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ALMOST SPLIT SEQUENCES WHOSE MIDDLE TERM HAS AT MOST TWO INDECOMPOSABLE SUMMANDS

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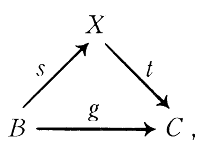
Introduction. Let Λ be an artin algebra, and denote by mod Λ the category of finitely generated Λ-modules. All modules we consider are finitely generated. We recall from [**6**] that a nonsplit exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in mod Λ is said to be *almost split* if A and C are indecomposable, and given a map $h: X \to C$ which is not an isomorphism and with X indecomposable, there is some $t: X \to B$ such that $gt = h$.

Almost split sequences have turned out to be useful in the study of representation theory of artin algebras. Given a nonprojective indecomposable Λ-module C (or an indecomposable noninjective Λ-module A), we know that there exists a unique almost split sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ [**6**, Proposition 4.3], [**5**, Section 3]. To an indecomposable nonprojective Λ-module C there is hence associated an invariant $\alpha(C)$, which denotes the number of summands in a direct sum decomposition of B into indecomposable summands. If for example Λ is a Nakayama (i.e. generalized uniserial) algebra, it is not hard to see that $\alpha(C) \leqq 2$ for each indecomposable nonprojective Λ-module C (see [**7**, Proposition 4.12]). It would be interesting to know if there is some integer K such that if C is an indecomposable non-projective module over an algebra Λ of finite representation type, then $\alpha(C) \leqq K$. Examples show that if there is some such K, then $K \geqq 4$. It would also be interesting to know if for a given artin algebra Λ, there is some integer $\alpha(\Lambda)$, such that $\alpha(C) \leqq \alpha(\Lambda)$ for each indecomposable nonprojective Λ-module C.

We recall from [**7**] that a map $g: B \to C$ in mod Λ is said to be *irreducible* if g is neither a split monomorphism nor a split epimorphism, and whenever there is some commutative diagram



then s is a split monomorphism or t is a split epimorphism. It is not hard to see that an irreducible map is either a proper epimorphism or a proper monomorphism (see [**7**, Proposition 2.6]). If C is an indecomposable nonprojective

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Λ-module, then a map $g'\colon B' \to C$ is irreducible if and only if g' is not zero and there is a map $g''\colon B'' \to C$ such that we have an almost split sequence

$$0 \to A \to B' \amalg B'' \xrightarrow{(g', g'')} C \to 0.$$

And if A is an indecomposable noninjective Λ-module, then a map $f'\colon A \to B'$ is irreducible if and only if f' is not zero and there is some map $f''\colon A \to B''$ such that we have an almost split sequence

$$0 \to A \xrightarrow{(f', f'')} B' \amalg B'' \to C \to 0$$

(see [**7**, Section 3]). We recall from [**2**, Section 6] that we have an equivalence relation $\sim$ on the indecomposable objects in mod Λ, where $A \sim B$ if and only if there is a finite sequence $A = A_0, A_1, \ldots, A_n = B$ of indecomposable modules such that there is some irreducible map from A_i to A_{i+1} or from A_{i+1} to A_i, for $i = 0, \ldots, n-1$. If C is an indecomposable Λ-module, $[C]$ denotes the corresponding equivalence class. If $0 \to A \to B \to C \to 0$ is an almost split sequence, it follows from the connection between almost split sequences and irreducible maps that $[A] = [C]$.

The main result in this paper is that for a certain class of artin algebras containing the algebras stably equivalent to hereditary algebras, in particular the hereditary algebras, we have $\alpha(C) \leqq 2$ for each indecomposable module C such that $[C]$ contains neither projectives nor injectives.

The organization of the paper is the following. In Section 1 we introduce a condition (A) on an artin algebra Λ, and show that under this condition $\alpha(C) \leqq 3$ when $[C]$ contains neither projectives nor injectives. We also investigate the special cases when $\alpha(C) \leqq 2$ and when there is some C such that $\alpha(C) = 3$ more closely.

In Section 2 we consider conditions which imply that $\alpha(C) \leqq 2$ when $[C]$ contains neither projectives nor injectives. In particular, we introduce a new condition (B) on artin algebras, which together with condition (A) will be sufficient to get our desired conclusion.

In Section 3 we show that an artin algebra stably equivalent to an hereditary algebra satisfies conditions (A) and (B). We also include a discussion of the two questions mentioned in the beginning of this introduction.

The main result of this paper was to a large extent inspired by a talk given by C. M. Ringel in Oberwolfach in June 1977, where he announced that $\alpha(C) \leqq 2$ if C is an indecomposable Λ-module such that $[C]$ contains neither projectives nor injectives, for a large class of hereditary algebras Λ. Since then Ringel has independent of our work extended his work to all hereditary algebras [**13**]. Whereas the proofs of our preliminary results in Section 1 are similar to those of Ringel, the proofs that $\alpha(C) \leqq 2$ for C indecomposable such that $[C]$ contains neither projectives nor injectives are completely different. We use for this, as in Section 1, only the theory of almost split sequences and irreducible maps, whereas Ringel assumes that the algebra is hereditary and uses in addition properties of the Coxeter transformations and earlier results by Dlab and himself on tame hereditary algebras.

If Λ is an artin algebra satisfying our conditions (A) and (B), and C is an indecomposable Λ-module such that $[C]$ contains neither projectives nor injectives, we have in addition to $\alpha(C) \leqq 2$ that if $0 \to A \to B_1 \amalg B_2 \to C \to 0$ is an almost split sequence where B_1 and B_2 are indecomposable and $L(B_1) \leqq L(C)$, then $L(B_2) > L(C)$. Here L denotes length. Much of the interest in our results lies in the fact that Ringel has shown (announced in Oberwolfach, see [**13**]) that if $[C]$ has the above properties, then the objects in $[C]$ behave much like uniserial objects. Hence it would be very interesting to find more general classes of artin algebras satisfying (A) and (B). Actually, we do not know of any artin algebra which does not satisfy these conditions.

§ **1.** Let Λ be an artin algebra and denote by $\mathscr{C}$ the full subcategory of mod Λ whose objects X are such that the indecomposable summands B of X have the property that $[B]$ contains no injectives and no projectives. We point out that $\mathscr{C}$ is empty if Λ is of finite type, and we conjecture that $\mathscr{C}$ is not empty otherwise. Denote by D the ordinary duality for artin algebras, and by Tr the transpose. For this we recall that if C is in mod Λ and $P_1 \to P_0 \to C \to 0$ is a minimal projective presentation of C in mod Λ, then the Λ^{op}-module TrC is given by the exactness of the sequence

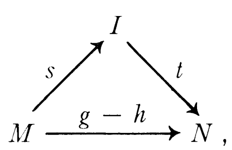
$$\mathrm{Hom}_\Lambda(P_0, \Lambda) \to \mathrm{Hom}_\Lambda(P_1, \Lambda) \to \mathrm{Tr}C \to 0.$$

Tr is not in general a functor from mod Λ to mod Λ^{op}, but is a functor, and even a duality, from $\underline{\mathrm{mod}}\,\Lambda$ to $\underline{\mathrm{mod}}\,\Lambda^{op}$, the module categories modulo projectives. Consequently DTr is not in general a functor from mod Λ to mod Λ, but from $\underline{\mathrm{mod}}\,\Lambda$ to $\overline{\mathrm{mod}}\,\Lambda$, and here even an equivalence, where $\overline{\mathrm{mod}}\,\Lambda$ denotes mod Λ modulo injectives (see [**6**]). For f in mod Λ we denote by $\underline{f}$ and $\bar{f}$ the corresponding morphisms in $\underline{\mathrm{mod}}\,\Lambda$ and $\overline{\mathrm{mod}}\,\Lambda$.

