C : K_2C = BZ : AZ$$

If k_1 k_2 k_3 denote the distances of K from BC CA AB

$$k_1^2 + k_2^2 + k_3^2 : a^2 + b^2 + c^2 = \tfrac{1}{3}\text{LMN} : \text{ABC} \qquad (31)\,*$$

If k_1' k_2' k_3' denote the distances of K_1
k_1'' k_2'' k_3'' ,, ,, ,, ,, K_2
k_1''' k_2''' k_3''' ,, ,, ,, ,, K_3
from BC CA AB

$$k_1'k_1''k_1''' : k_2'k_2''k_2''' : k_3'k_3''k_3''' = a^3 : b^3 : c^3 \qquad (32)\,*$$

$$K_2K_3 \cdot K_3K_1 \cdot K_1K_2 : BC \cdot CA \cdot AB = 2\text{circle}K_1K_2K_3 : \text{circle}ABC \qquad (33)\,*$$

FIGURE 26

If the sides of triangle DEF be denoted by d e f, the following formula is obtained by comparison of the similar triangles $I_1I_2I_3$ DEF

$$\frac{def}{r^2} = \frac{abc}{2R^2}$$

Now, in Fig. 26, triangles DEF ABC stand in the same relation to each other as ABC $K_1K_2K_3$

* Dr Wetzig in Schlömilch's *Zeitschrift*, XII. 298, 296, 292 (1867)

$$abc : R^2 = 2\triangle K_1K_2K_3 : \text{radius of circle } K_1K_2K_3 \qquad (34)$$

This follows from the preceding since

$$\frac{abc}{2R^2} = \frac{2\triangle}{R}$$

COSINE CIRCLE OR SECOND LEMOINE CIRCLE

FIGURE 19

$$\left.\begin{aligned} AE &= \frac{2bc^2}{a^2+b^2+c^2} & AF' &= \frac{2b^2c}{a^2+b^2+c^2} \\ BF &= \frac{2ca^2}{a^2+b^2+c^2} & BD' &= \frac{2c^2a}{a^2+b^2+c^2} \\ CD &= \frac{2ab^2}{a^2+b^2+c^2} & CE' &= \frac{2a^2b}{a^2+b^2+c^2} \end{aligned}\right\} \qquad (35)$$

For triangles AEF′ ABC are similar and AK is a median of AEF′

therefore $\qquad AE : AB = AK : m_1$

$$\left.\begin{aligned} AE' &= \frac{b(-a^2+b^2+c^2)}{a^2+b^2+c^2} & AF &= \frac{c(-a^2+b^2+c^2)}{a^2+b^2+c^2} \\ BF' &= \frac{c(a^2-b^2+c^2)}{a^2+b^2+c^2} & BD &= \frac{a(a^2-b^2+c^2)}{a^2+b^2+c^2} \\ CD' &= \frac{a(a^2+b^2-c^2)}{a^2+b^2+c^2} & CE &= \frac{b(a^2+b^2-c^2)}{a^2+b^2+c^2} \end{aligned}\right\} \qquad (36)$$

$$DD' = \frac{a(-a^2+b^2+c^2)}{a^2+b^2+c^2} \quad EE' = \frac{b(a^2-b^2+c^2)}{a^2+b^2+c^2} \quad FF' = \frac{c(a^2+b^2-c^2)}{a^2+b^2+c^2} \qquad (37)$$

$$FD = \frac{4a\triangle}{a^2+b^2+c^2} \qquad DE = \frac{4b\triangle}{a^2+b^2+c^2} \qquad EF = \frac{4c\triangle}{a^2+b^2+c_2} \qquad (38)$$

