Canad. Math. Bull. Vol. **22** (1), 1979

A LOWER BOUND FOR THE NUMBER OF NEGATIVE ZEROS OF POWER SERIES

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0. In this paper we are concerned with power series of the type

$$f(z)=\sum_0^\infty a(n)z^n, \tag{1}$$

which admit unique analytic extension onto a domain containing the negative real axis. Our primary object is to establish a general theorem giving a lower estimate for the number of different zeros of (1) on the negative real axis. W. Jurkat and A. Peyerimhoff showed that for a certain class of coefficient functions $a(z)$ the number of negative zeros of (1) is closely related to the behaviour of $a(z)$ at $z=0$. In particular they proved the following theorem [4, p. 219, Theorem 4].

THEOREM JP. *Let $a\in C_p[0,\infty)$ for some $p=0,1,\ldots,k-1$ $(k\geq 1)$ be a real solution of the differential equation*

$$\left\{\prod_1^k\left(\frac{d}{dx}-\xi_i\right)\right\}a(x)=\phi(x),\qquad x>0,\quad \xi_i\leq 0, \tag{2}$$

$\phi(x)$ being completely monotone for $x>0$. Moreover let

$$a(0)=a'(0)=\cdots=a^{(p)}(0)=0.$$

Then

$$f(z)=\sum_0^\infty a(n+\tau)z^n,\qquad \tau\in[0,1),$$

defines on $\mathbb{C}^*=\{z=x+iy\mid y\neq 0 \text{ if } x\geq 1\}$ uniquely a holomorphic function which has at most k zeros (unless $f\equiv 0$) and at least $p+1$ different zeros which are ≤ 0.

Their proof for the upper estimate as well as for the lower estimate essentially is based on condition (2).

Received by the editors July 20, 1977 and in revised form, February 13, 1978 and March 31, 1978.

This research was supported by the National Research Council of Canada, Grants A-2983 and A-4055.

Since $\phi(x)$ is completely monotone, we have $\phi(x)=\int_{+0}^{1} w^x\,dg(w)$, $x>0$, for some monotonically increasing $g(w)$. Hence every solution of (2) is holomorphic for $\operatorname{Re} x>0$. It is the main purpose of this paper to replace condition (2) by much weaker assumptions on the growth and the analytic behaviour of $a(z)$ in the right half-plane. Then we show that the lower estimate for the number of negative zeros remains true. We remark that our functions neither need to be holomorphic in $\mathbb{C}^*$ (see example V in Section 2) nor have to have a finite number of negative zeros (see examples II and III in Sec. 2) like those in [4] so that we are in a position to discuss power series which cannot be treated by known methods.

1. Before stating our main theorem we recall a well-known general result concerning analytic continuation of power series due to Lindelöf [5, chapitre V, p. 109].

If $a(z)$ is holomorphic in a right half-plane, $\operatorname{Re} z\geq\alpha$ say, and if there exists a number $\theta<\pi$ such that for every $\varepsilon>0$ and sufficiently large r

(3) $$|a(\alpha+re^{i\phi})|<e^{(\theta+\varepsilon)r}, \qquad |\phi|\leq\pi/2,$$

then the power series (1) defines (uniquely) a holomorphic function in the angle

(4) $$\theta<\arg z<2\pi-\theta.$$

(3) means that $a(z)$ is of exponential type and possesses a conjugate diagram whose width is less than 2π. Further if (3) holds for α, so it does for every $\beta\geq\alpha$ [cf. 3, Sec. 11.3].

THEOREM. *Suppose that the function* $a(z)$ *is holomorphic throughout* $\operatorname{Re} z>0$, *continuous for* $\operatorname{Re} z\geq 0$, *real-valued for real* $z\geq 0$, *and that* (3) *is satisfied for* $\alpha\geq 0$. *Moreover, assume that* $a^*\in C_p[0,\infty)$ ($a^*(t)=a(it)$ *for real* t) *for some integer* $p\geq 0$,