If f: $M \to N$ is a map in mod Λ, there is some f': DTr$M \to$ DTrN such that DTr $\underline{f} = \bar{f}'$, but f' is not in general uniquely determined. However, whether f' is a monomorphism is independent of the choice of f', as follows from the following lemma.

LEMMA 1.1. *Let Λ be an artin algebra and M and N objects in* mod Λ *with no nonzero injective summands. If g: $M \to N$ is a monomorphism and h: $M \to N$ is such that $\bar{g} = \bar{h}$, then h is also a monomorphism.*

Proof. Since $\bar{g} = \bar{h}$, we have a commutative diagram



where I is an injective Λ-module. Since N has no nonzero injective summands,

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we must have $t(\text{soc } I) = 0$, and consequently $(g - h)$ $(\text{soc } M) = 0$, where soc X denotes the socle of X. If x is a nonzero element in soc M, then $g(x) \neq 0$, since g is a monomorphism. Consequently we have $h(x) = g(x) \neq 0$, which shows that h: $M \to N$ is a monomorphism.

It is a consequence of Lemma 1.1 that if f: $M \to N$ is a map in mod Λ, it is well defined to say that $\mathrm{DTr}f$: $\mathrm{DTr}M \to \mathrm{DTr}N$ is a monomorphism. Similarly it is well defined to say that $\mathrm{TrD}f$: $\mathrm{TrD}M \to \mathrm{TrD}N$ is an epimorphism.

We shall now introduce the following conditions on an artin algebra Λ.

(A) If f: $M \to N$ is an irreducible monomorphism and M or N is indecomposable and in $\mathscr{C}$, then $\mathrm{DTr}f$: $\mathrm{DTr}M \to \mathrm{TrD}N$ is a monomorphism.

(A') If f: $M \to N$ is an irreducible epimorphism and M or N is indecomposable and in $\mathscr{C}$, then $\mathrm{TrD}f$: $\mathrm{TrD}M \to \mathrm{TrD}N$ is an epimorphism.

We have the following relationship between these conditions.

LEMMA 1.2. *For an artin algebra Λ, the conditions (A) and (A') are equivalent.*

Proof. Assume first that (A) holds, and let f: $M \to N$ be an irreducible epimorphism where M or N is indecomposable and in $\mathscr{C}$. Choose f': $\mathrm{TrD}M \to \mathrm{TrD}N$ such that $\underline{f'} = \mathrm{TrD}\bar{f}$. Assume to the contrary that f' is not an epimorphism. Since we know from [**8**, Proposition 1.2] that f' is irreducible, f' must then be a monomorphism. We then have $\mathrm{DTr}\underline{f'} = \mathrm{DTrTrD}\bar{f} = \bar{f}$, so that we conclude by condition (A) that f: $M \to N$ is a monomorphism. This is a contradiction to the fact that f: $M \to N$ is an irreducible epimorphism. Hence (A') holds.

It follows similarly that (A') implies (A).

We shall in this section show that if Λ satisfies (A) and C is indecomposable and in $\mathscr{C}$, then we have $\alpha(C) \leqq 3$, where $\alpha(C)$ denotes the number of indecomposable summands of B in an almost split sequence $0 \to A \to B \to C \to 0$. And we shall further get some more information in the case when $\alpha(C) \leqq 2$ for all indecomposable C in $\mathscr{C}$ and in the case when $\alpha(C) = 3$ for some indecomposable C in $\mathscr{C}$.

We start out with some preliminary results.

LEMMA 1.3. *Let Λ be an artin algebra, and assume that f: $M \to N$ and g: $N \to \mathrm{TrD}M$ are irreducible maps and that M or N is indecomposable and in $\mathscr{C}$. If Λ satisfies (A), we have the following.*

(a) f and g are not both monomorphisms.

(b) f and g are not both epimorphisms.

Proof. (a) If f: $M \to N$ and g: $N \to \mathrm{TrD}M$ are monomorphisms, then $\mathrm{DTr}f$: $\mathrm{DTr}M \to \mathrm{DTr}N$ and $\mathrm{DTr}g$: $\mathrm{DTr}N \to M$ are monomorphisms since Λ satisfies (A). Since M or N is indecomposable and in $\mathscr{C}$, we have that $\mathrm{DTr}M$ or $\mathrm{DTr}N$ is indecomposable, and is in $\mathscr{C}$, since we know that if $0 \to A \to B \to C \to 0$ is an almost split sequence, then $A \cong \mathrm{DTr}C$ [**6**, Proposition 4.3] and consequently $[C] = [\mathrm{DTr}C]$. It then follows that for all $r > 0$, $\mathrm{DTr}^r f$ and $\mathrm{DTr}^r g$ are nonzero monomorphisms, so that we get a chain of proper nonzero monomorphisms $\mathrm{DTr}^r N \to \mathrm{DTr}^{r-1} N \to \ldots \to \mathrm{DTr}N \to N$. Since r can be chosen arbitrarily large, this is a contradiction.

(*b*) Follows in an analogous way.

LEMMA 1.4. *Let* Λ *be an artin algebra satisfying* (A), *and assume that* B *is indecomposable in* $\mathscr{C}$. *Then we have the following.*

(*a*) *If* $f_1: B_1 \to B$ *and* $f_2: B_2 \to B$ *are such that* $(f_1, f_2): B_1 \amalg B_2 \to B$ *is irreducible, then* f_2 *is a monomorphism if* f_1 *is an epimorphism.*

(*b*) *If* $f_1: B \to B_1$ *and* $f_2: B \to B_2$ *are such that* $(f_1, f_2): B \to B_1 \amalg B_2$ *is irreducible, then* f_2 *is an epimorphism if* f_1 *is a monomorphism.*

Proof. (*a*) Assume to the contrary that $f_1: B_1 \to B$ and $f_2: B_2 \to B$ are both epimorphisms, and let

$$0 \to \mathrm{DTr}B \to B_1 \amalg B_2 \amalg K \to B \to 0$$

be an almost split sequence. Letting L denote length, we have that $L(B_1) > L(B)$, and consequently $L(\mathrm{DTr}B) > L(B_2)$, so that the irreducible map $\mathrm{DTr}B \to B_2$ is an epimorphism. Since also $f_2: B_2 \to B$ is an irreducible epimorphism and B is in $\mathscr{C}$, we have a contradiction by Lemma 1.3(*b*).

(*b*) is proved in a similar way.

LEMMA 1.5. *Let* Λ *be an artin algebra satisfying* (A). *Let* B *be indecomposable in* $\mathscr{C}$ *and* $f_i: B_i \to B$, $i = 1, \ldots, n$ *nonzero maps such that the induced map*

$$f: B_1 \amalg B_2 \amalg \ldots \amalg B_n \to B$$

is irreducible. If $f_1: B_1 \to C$ *is an epimorphism, then* $n \leqq 2$.

Proof. Assume that the conclusion is not true. We can then clearly assume that $n = 3$ and that B_2 and B_3 are indecomposable. Consider the almost split sequence $0 \to B_i \to C_i \to \mathrm{TrD}B_i \to 0$ for $i = 2, 3$. Since $f_i: B_i \to B$ is irreducible, we know that B is a summand of C_i. Hence

$$L(B_i) + L(\mathrm{TrD}B_i) = L(B) + r_i,$$

where $r_i \geqq 0$. Consider the almost split sequence

$$0 \to \mathrm{DTr}B \to B_1 \amalg B_2 \amalg B_3 \amalg K \to B \to 0.$$

We know from [**8**, Section 2] that we then have an almost split sequence

$$0 \to B \to \mathrm{TrD}B_1 \amalg \mathrm{TrD}B_2 \amalg \mathrm{TrD}B_3 \amalg \mathrm{TrD}K \to \mathrm{TrD}B \to 0.$$

Here we use that $\mathrm{TrD}B$ is in $\mathscr{C}$, so that the middle term in the above almost split sequence has no nonzero projective summands.