For $\qquad FD^2 = E'D^2 - E'F^2 = E'D^2 - D'D^2$

https://doi.org/10.1017/S0013091500031758 Published online by Cambridge University Press

$$BD' \cdot CD = CE' \cdot AE = AF' \cdot BF \qquad (39)$$

$$\left.\begin{aligned} BD' : CE' : AF' &= \frac{c}{b} : \frac{a}{c} : \frac{b}{a} \\ BF : CD : AE &= \frac{a}{b} : \frac{b}{c} : \frac{c}{a} \end{aligned}\right\} \qquad (40)$$

Triplicate Ratio or First Lemoine Circle

The whole of the subsequent results are taken from two of Mr R. Tucker's papers in the *Quarterly Journal of Mathematics*, XIX. 342-348 (1883) and XX. 57-59 (1885). The proofs are sometimes different from Mr Tucker's

Figure 32

$$\left.\begin{aligned} AF &= \frac{b^2c}{a^2+b^2+c^2} & AE' &= \frac{bc^2}{a^2+b^2+c^2} \\ BD &= \frac{c^2a}{a^2+b^2+c^2} & BF' &= \frac{ca^2}{a^2+b^2+c^2} \\ CE &= \frac{a^2b}{a^2+b^2+c^2} & CD' &= \frac{ab^2}{a^2+b^2+c^2} \end{aligned}\right\} \qquad (41)$$

For $$\frac{ABC}{AFE'} = \frac{AB \cdot AC}{AF \cdot AE'}$$

therefore $$\frac{2\triangle}{AE' \cdot KM} = \frac{bc}{AF \cdot AE'}$$

therefore $$AF = \frac{bc\ KM}{2\triangle}$$

$$= \frac{bc \cdot b}{a^2+b^2+c^2}$$

$$BD \cdot CD' = CE \cdot AE' = AF \cdot BF' \qquad (42)$$

https://doi.org/10.1017/S0013091500031758 Published online by Cambridge University Press

$$
\left.\begin{aligned}
\mathrm{AF}' &= \frac{c(b^2+c^2)}{a^2+b^2+c^2} & \mathrm{AE} &= \frac{b(b^2+c^2)}{a^2+b^2+c^2} \\
\mathrm{BD}' &= \frac{a(c^2+a^2)}{a^2+b^2+c^2} & \mathrm{BF} &= \frac{c(c^2+a^2)}{a^2+b^2+c^2} \\
\mathrm{CE}' &= \frac{b(a^2+b^2)}{a^2+b^2+c^2} & \mathrm{CD} &= \frac{a(a^2+b^2)}{a^2+b^2+c^2}
\end{aligned}\right\} \quad (43)
$$

$$
\mathrm{DD}' = \frac{a^3}{a^2+b^2+c^2} \qquad \mathrm{EE}' = \frac{b^3}{a^2+b^2+c^2} \qquad \mathrm{FF}' = \frac{c^3}{a^2+b^2+c^2} \quad (44)\,*
$$

$$
\left.\begin{aligned}
\mathrm{BD} : \mathrm{DD}' : \mathrm{D'C} &= c^2 : a^2 : b^2 \\
\mathrm{CE} : \mathrm{EE}' : \mathrm{E'A} &= a^2 : b^2 : c^2 \\
\mathrm{AF} : \mathrm{FF}' : \mathrm{F'B} &= b^2 : c^2 : a^2
\end{aligned}\right\} \quad (45)
$$

$$
\mathrm{EF}' = \frac{a(b^2+c^2)}{a^2+b^2+c^2} \qquad \mathrm{FD}' = \frac{b(c^2+a^2)}{a^2+b^2+c^2} \qquad \mathrm{DE}' = \frac{c(a^2+b^2)}{a^2+b^2+c^2} \quad (46)
$$

$$
\mathrm{E'F} = \mathrm{F'D} = \mathrm{D'E} = \frac{abc}{a^2+b^2+c^2} \quad (47)
$$