(5) $$|a^{*(p)}(t)|<e^{(\theta+\varepsilon)|t|}$$

for sufficiently large $|t|$, *and that*

(6) $$a^*(0)=a^{*\prime}(0)=\cdots=a^{*(p)}(0)=0.$$

Then

$$f(z)=\sum_{0}^{\infty} a(n+\tau)z^n, \qquad \tau\in[0,1),$$

defines (*uniquely*) *a holomorphic function in the domain* (4) *and has at least* $p+1$ *different zeros which are* ≤ 0. *The zeros of those being on the negative real axis have odd multiplicities.*

REMARK. Obviously (5) holds for $a^{*(\nu)}(t)$, $\nu=0,\ldots,p$. Further, since $a(z)$ is real-valued for real $z\geq 0$, actually, by (6), we have $a^*\in C_p(-\infty,\infty)$.

Proof of the Theorem. It remains to show the assertion concerning the zeros. First we use standard residue technique. By (3), we have the following representation, $0<\delta+\tau<1, \delta>0$,

$$f(z)=-\int_{\delta-i\infty}^{\delta+i\infty}\frac{a(\zeta+\tau)}{e^{2\pi i\zeta}-1}e^{\zeta\log z}\,d\zeta+a(\tau), \tag{7}$$

valid throughout the region (4). The contour of integration is the oriented line $\operatorname{Re}\zeta=\delta$ and $\log z$ is defined by $\log z=\log|z|+i\arg z$, $0<\arg z<2\pi$. Now we put $\xi=\zeta+\tau$ and shift the contour by $\delta+\tau$ to the left. Using (3) and the continuity of $a(\xi)$ on $\operatorname{Re}\xi\geq 0$ we obtain (If $\tau=0$, then observe that $a(0)=0$)

$$f(z)=-\int_{-i\infty}^{i\infty}\frac{a(\xi)}{e^{2\pi i(\xi-\tau)}-1}e^{(\xi-\tau)\log z}\,d\xi, \tag{8}$$

which again is valid throughout (4). Introducing a new variable in (8) by $\xi=it$ it follows that

$$f(z)=-i\int_{-\infty}^{\infty}\frac{a(it)}{e^{-2\pi t-2\pi i\tau}-1}e^{(it-\tau)\log z}\,dt, \tag{9}$$

and so for $z=-x$, $x>0$, on the negative real axis

$$x^{\tau}f(-x)=\frac{i}{2}\int_{-\infty}^{\infty}\frac{a(it)}{\sinh(\pi t+\pi i\tau)}e^{it\log x}\,dt. \tag{10}$$

Next, we put

$$g(t)=\frac{i}{2}\frac{a(it)}{\sinh(\pi t+\pi i\tau)}\quad\text{and}\quad \hat{g}(\xi)=e^{\tau\xi}f(-e^{\xi}).$$

Then (10) can be rewritten as

$$\hat{g}(\xi)=\int_{-\infty}^{\infty}g(t)e^{it\xi}\,dt. \tag{11}$$

Since the case $\tau=p=0$ is trivial, we may confine ourselves to the case $p\geq 1$, when $\tau=0$. Now it suffices to show that $\hat{g}(\xi)$ has at least $p+1$ or p different real zeros, when $\tau>0$ or $\tau=0$ respectively.

Suppose $\tau=0$ and $p\geq 1$. In view of the differentiability properties of a^* and (6) we have, by Taylor's theorem, that

$$a^{*(\nu)}(t)=\frac{1}{(p-\nu-1)!}\int_0^t(t-x)^{p-\nu-1}a^{*(p)}(x)\,dx,\qquad 0\leq\nu<p,\quad p\geq 1.$$

By the continuity of $a^{*(p)}(t)$ a simple computation leads to the estimate

$$|g^{(\mu)}(t)|\leq Kt^{p-\mu-1}\max_{0\leq x\leq t}|a^{*(p)}(x)|=o(1),$$

https://doi.org/10.4153/CMB-1979-007-5 Published online by Cambridge University Press

as $t\to 0$, and so we get that $g^{(\mu)}(0)=0$, $\mu=0,\ldots,p-1$. Next it follows from (5) that $g^{(\mu)}\in L^1(-\infty,\infty)$. Hence [see e.g. 1, Prop. 5.1.14, p. 194]

$$\xi^\mu\hat{g}(\xi)=C\int_{-\infty}^{\infty} g^{(\mu)}(t)e^{it\xi}\,dt, \qquad \mu=0,\ldots,p-1.$$