From these exact sequences we obtain

$$L(\amalg_{i=1}^{3} B_i) + L(\amalg_{i=1}^{3} \mathrm{TrD}B_i) + L(K) + L(\mathrm{TrD}K) =$$
$$= L(\mathrm{DTr}B) + L(\mathrm{TrD}B) + 2L(B) = L(B_1) + L(\mathrm{TrD}B_1)$$
$$+ 2L(B) + r_2 + r_3 + L(K) + L(\mathrm{TrD}K).$$

Hence $L(B_1) + L(\mathrm{TrD}B_1) \leqq L(\mathrm{DTr}B) + L(\mathrm{TrD}B)$. But since $f_1: B_1 \to B$ is an epimorphism, it follows by Lemma 1.3 that $\mathrm{DTr}B \to B_1$ is an irreducible monomorphism, and consequently $L(\mathrm{DTr}B) < L(B_1)$. Further it follows by condition (A') that $\mathrm{TrD}f_1: \mathrm{TrD}B_1 \to \mathrm{TrD}B$ is an epimorphism, so that $L(\mathrm{TrD}B) \leqq L(\mathrm{TrD}B_1)$. We then have a contradiction, and can conclude that $n \leqq 2$.

We are now ready to prove the main result in this section. We point out that by definition $(\mathrm{DTr})^0C = C = (\mathrm{TrD})^0C$, and for $r > 0$ $(\mathrm{DTr})^{-r}C = (\mathrm{TrD})^rC$.

THEOREM 1.6. *Let Λ be an artin algebra satisfying condition (A), and let B be indecomposable in $\mathscr{C}$. Then we have the following.*

(a) $\alpha(B) \leqq 3$.

(b) *If $\alpha(B) = 2$, and* $0 \to \mathrm{DTr}B \to E_1 \amalg E_2 \xrightarrow{(f_1, f_2)} B \to 0$ *is an almost split sequence with E_1 and E_2 indecomposable, then one of the maps $f_1: E_1 \to B$ and $f_2: E_2 \to B$ is a monomorphism and the other is an epimorphism.*

(c) *If $\alpha(B) = 3$ and $0 \to \mathrm{DTr}B \to E_1 \amalg E_2 \amalg E_3 \to B \to 0$ is an almost split sequence, then all the induced maps $E_i \to B$, $i = 1, 2, 3$, are monomorphisms, and all induced maps $\mathrm{DTr}B \to E_i$ are epimorphisms.*

Proof. Assume first that we have an almost split sequence

$$0 \to \mathrm{DTr}B \to \amalg_{i=1}^{n} E_i \to B \to 0,$$

where E_i is indecomposable and $n \geqq 2$. If $f_1: E_1 \to B$ is an epimorphism, we have that $n = 2$ by Lemma 1.5 and that $f_2: E_2 \to B$ is a monomorphism by Lemma 1.3. For if $f_2: E_2 \to B$ was an epimorphism, $\mathrm{DTr}B \to E_1$ would be an epimorphism.

If $n = 2$, then $f_1: E_1 \to B$ and $f_2: E_2 \to B$ can not both be monomorphisms, since then $\mathrm{DTr}B \to E_1$ would be a monomorphism and we would get a contradiction to Lemma 1.3.

We can now assume that $f_i: E_1 \to B$ is a monomorphism for $i = 1, \ldots, n$. Assume that $n \geqq 4$. By Lemma 1.5 we then have that $(f_1, f_2): E_1 \amalg E_2 \to B$ is not an epimorphism, and hence a monomorphism since it is irreducible. Similarly,

$$(f_3, \ldots, f_n): E_3 \amalg \ldots \amalg E_n \to B$$

is also a monomorphism, so that $\mathrm{DTr}B \to E_1 \amalg E_2$ is a monomorphism. We then have a contradiction by Lemma 1.3, so that $n \leqq 3$. We finally point out that if $n = 3$, then the induced maps $\mathrm{DTr}B \to E_i$ must be epimorphisms by Lemma 1.3. This finishes the proof of the theorem.

We shall now study more closely what we can say when there is some indecomposable B in $\mathscr{C}$ such that $\alpha(B) = 3$.

PROPOSITION 1.7. *Let Λ be an artin algebra satisfying condition (A), and assume that B is an indecomposable module in $\mathscr{C}$ and*

$$0 \to \mathrm{DTr}B \to E_1 \amalg E_2 \amalg E_3 \to B \to 0$$

is almost split with the E_i indecomposable. Then we have the following.

(a) *There is a sequence of irreducible monomorphisms between indecomposable modules $E_{i,k_i} \to \ldots \to E_{i,1} = E_i \to B$, $i = 1, 2, 3$, such that $\alpha(E_{i,k_i}) = 1$, $\alpha(E_{i,j}) = 2$ for $j < k_i$.*

(b) *There is a sequence of irreducible epimorphisms between indecomposable modules*

$$B \to \mathrm{TrD}E_i \to (\mathrm{TrD})^2 E_{i,2} \to \ldots \to (\mathrm{TrD})^{k_i} E_{i,k_i},$$

where $\alpha(\mathrm{TrD}^{k_i}E_{i,k_i}) = 1$ and $\alpha(\mathrm{TrD}^j E_{i,j}) = 2$ for $j < k_i$; $i = 1, 2, 3$.

(c) *All modules in $[B]$ are of the form $\mathrm{DTr}^t E_{i,j}$, $i = 1, 2, 3$, $1 \leqq j \leqq k_i$, $t \in Z$, or of the form $\mathrm{DTr}^t B$, $t \in Z$.*

(d) *If X is in $[B]$ and there is some finite chain of irreducible maps between indecomposable modules*

$$E = E_{i,k_i} \to X_1 \to X_2 \to \ldots \to X_n = X,$$

then X is isomorphic to some $\mathrm{TrD}^t E_{i,j}$, $t \geqq 0$ or to some $\mathrm{TrD}^t B$, $t \geqq 0$.

Proof. (a) Since there is an irreducible monomorphism $E_i \to B$, we know by Theorem 1.6 (a) and (c) that $\alpha(E_i) \leqq 2$. If $\alpha(E_i) = 2$, we know by Theorem 1.6 (b) that there is an irreducible monomorphism $X \to E_i$. Continuing this way, we get our desired chain.

(b) We first observe that we have an irreducible epimorphism $B \to \mathrm{TrD}E_i$. Analogous to (a) we get a sequence of irreducible epimorphisms

$$B \to \mathrm{TrD}E_i = K_{i,1} \to \ldots \to K_{i,n_i - 1} \to K_{i,n_i}$$

with $\alpha(K_{i,n_i}) = 1$, $\alpha(K_{i,j}) = 2$ for $j < n_i$. So we want to show that $n_i = k_i$ and $(\mathrm{TrD})^j E_{i,j} \cong K_{i,j}$. Assume that for a fixed i we have shown this for $j = j_0 < k_i$. We have an irreducible monomorphism $E_{i,j_0+1} \to E_{i,j_0}$, hence an irreducible epimorphism $E_{i,j_0} \to \mathrm{TrD}E_{i,j_0+1}$, so that we get an irreducible epimorphism

$$\mathrm{TrD}^{j_0}E_{i,j_0} \to \mathrm{TrD}^{j_0+1}E_{i,j_0+1}.$$

Since $\alpha(\mathrm{TrD}^{j_0}E_{i,j_0}) = 2$, we can now conclude that

$$\mathrm{TrD}^{j_0+1}E_{i,j_0+1} \cong K_{i,j_0+1}.$$

By considering the values of α we now get the desired conclusion.