For DE′FF′ is a symmetrical trapezium

therefore $\mathrm{E'F}^2 = \frac{1}{4}(\mathrm{DE}' - \mathrm{FF}')^2 + \mathrm{KN}^2$

$$
\begin{aligned}
&= \tfrac{1}{4}\left\{\frac{c(a^2+b^2)}{a^2+b^2+c^2} - \frac{c^3}{a^2+b^2+c^2}\right\}^2 + \left\{\frac{2c\Delta}{a^2+b^2+c^2}\right\}^2 \\
&= \frac{c^2}{4(a^2+b^2+c^2)^2}\{(a^2+b^2-c^2)^2 + 16\Delta^2\} \\
&= \frac{4a^2b^2c^2}{4(a^2+b^2+c^2)^2}
\end{aligned}
$$

$$
\left.\begin{aligned}
\mathrm{DF} &= \frac{c}{a^2+b^2+e^2}\sqrt{b^2c^2+c^2a^2+a^2b^2} \\
\mathrm{FE} &= \frac{b}{a^2+b^2+c^2}\sqrt{b^2c^2+c^2a^2+a^2b^2} \\
\mathrm{ED} &= \frac{a}{a^2+b^2+c^2}\sqrt{b^2c^2+c^2a^2+a^2b^2}
\end{aligned}\right\} \quad (48)
$$

* It was this property which suggested to Mr Tucker the name "triplicate-ratio circle."

https://doi.org/10.1017/S0013091500031758 Published online by Cambridge University Press

For D E′ F F′ are concyclic
and DE′FF′ is a symmetrical trapezium ;

therefore $$DF \cdot E'F' = DE' \cdot FF' + DF' \cdot FE'$$

that is $$DF^2 = DE' \cdot FF' + DF'^2$$

$$\left.\begin{aligned} KE \cdot KF' = KF \cdot KD' = KD \cdot KE' \\ = E'F^2 = F'D^2 = D'E^2 \end{aligned}\right\} \qquad (49)$$

For $$KE \cdot KF' = CD' \cdot BD$$

$$\left.\begin{aligned} &\text{The minimum chord through K} \\ &= 2E'F = 2F'D = 2D'E \end{aligned}\right\} \qquad (50)$$

For $$KE \cdot KF' = E'F^2$$

$$\left.\begin{aligned} DD' \cdot EE' \cdot FF' = E'F \cdot F'D \cdot D'E \\ = \frac{a^3b^3c^3}{(a^2+b^2+c^2)^3} \end{aligned}\right\} \qquad (51)$$

The hexagon DD′EE′FF′ has its

perimeter $$= \frac{a^3+b^3+c^3+3abc}{a^2+b^2+c^2} \qquad (52)$$

area $$= \frac{\triangle(a^4+b^4+c^4+b^2c^2+c^2a^2+a^2b^2)}{(a^2+b^2+c^2)^2} \qquad (53)$$

$$\left.\begin{aligned} &= AFE + BDF + CED \\ &= AE'F' + BF'D' + CD'E' \end{aligned}\right\} \qquad (54)$$

The circle DD′EE′FF′ has its

radius $$= \frac{\sqrt{b^2c^2+c^2a^2+a^2b^2}}{a^2+b^2+c^2} \cdot R \qquad (55)$$

If EF′	cuts	FD	D′E′	at	L	L′
FD′	,,	DE	E′F′	,,	M	M′
DE′	,,	EF	F′D′	,,	N	N′

then
$$\left.\begin{aligned}
FL &: DL = c^2 : a^2 \\
D'L' &: E'L' = a^2 : b^2 \\
DM &: EM = a^2 : b^2 \\
E'M' &: F'M' = b^2 : c^2 \\
EN &: FN = b^2 : c^2 \\
F'N' &: D'N' = c^2 : a^2
\end{aligned}\right\} \quad (56)$$

FIGURE 33

For $\quad FL : DL = FK : D'K = AE' : CE$

and $\quad D'L' : E'L' = CE : EE'$

https://doi.org/10.1017/S0013091500031758 Published online by Cambridge University Press