Now, since $g^{(\mu)}(t)$ is continuous, an application of Fourier's theorem yields

$$g^{(\mu)}(t)=\frac{C}{2\pi}\int_{-\infty}^{\infty} \xi^\mu\hat{g}(\xi)e^{-it\xi}\,d\xi, \qquad \mu=0,\ldots,p-1,$$

(The integrals exist (at least) as a principal value) and thus

$$\int_{-\infty}^{\infty} \xi^\mu\hat{g}(\xi)\,d\xi=0, \qquad \mu=0,\ldots,p-1.$$

Hence $\hat{g}(\xi)$ changes sign at least p times [8, prob. 140, p. 65] and this is equivalent to the fact that $f(z)$ has at least p different zeros which are negative and possess odd multiplicities. Since $z=0$ is a zero, in this case $(\tau=0)$ the proof is complete. (For a similar method see [7, p. 187].)

If $\tau\in(0,1)$, then direct application of our preceding arguments to (11) leads to

$$\int_{-\infty}^{\infty} \xi^\mu\hat{g}(\xi)\,d\xi=0, \qquad \mu=0,\ldots,p,$$

and so $\hat{g}(\xi)$ changes sign at least $p+1$ times. This completes the proof.

REMARK. Actually the proof shows that the number of zeros being negative is at least $p+1$, when $\tau\in(0,1)$.

2. In this section we give various applications of the preceding results. For some of the following examples, where $f(z)$ can be extended analytically onto $\mathbb{C}^*$ (this corresponds to the case $\theta=0$), upper estimates are given in [2, 4, 6]. Most verifications of the assumptions in the theorem are very simple and so we omit them.

(I) The choices [4, p. 219]

$$a_1(z)=z^\kappa; \qquad a_2(z)=(1-c^z)^\kappa,\quad 0<c<1; \qquad a_3(z)=\int_0^z t^{\kappa-1}e^{-t}\,dt,$$

where $k<\kappa\le k+1$, $k=0,1,2,\ldots,$ lead to functions $f_i(z)=\sum_0^\infty a_i(n+\tau)z^n$, $\tau\in[0,1)$, being holomorphic in $\mathbb{C}^*$. They have at least $k+1$ different zeros which are ≤ 0. It follows from Theorem JP that $k+1$ is the exact number of zeros of $f_i(z)$ in $\mathbb{C}^*$.

Note that $a_2(z)$ has branch points at $z=2m\pi i/\log c$, $m\in\mathbb{Z}$, when κ is not an integer, but $a_2(z)$ is k times continuously differentiable on $\operatorname{Re} z\ge 0$, and $a_2^{*(k)}(t)=C(1-c^{it})^{\kappa-k}$ satisfies the growth condition (5).

For the following two examples the theorem guarantees an infinite number of negative zeros.

(II) $f(z)=\sum_0^\infty e^{-(n+\tau)^{-\alpha}}z^n$, $0<\alpha<1$, $\tau\in[0,1)$, defines a holomorphic function on $\mathbb{C}^*$. Since the theorem applies with every $p\geq 0$, $f(z)$ has an infinite number of zeros on the negative real axis. (Interpret $\exp(-0^{-\alpha})=\lim_{t\to 0+}\exp(-t^{-\alpha})=0$.)

(III) $f(z)=\sum_0^\infty (n+\tau)^{-\log(n+\tau)}z^n$, $\tau\in[0,1)$, furnishes an analytic function on $\mathbb{C}^*$ which has infinitely many zeros on the negative real axis. (Again interpret $0^{-\log 0}=\lim_{t\to 0+}\exp(-\log^2 t)=0$.) In connection with this example it should be mentioned that

$$g_1(z)=\sum_1^\infty \frac{z^n}{n^\alpha}=\frac{z}{\Gamma(\alpha)}\int_0^1 \frac{(\log(1/t))^{\alpha-1}}{1-zt}\,dt \qquad (\alpha>0)$$

has no zero in $\mathbb{C}^*$ except for $z=0$ [6, Lemma 1, p. 194], and the entire function

$$g_2(z)=\sum_1^\infty \frac{z^n}{n^n}=z\int_0^1 e^{zt\log(1/t)}\,dt$$

has no real zero except for $z=0$.