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(c) Let M be in $[B]$. Then $M \cong B$ or there is a chain of irreducible maps between indecomposable modules $M_0 = M - M_1 - \ldots - M_n = B$, where $M_i - M_{i+1}$ indicates that there is an irreducible map $M_i \to M_{i+1}$ or an irreducible map $M_{i+1} \to M_i$. We shall prove our claim by induction on the length of the chain. If we have an irreducible map $M \to B = M_1$, then $M \cong E_i$, and if we have an irreducible map $B \to M$, then $M \cong \mathrm{TrD}E_i$. Let now $n > 1$. By induction hypothesis we have that $M_1 \cong (\mathrm{TrD})^r E_{i,j}$ for some i, j, r or $M_1 \cong \mathrm{TrD}^r B$ for some r. To finish the proof we explain what certain almost split sequences look like.

Since we have an almost split sequence

$$0 \to \mathrm{DTr}B \to E_1 \amalg E_2 \amalg E_3 \to B \to 0,$$

we have almost split sequences

$$0 \to \mathrm{TrD}^{r-1}B \to \mathrm{TrD}^r E_1 \amalg \mathrm{TrD}^r E_2 \amalg \mathrm{TrD}^r E_3 \to \mathrm{TrD}^r B \to 0.$$

We have seen that $\alpha(E_{i,j}) = 2$ if $j < k_i$ and that if also $j > 1$ we have irreducible maps $E_{i,j} \to E_{i,j-1}$ and $E_{i,j} \to \mathrm{TrD}E_{i,j+1}$, so that we have an almost split sequence

$$0 \to E_{i,j} \to E_{i,j-1} \amalg \mathrm{TrD}E_{i,j+1} \to \mathrm{TrD}E_{i,j} \to 0.$$

Similarly we have an almost split sequence

$$0 \to E_i \to B \amalg \mathrm{TrD}E_{i,2} \to \mathrm{TrD}E_i \to 0,$$

and an almost split sequence

$$0 \to E_{i,k_i} \to E_{i,k_i-1} \to \mathrm{TrD}E_{i,k_i} \to 0.$$

Applying $(\mathrm{TrD})^r$ for $r \in \mathbf{Z}$ to the above almost split sequences, we get our desired result.

(d) Assume that we have a chain of irreducible maps between indecomposable modules $E_{i,k_i} = X_0 \to X_1 \to \ldots \to X_n = X$. If $n = 1$, we know by the proof of (c) that X is of the desired type. It is then easy to see that we get our desired result by induction, by using the computation of the almost split sequences in the proof of (c).

We shall illustrate the above theorem by drawing a diagram of the irreducible maps between the indecomposable modules in $[B]$. We only draw that part of the diagram corresponding to $i = 1$. $\hookrightarrow$ indicates monomorphism and $\twoheadrightarrow$ epimorphism.

In the case when there is no C in $[B]$ with $\alpha(C) = 3$, we have the following result.

PROPOSITION 1.8. *Let Λ be an artin algebra satisfying condition (A). Assume that B is indecomposable in $\mathscr{C}$, and that $\alpha(C) \leqq 2$ for all C in $[B]$. Then we have the following.*

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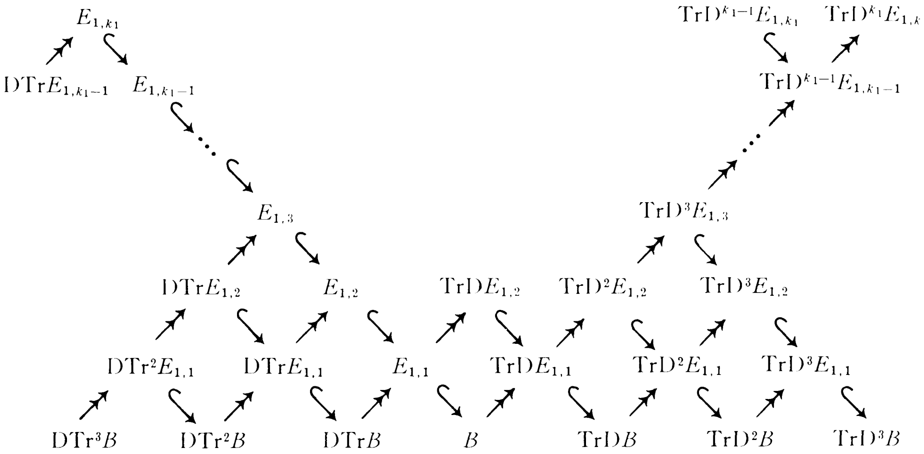


(a) *There is some* B_1 *in* $[B]$ *such that* $\alpha(B_1) = 1$ *and an infinite chain of irreducible monomorphisms between indecomposable modules* $B_1 \to B_2 \to B_3 \to \ldots \to B_i \to \ldots$, *where* $\alpha(B_i) = 2$ *for* $i > 1$.

(b) *For each i there is a chain of irreducible epimorphisms*

$$B_i \to \mathrm{TrD}B_{i-1} \to \ldots \to \mathrm{TrD}^{i-1}B_1.$$

(c) *Every* C *in* $[B]$ *is of the type* $\mathrm{DTr}^r B_i$ *for some* $i \geqq 1$, $r \in \mathbf{Z}$.

(d) *The almost split sequences are of the form*

$$0 \to \mathrm{DTr}^r B_1 \to \mathrm{DTr}^r B_2 \to \mathrm{DTr}^{r-1} B_1 \to 0 \text{ and}$$

$$0 \to \mathrm{DTr}^r B_i \to \mathrm{DTr}^r B_{i+1} \amalg \mathrm{DTr}^{r-1} B_{i-1} \to \mathrm{DTr}^{r-1} B_i \to 0, \text{ for } i > 1.$$

Proof. (a) If C is in $[B]$, then either $\alpha(C) = 1$, or there is some irreducible monomorphism $C' \to C$. This shows the existence of some B_1 in $[B]$ with $\alpha(B_1) = 1$. Given C in $[B]$, there is always an irreducible monomorphism $C \to C'$. Hence we get our desired chain of irreducible monomorphisms $B_1 \to B_2 \to B_3 \to \ldots B_i \to \ldots$. Clearly $\alpha(B_i) = 2$ for $i > 1$.

(b) Since we have an irreducible monomorphism $B_{i-1} \to B_i$ for $i > 1$, we have an irreducible epimorphism $B_i \to \mathrm{TrD}B_{i-1}$. Our claim is then easily proved by induction.

(c) follows easily by induction.

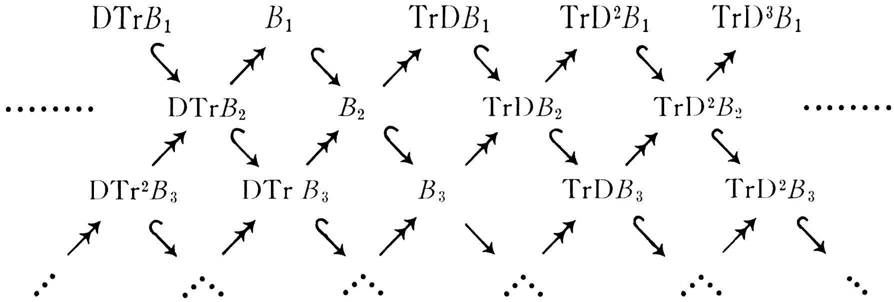
(d) is trivial.

We shall illustrate this result in the diagram below of irreducible maps.

We point out the following result, which gives some extra information on algebras satisfying (A), and which follows from our discussion so far.

PROPOSITION 1.9. *Let* Λ *be an artin algebra satisfying condition* (A). *If* $f: M \to N$ *is an irreducible epimorphism where* M *or* N *is indecomposable and in* $\mathscr{C}$, *then* $\mathrm{DTr}f$: $\mathrm{DTr}M \to \mathrm{DTr}N$ *is an epimorphism* (*for any choice of* $\mathrm{DTr}f$).

If $f\colon M \to N$ *is an irreducible monomorphism where* M *or* N *is indecomposable and in* $\mathscr{C}$, *then* $\mathrm{TrD}f\colon \mathrm{TrD}M \to \mathrm{TrD}N$ *is a monomorphism (for any choice of* $\mathrm{TrD}f$).



We end this section with the following interesting observation.

PROPOSITION 1.10. *Let* Λ *be an artin algebra satisfying condition* (A). *Assume that* X *is indecomposable in* $\mathscr{C}$, *and that* $\alpha(Y) \leqq 2$ *for all* Y *in* $[X]$. *If* $f\colon A \to B$ *is an irreducible monomorphism with* A *and* B *indecomposable and in* $[X]$, *then* $C = \mathrm{Coker}\, f$ *is in* $[X]$.

Proof. Let C be an indecomposable Λ-module such that $C \cong \mathrm{Coker}\, f$ for some irreducible monomorphism $f\colon A \to B$, where A and B are indecomposable and in $[X]$. Choose $f\colon A \to B$ such that B has as short length as possible. If

$$0 \to A \xrightarrow{f} B \to C \to 0$$

is an almost split sequence, then C is in $[X]$. If

$$0 \to A \xrightarrow{f} B \to C \to 0$$

is not an almost split sequence, we know by our assumption and Lemma 1.4 that we have an almost split sequence

$$0 \to A \xrightarrow{(f, f')} B \amalg B' \xrightarrow{(g, g')} \mathrm{TrD}A \to 0,$$

where $f'\colon A \to B'$ is an irreducible epimorphism with B' indecomposable. It is then not hard to see that we have an exact sequence

$$0 \to B' \xrightarrow{g'} \mathrm{TrD}A \to C \to 0,$$

where $g'\colon B' \to \mathrm{TrD}A$ is an irreducible monomorphism and B' and $\mathrm{TrD}A$ are indecomposable and in $[X]$. Since we have $L(\mathrm{TrD}A) < L(B)$, we then get a contradiction to the minimality of the length of B. This finishes the proof.

§ **2.** Let Λ be an artin algebra. In this section we shall give sufficient conditions for $\alpha(C) \leqq 2$, for all indecomposable objects C in $\mathscr{C}$, by using our results from Section 1.

We shall start out with some preliminary results.

LEMMA 2.1. *Let Λ be an artin algebra, and K a positive integer. Then there are integers K' and K'' such that if $g: B \to C$ and $h: C \to A$ are irreducible maps between indecomposable nonprojective noninjective modules, and $L(C) < K$, then $L(B) < K'$ and $L(A) < K''$.*

Proof. Let C be an indecomposable nonprojective Λ-module such that $L(C) < K$, and consider the almost split sequence $0 \to \mathrm{DTr}C \to E \to C \to 0$. Let $P_1 \to P_0 \to C \to 0$ be a minimal projective presentation of C in mod Λ, and consider the exact sequence

$$\mathrm{Hom}_\Lambda(P_0, \Lambda) \to \mathrm{Hom}_\Lambda(P_1, \Lambda) \to \mathrm{Tr}C \to 0.$$

Then there is clearly some integer K_1 such that $L(\mathrm{Tr}C) < K_1$, and consequently $L(\mathrm{DTr}C) < K_1$. Since $g: B \to C$ is irreducible, we know that B is a summand of E [**7**, Theorem 2.4, Proposition 3.1]. Letting $K' = K + K_1$ we then have that $L(B) < K'$.

The second half of the lemma is proved similarly.

LEMMA 2.2. *Let Λ be an artin algebra satisfying condition (A). Assume that C is an indecomposable Λ-module in $\mathscr{C}$ such that there is some integer K such that $L(\mathrm{DTr}^r C) < K$ for all $r \in \mathbf{Z}$. (This is the case for example if C is* DTr-*periodic.) Then $\alpha(X) \leqq 2$ for all indecomposable X in $[C]$.*

Proof. Assume to the contrary that there is some B in $[C]$ such that $\alpha(B) = 3$. By Proposition 1.7(c) and repeated application of Lemma 2.1 there is then some integer K' such that $L(X) < K'$ for all X in $[C]$. Since we know from [**2**, Section 6] that $[B]$ contains indecomposable modules of arbitrarily large length, we have a contradiction, and the proof is complete.

PROPOSITION 2.3. *Let Λ be an artin algebra satisfying condition (A). Assume that there is an indecomposable Λ-module C in $\mathscr{C}$ such that there is some integer K such that $L(\mathrm{DTr}^r C) < K$ for an infinite number of $r \in \mathbf{Z}$. Then $\alpha(X) \leqq 2$ for all X in $[C]$.*

Proof. Assume to the contrary that there is some indecomposable B in $[\mathscr{C}]$ such that $\alpha(B) = 3$. Choose B such that $B = C$ or there is a finite chain of irreducible maps between indecomposable modules $C \to C_1 \to \ldots \to C_n = B$, and let

$$0 \to \mathrm{DTr}B \to E_1 \amalg E_2 \amalg E_3 \to B \to 0$$

be an almost split sequence. Since we know by Theorem 1.6 (c) that $E_i \to B$ is a monomorphism for $i = 1, 2, 3$, $\mathrm{DTr}B \to E_1 \amalg E_2$ is a monomorphism, and the composite map

$$\mathrm{DTr}B \to E_1 \amalg E_2 \to B \amalg B = B^2$$

is a monomorphism. By considering the almost split sequences

$$0 \to \mathrm{TrD}^i B \to \mathrm{TrD}^{i+1}E_1 \amalg \mathrm{TrD}^{i+1}E_2 \amalg \mathrm{TrD}^{i+1}E_3 \to \mathrm{TrD}^{i+1}B \to 0,$$

we get in the same way an induced monomorphism $\mathrm{TrD}^i B \to (\mathrm{TrD}^{i+1}B)^2$. Hence we get a sequence of monomorphisms

$$B \to (\mathrm{TrD}B)^2 \to \ldots \to ((\mathrm{TrD})^i B)^{2i} \to \ldots,$$

and similarly for each $i > 0$ a sequence of monomorphisms

$$\mathrm{DTr}^i B \to (\mathrm{DTr}^{i-1}B)^2 \to \ldots \to (\mathrm{DTr}B)^{2i} \to B^{2i+1}.$$

By Lemma 2.2 we can assume that B is not DTr-periodic. Then for any $i > 0$ we have sequences of maps which are not isomorphisms between indecomposable modules

$$(\mathrm{DTr})^i B \to (\mathrm{DTr})^{i-1}B \to \ldots \to B \quad \text{and} \quad B \to \mathrm{TrD}B \to \ldots \to (\mathrm{TrD})^i B,$$

such that the composite is not zero.

We now use the following result [**11**, Lemma 12].