(IV) $f(z)=\sum_0^\infty (n+\tau)^\kappa(\log(n+1+\tau))^\lambda z^n$, $\kappa+\lambda>0$, $k<\kappa+\lambda\leq k+1$, $k=0,1,2,\ldots$, $\tau\in[0,1)$. The analytic extension of this power series (onto $\mathbb{C}^*$) has at least $k+1$ different zeros which are ≤ 0.

Taking $\kappa=0$, $\tau=\lambda=1$, it follows from the formula

$$\log(n+2)=\int_0^1 \frac{1-t^{n+1}}{\log(1/t)}\,dt, \qquad n=0,1,2,\ldots,$$

and a simple computation that

$$(1-z)f(z)=\int_0^1 \frac{1-t}{\log(1/t)}\frac{dt}{1-zt}.$$

Hence [6, p. 194] $f(z)$ has no zero in $\mathbb{C}^*$. This proves that the theorem cannot be extended to $\tau=1$.

(V) $f(z)=\sum_0^\infty n^\kappa\sin(\alpha n)z^n$, $\kappa>0$, $0<\alpha<\pi$, gives an example having no analytic extension onto $\mathbb{C}^*$. The Theorem ensures analytic continuation into the angle $\alpha<\arg z<2\pi-\alpha$. But noting that $f_\kappa(z)=\sum_0^\infty n^\kappa z^n$ is holomorphic in $\mathbb{C}^*$, actually we have

$$f(z)=\frac{1}{2i}(f_\kappa(e^{i\alpha}z)-f_\kappa(e^{-i\alpha}z)).$$

Thus $f(z)$ is holomorphic in the slit plane $\mathbb{C}-\{z \mid |z|\geq 1, \arg z=\pm\alpha\}$. Defining the non-negative integer k by $k<\kappa\leq k+1$ we get that $f(z)$ has at least $k+2$ zeros which are ≤ 0.

By the last example we illustrate a case for which the theorem implies the existence of a non-simple zero.

(VI) $f(z)=\sum_0^\infty P(n)z^n$, where

$$P(z)=\tfrac{1}{6}(1+\alpha)^3(z+3)(z+2)(z+1)$$
$$-\tfrac{3}{2}(1+\alpha)^2(z+2)(z+1)+3(1+\alpha)(z+1)-1, \qquad \alpha>0.$$

Since $P(-1)=-1$ and $P(0)=\alpha^3$, there exists $\tau\in(0,1)$ such that $P(-\tau)=0$. Put $a(z)=P(z-\tau)$. Hence, since $a(\tau)\neq 0$, the Theorem implies that

$$f(z)=\sum_0^\infty a(n+\tau)z^n$$

has at least one zero on the negative real axis, and a simple computation yields

$$f(z)=\frac{(z+\alpha)^3}{(1-z)^4}.$$

References

1. P. L. Butzer and R. J. Nessel, *Fourier Analysis and Approximation*, vol. **I**, Academic Press, New York, London, 1971.

2. W. Gawronski and A. Peyerimhoff, *On the zeros of power series with rational coefficients*, Arch. Math., **XXIX** (1977), Fasc. 2, 173–186.

3. E. Hille, *Analytic Function Theory*, vol. **II**, Blaisdell Pub. Comp., Waltham Mass., Toronto, London, 1962.

4. W. B. Jurkat and A. Peyerimhoff, *On power series with negative zeros*, Tôhoku Math. J., vol. **24**, no. 2, 207–221.

5. E. Lindelöf, *Le calcul des résidus*, Chelsea Pub. Comp., New York, 1947.

6. A. Peyerimhoff, *On the zeros of power series*, Mich. Math. J., vol. **13** (1966), 193–214.

7. G. Polya, *On the zeros of an integral function represented by Fourier's Integral*, Messenger of Math., **3** (1923), 185–188.

8. G. Polya und G. Szegö, *Aufgaben und Lehrsätze aus der Analysis I*, Springer Verlag, Berlin, Heidelberg, New York, 1970.

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