LEMMA 2.4. *Let R be a ring and $\{M_i\}_{i\geqq 0}$ a family of indecomposable R-modules of finite length, f_i; $M_i \to M_{i+1}$ for $i \geqq 0$ maps which are not isomorphisms, n an integer such that $L(M_i) \leqq n$ for all i. Then there is some integer n_0 such that the composition $f_{n_0} \ldots f_1$ is zero.*

Since $C \simeq B$ or there is a finite sequence of irreducible maps between indecomposable modules $C \to \ldots \to B$, we get by Lemma 2.1 that there exists an integer K' such that $L(\mathrm{DTr}^r B) < K'$ for an infinite number of $r \in \mathbf{Z}$. By considering the two sequences of maps

$$(\mathrm{DTr})^i B \to \ldots \to B \quad \text{and} \quad B \to \mathrm{TrD}B \to \ldots \to (\mathrm{TrD})^i B,$$

we then get a contradiction using Lemma 2.4. This finishes the proof of Proposition 2.3.

The following lemma will be useful.

LEMMA 2.5. *Let Λ be an artin algebra, and assume that C is an indecomposable module such that $[C]$ contains no injectives. If X is in $[C]$, there is an infinite number of Y in $[C]$ such that there is a nonzero map from X to Y.*

Proof. Assume to the contrary that there is some X in $[C]$ such that there is only a finite number of Y in $[C]$ such that there is a nonzero map from X to Y. We then know from [**9**, Section 1] that if there is some nonzero map f: $X \to Z$ with Z indecomposable, then $X \cong Z$ or there is a finite chain of irreducible maps between indecomposable modules from X to Z, and consequently Z is in $[X] = [C]$. Since there is some nonzero map g: $X \to I$ for an indecompoasble injective module I, we then get a contradiction to our hypothesis. This finishes the proof of the lemma.

To prove our main theorem we need to introduce a new condition on an artin algebra Λ. But first we shall recall some results from [**5**, Section 3].

Consider the category of additive covariant functors $(\text{mod}\,\Lambda, \text{Ab})$ from $\text{mod}\,\Lambda$ to abelian groups. For X in $\text{mod}\,\Lambda$, denote by $(X, \ )$ the corresponding representable functor. A functor F is finitely presented if we have an exact sequence $(X, \ ) \to (Y, \ ) \to F \to 0$ with X and Y in $\text{mod}\,\Lambda$. For a finitely presented functor F, $\underline{r}F$ is defined to be the intersection of the maximal subfunctors of F. $\underline{r}F$ is again finitely presented, and we define inductively $\underline{r}^{i+1}F = \underline{r}(\underline{r}^iF)$. We denote $\bigcap_{i=1}^{\infty} \underline{r}^iF$ by $\underline{r}^{\infty}F$.
We now introduce the following condition on an artin algebra.

(B) If f: $A = A_0 \to A_1 \to \ldots \to A_n = B$ is a composition of irreducible maps between indecomposable modules in $\mathscr{C}$ such that f is a monomorphism and $\text{Im}(f, \ ) \not\subset \underline{r}^{\infty}(A, \ )$, then $L(\text{DTr}^iA) \leqq L(\text{DTr}^iB)$ for all $i > 0$.

Before we go on we point out some conditions which easily imply (B) and whose statements do not involve any functor category.

(B_1) If f: $A = A_0 \to A_1 \to \ldots \to A_n = B$ is a composition of irreducible maps between indecomposable modules in $\mathscr{C}$ such that f is a monomorphism, then $\text{DTr}f$: $\text{DTr}A \to \text{DTr}B$ is a monomorphism.

(B_2) If A and B are indecomposable in $\mathscr{C}$ and there is some monomorphism f: $A \to B$, then there is some monomorphism f': $\text{DTr}A \to \text{DTr}B$.

We also have the following dual condition to (B).

(B^*) If f: $A = A_0 \to A_1 \to \ldots \to A_n = B$ is a composition of irreducible maps between indecomposable modules in $\mathscr{C}$ such that f is an epimorphism and $\text{Im}(f, \ ) \not\subset \underline{r}^{\infty}(A, \ )$, then $L(\text{TrD}^iA) \geqq L(\text{TrD}^iB)$ for all $i > 0$.

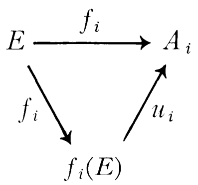
It is not hard to see that in the following main result (B) can be replaced by (B^*).

THEOREM 2.6. *Let Λ be an artin algebra satisfying conditions (A) and (B). Then $\alpha(C) \leqq 2$ for each indecomposable C in $\mathscr{C}$.*

Proof. Assume to the contrary that there is some indecomposable B in $\mathscr{C}$ such that $\alpha(B) = 3$, and consider, in the notation of Section 1, the sequences of irreducible monomorphisms

$$E_{i,k_i} \to E_{i,k_i-1} \to \ldots \to E_{i,1} \to B \quad \text{for} \quad i = 1, 2, 3.$$

By possibly replacing B by some DTr^jB, $j \in \mathbf{Z}$, and possibly changing the numbering, we may assume that $E = E_{1,k_1}$ has minimal length in $[B]$. Since $[E]$ contains no injectives, we know by Lemma 2.5 that there is an infinite number of indecomposable modules Y such that there is some nonzero map g: $E \to Y$. We then know that there is an infinite number of indecomposable modules A_i together with maps f_i: $E \to A_i$ such that f_i is a composition of irreducible maps between indecomposable modules and $\text{Im}(f_i, \ ) \not\subset \underline{r}^{\infty}(E, \ )$ ([**1**], [**9**, Proposition 1.5]). For each f_i: $E \to A_i$, consider the diagram



Since $\mathrm{Im}(f_i,) \not\subset \underline{r}^{\infty}(E,)$, we have $\mathrm{Im}(u_i,) \not\subset \underline{r}^{\infty}(f_i(E),)$ by elementary properties of the radical (see [**5**, Section 3]). Hence for some indecomposable summand X of $f_i(E)$ there is some chain of irreducible maps between indecomposable modules $X \to X_1 \to X_2 \to \ldots \to X_n = A_i$ [**9**, Proposition 1.5], so that $X \in [B]$. Since E has minimal length in $[B]$, f_i: $E \to A_i$ must be a monomorphism. By Proposition 1.7(d) we have for a B_0 which is either one of the modules $E_{i,j}$ or B that an infinite number of A_i is of the form $(\mathrm{TrD})^{r_i}B_0$, $r_i \geqq 0$. Hence we have monomorphisms f_{r_i}: $E \to (\mathrm{TrD})^{r_i}B_0$ such that f_{r_i} is a composite of irreducible maps between indecomposable modules and $\mathrm{Im}(f_{r_i},) \not\subset \underline{r}^{\infty}(E,)$. By condition (B) we then have that $L(\mathrm{DTr}^{r_i}E) \leqq L(B_0)$. We are then done by applying Proposition 2.3.

We end this section with another sufficient condition.

PROPOSITION 2.7. *Let Λ be an artin algebra satisfying condition (A). Let B be indecomposable in $\mathscr{C}$ and assume that there is some $r \in \mathbf{Z}$ such that*

$$L(\mathrm{DTr}^{r}B) \geqq 2L(\mathrm{DTr}^{r-1}B) \quad \text{or} \quad L(\mathrm{DTr}^{r-1}B) \geqq 2L(\mathrm{DTr}^{r}B).$$

Then we have $\alpha(B) \leqq 2$.

Proof. Assume to the contrary that $\alpha(B) = 3$. Then we have an almost split sequence

$$0 \to \mathrm{DTr}B \to E_1 \amalg E_2 \amalg E_3 \to B \to 0,$$

and consequently an almost split sequence

$$0 \to \mathrm{DTr}^{r}B \to \mathrm{DTr}^{r-1}E_1 \amalg \mathrm{DTr}^{r-1}E_2 \amalg \mathrm{DTr}^{r-1}E_3 \to \mathrm{DTr}^{r-1}B \to 0.$$

We know by Theorem 1.6(c) that the induced maps $\mathrm{DTr}^{r}B \to \mathrm{DTr}^{r-1}E_i$, $i = 1, 2, 3$ are epimorphisms, so that $L(\mathrm{DTr}^{r}B) > L(\mathrm{DTr}^{r-1}E_i)$, and the maps $\mathrm{DTr}^{r-1}E_i \to \mathrm{DTr}^{r-1}B$ are monomorphisms, so that

$$L(\mathrm{DTr}^{r-1}B) > L(\mathrm{DTr}^{r-1}E_i).$$

We further know that

$$\mathrm{DTr}^{r}B \to \mathrm{DTr}^{r-1}E_1 \amalg \mathrm{DTr}^{r-1}E_2$$

is a monomorphism. It follows that

$$L(\mathrm{DTr}^{r}B) < L(\mathrm{DTr}^{r-1}E_1) + L(\mathrm{DTr}^{r-1}E_2) < 2L(\mathrm{DTr}^{r-1}B)$$

and similarly

$$L(\mathrm{DTr}^{r-1}B) < L(\mathrm{DTr}^{r-1}(E_1 \amalg E_2)) < 2L(\mathrm{DTr}^{r}B).$$

This contradicts the hypothesis, so we conclude that $\alpha(B) \leqq 2$.

§ 3. In this section we shall discuss which algebras satisfy conditions (A) and (B). If Λ is an indecomposable algebra of finite representation type, then there is only one equivalence class of indecomposable modules, which hence must contain projectives [**2**, Section 6]. We do not know of any artin algebras which do not satisfy the conditions, and it would be interesting to know if they all do. (J. Alperin has found an algebra which does not satisfy (A) or (B), and C. M. Ringel has shown that $\alpha(C) > 2$ in $\mathscr{C}$ can occur.)

An important class of artin algebras which we can show satisfy the conditions is the artin algebras stably equivalent to hereditary algebras, in particular the hereditary algebras. We recall that two artin algebras Λ and Λ' are *stably equivalent* if $\underline{\mathrm{mod}}\,\Lambda$ and $\underline{\mathrm{mod}}\,\Lambda'$ are equivalent categories (see [**4**]). For hereditary algebras this is a direct consequence of the following well known result.

LEMMA 3.1. *Let* Λ *be an hereditary artin algebra and* $f: A \to B$ *a monomorphism in* $\mathrm{mod}\,\Lambda$, $g: E \to F$ *an epimorphism in* $\mathrm{mod}\,\Lambda$. *Then* $\mathrm{DTr}f: \mathrm{DTr}A \to \mathrm{DTr}B$ *is a monomorphism and* $\mathrm{TrD}g: \mathrm{TrD}E \to \mathrm{TrD}F$ *is an epimorphism.*

Proof. Considering the definition of Tr, it is not hard to see that for an hereditary algebra Λ, Tr is a functor from $\mathrm{mod}\,\Lambda$ to $\mathrm{mod}\,\Lambda^{op}$, which is isomorphic to $\mathrm{Ext}^1(\ ,\Lambda)$. Tr is hence right exact, so that DTr is left exact and TrD is right exact. This gives our desired result.

To get the result for algebras stably equivalent to hereditary algebras we shall also need the following.

LEMMA 3.2. *Let* Λ *be an artin algebra stably equivalent to an hereditary algebra. If* $f: A \to B$ *is a monomorphism with* A *and* B *in* $\mathscr{C}$, *then there is some monomorphism* $\mathrm{DTr}A \to \mathrm{DTr}B$.

Proof. Let Γ be an hereditary algebra such that we have an equivalence of categories $\gamma: \underline{\mathrm{mod}\,\Lambda} \to \underline{\mathrm{mod}\,\Gamma}$. We also denote by γ the induced correspondence between the modules with no nonzero injective summands. Let $f: A \to B$ be a monomorphism, where A and B are in $\mathscr{C}$. Since A has no nonzero injective summands, we know from [**4**, Chapter IV, Proposition 1.2] that if $f': \gamma(A) \to \gamma(B)$ is such that $\gamma(\bar{f}) = \bar{f}'$, then f' is a monomorphism in $\mathrm{mod}\,\Gamma$. Since Γ is hereditary, $\mathrm{DTr}f': \mathrm{DTr}\gamma(A) \to \mathrm{DTr}\gamma(B)$ is a monomorphism by Lemma 3.1. Since A and B are in $\mathscr{C}$, it follows from [**9**, Theorem 3.1] that $\mathrm{DTr}\gamma(A) \cong \gamma(\mathrm{DTr}A)$ and $\mathrm{DTr}\gamma(B) \cong \gamma(\mathrm{DTr}B)$. Hence we have a monomorphism $s: \gamma(\mathrm{DTr}A) \to \gamma(\mathrm{DTr}B)$. Letting $t: \mathrm{DTr}A \to \mathrm{DTr}B$ in $\mathrm{mod}\,\Lambda$ be such that $\gamma(\bar{t}) = \bar{s}$, we get again by [**4**, Chapter IV, Proposition 1.2] that $t: \mathrm{DTr}A \to \mathrm{DTr}B$ is a monomorphism.

As a direct consequence of these preliminary results we now get the following, using that if $f: X \to Y$ with X, Y in $\mathscr{C}$ is irreducible and $L(\mathrm{DTr}X) \leqq L(\mathrm{DTr}Y)$, then $\mathrm{DTr}f: \mathrm{DTr}X \to \mathrm{DTr}Y$ is a monomorphism.

PROPOSITION 3.3. *If* Λ *is an artin algebra stably equivalent to an hereditary algebra, then* Λ *satisfies conditions* (A) *and* (B).

Another sufficient condition for an artin algebra to satisfy our conditions is given in the following result.

PROPOSITION 3.4. *Let* Λ *be an hereditary artin algebra and* $\mathfrak{A}$ *a twosided ideal in* Λ *such that* $\mathrm{DTr}_{\Lambda/\mathfrak{A}}M \cong \mathrm{Hom}_\Lambda(\Lambda/\mathfrak{A}, \mathrm{DTr}M)$ *for all* $\Lambda/\mathfrak{A}$*-modules* M *in* $\mathscr{C}_{\Lambda/\mathfrak{A}}$. *Then* $\Lambda/\mathfrak{A}$ *satisfies conditions* (A) *and* (B).

Proof. Let $f: M \to N$ be a monomorphism where M and N are in $\mathscr{C}_{\Lambda/\mathfrak{A}}$. Since Λ is hereditary, we have by Lemma 3.1 a monomorphism $\mathrm{DTr}f$: $\mathrm{DTr}_\Lambda M \to \mathrm{DTr}_\Lambda N$. Hence we have a monomorphism $\mathrm{Hom}_\Lambda(\Lambda/\mathfrak{A}, \mathrm{DTr}M) \to \mathrm{Hom}_\Lambda(\Lambda/\mathfrak{A}, \mathrm{DTr}N)$. By our assumption it then follows that we have a monomorphism $\mathrm{DTr}_{\Lambda/\mathfrak{A}}M \to \mathrm{DTr}_{\Lambda/\mathfrak{A}}N$. It follows from this that $\Lambda/\mathfrak{A}$ satisfies (A) and (B).

We point out that in [**8**, Corollary 4.4] are given several statements equivalent to the condition

$$\mathrm{DTr}_{\Lambda/\mathfrak{A}}M \cong \mathrm{Hom}_\Lambda(\Lambda/\mathfrak{A}, \mathrm{DTr}M)$$

for a $\Lambda/\mathfrak{A}$-module M.

We shall now show that we can use our results to get some information on the following two questions which we mentioned in the introduction.

(1) If Λ is an artin algebra, is there an integer $N = \alpha(\Lambda)$ such that $\alpha(C) \leqq N$ for each indecomposable nonprojective Λ-module C?

(2) Is there some integer K such that if Λ is an artin algebra of finite representation type, then $\alpha(\Lambda) \leqq K$? And if there is such a K, what is the least possible value for K?

Trivially, (1) holds for an artin algebra Λ of finite representation type. We shall now show that (1) also holds for an artin algebra stably equivalent to an hereditary algebra. For this the following two lemmas will be useful.

LEMMA 3.5. *Let* Λ *be an artin algebra and* $\mathscr{A}$ *a full subcategory of* mod Λ, *such that there is a finite number of indecomposable modules* $A_1, \ldots, A_n$, *with the property that every indecomposable module in* $\mathscr{A}$ *is of the form* $\mathrm{TrD}^i A_j$ *for some* $i \in \mathbf{Z}$, $j = 1, \ldots, n$. *Then the set of* $\alpha(A)$ *with* A *indecomposable in* $\mathscr{A}$ *is bounded.*

Proof. Since for an indecomposable Λ-module X there is only a finite number of indecomposable Λ-modules Y such that there is an irreducible map $X \to Y$, there are only a finite number of almost split sequences in mod Λ whose middle term has a nonzero projective or a nonzero injective summand. We further know that if $0 \to A \to B \to C \to 0$ is an almost split sequence and $0 \to A' \to B' \to \mathrm{DTr}C \to 0$ is an almost split sequence, the number of non-

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projective indecomposable summands of B is the same as the number of noninjective indecomposable summands of B' [**8**, Section 2]. If $0 \to A'' \to B'' \to \operatorname{TrD}C \to 0$ is an almost split sequence, the number of indecomposable noninjective summands of B is the same as the number of indecomposable nonprojective summands of B''. From these observations our claim follows.

LEMMA 3.6. *Let Λ and Λ' be stably equivalent artin algebras. If $\alpha(\Lambda)$ exists, then $\alpha(\Lambda')$ exists.*

Proof. Let β: $\underline{\operatorname{mod}\Lambda} \to \underline{\operatorname{mod}\Lambda'}$ be an equivalence, and denote also by β the induced correspondence between the modules with no nonzero projective summands. If $0 \to A \to B \to C \to 0$ is an almost split sequence in $\operatorname{mod}\Lambda$ and $0 \to A' \to B' \to \beta C \to 0$ an almost split sequence in $\operatorname{mod}\Lambda'$, we know from [**8**, Sections 1 and 2] that the number of indecomposable nonprojective summands is the same for B and B'. Since we have already seen that there is only a finite number of almost split sequences whose middle term has a nonzero projective summand, we are done.

We can now prove the following result.

PROPOSITION 3.7. *If Λ is an artin algebra stably equivalent to an hereditary algebra, then $\alpha(\Lambda)$ exists.*

Proof. By Lemma 3.6 we can assume that Λ is hereditary. Let C be an indecomposable Λ-module. If C is in $\mathscr{C}$ we know by Theorem 2.6 and Lemma 3.1 that $\alpha(C) \leqq 2$. If C is not in $\mathscr{C}$, then $[C]$ contains a projective or an injective module. If $[C]$ contains a projective module, it follows as in [**3**, Section 1] that $C \cong \operatorname{TrD}^i P$ for some indecomposable projective Λ-module P and some $i \geqq 0$, and if $[C]$ contains an injective module that $C \cong \operatorname{DTr}^j I$ for some indecomposable injective Λ-module I and some $j \geqq 0$. We are then done by using Lemma 3.5.

We remark that it is possible to prove that if Λ is an artin algebra stably equivalent to an hereditary algebra Λ', then $\alpha(C) \leqq 2$ for all indecomposable C in $\mathscr{C}$, by using the corresponding result for hereditary algebras. For denote by β: $\underline{\operatorname{mod}\Lambda} \to \underline{\operatorname{mod}\Lambda'}$ a stable equivalence and also the induced correspondence between the modules with no nonzero projective summands. It can be proved by using [**4**, Proposition 1.2] and [**8**] that if for an indecomposable nonprojective Λ-module C, $[C]$ contains no projectives or injectives, then the same is the case for $[\beta C]$. As in the proof of Lemma 3.6 we then get $\alpha(C) = \alpha(\beta C) \leqq 2$.

With respect to question (2), we list the following information.

PROPOSITION 3.8. *If Λ is an hereditary artin algebra of finite representation type, then $\alpha(\Lambda) \leqq 3$.*

Proof. Assume that Λ is an hereditary algebra of finite type, and let C be an indecomposable nonprojective Λ-module. If P is an indecomposable projective noninjective Λ-module, we know from [**8**, Proposition 2.4] that we have an almost split sequence

$$0 \to P \to \mathrm{TrD}(\underline{r}P) \amalg Q \to \mathrm{TrD}P \to 0,$$

where Q is projective, and $\underline{r}$ denotes the radical of Λ. By considering the diagrams associated with the hereditary algebras of finite type listed in [**10**], it is not hard to see that $\alpha(\mathrm{TrD}P) \leqq 3$. Since C must be of the type $\mathrm{TrD}^i P$ for some $i \geqq 0$ and some indecomposable projective Λ-module P [**3**, 10], and since in an almost split sequence $0 \to A \to B \to C \to 0$, B has no nonzero projective summands unless A is projective, we get that $\alpha(\Lambda) \leqq 3$.

We point out that there are artin algebras Λ of finite type such that $\alpha(\Lambda) = 4$, but we do not know if any higher value of $\alpha(\Lambda)$ can occur for algebras of finite type.

Example. Let Λ be a selfinjective algebra with $\underline{r}^3 = 0$ of finite type such that $\Lambda/\underline{r}^2$ is hereditary, and such that there is an indecomposable projective Λ-module P such that $\underline{r}P/\mathrm{soc}P = S_1 \amalg S_2 \amalg S_3$, S_i simple. It is not hard to find such an algebra (see [**12**]). We know from [**7**, Proposition 4.11] that we have an almost split sequence

$$0 \to \underline{r}P \to P \amalg \underline{r}P/\mathrm{soc}P \to P/\mathrm{soc}P \to 0$$

in $\mathrm{mod}\,\Lambda$, and consequently $\alpha(P/\mathrm{soc}P) = 4$. If Q is an arbitrary indecomposable projective Λ-module, we have an almost split sequence

$$0 \to \underline{r}Q \to Q \amalg \underline{r}Q/\mathrm{soc}Q \to Q/\mathrm{soc}Q \to 0.$$

Since $\Lambda/\underline{r}^2$ is hereditary of finite type, we know from [**10**][**12**] that $\underline{r}Q/\mathrm{soc}Q$ has at most three indecomposable summands, so that $\alpha(Q/\mathrm{soc}Q) \leqq 4$. Let now C be indecomposable in $\mathrm{mod}\,\Lambda$ and not isomorphic to $Q/\mathrm{soc}Q$ for any indecomposable projective Λ-module Q. Then C is an indecomposable nonprojective $\Lambda/\underline{r}^2$-module, so we have an almost split sequence $0 \to A \to B \to C \to 0$ in $\mathrm{mod}\,\Lambda/\underline{r}^2$. Since the only indecomposable Λ-modules which are not $\Lambda/\underline{r}^2$-modules are the indecomposable projective Λ-modules, it follows that the above sequence is almost split also in $\mathrm{mod}\,\Lambda$. Since $\Lambda/\underline{r}^2$ is hereditary, it follows from Proposition 3.8 that $\alpha(C) \leqq 3$. We have now shown that $\alpha(\Lambda) = 4$.

We end this section by stating without proof two other conditions which can replace condition (B) in our main result Theorem 2.6. One condition is based upon the following result whose proof we omit.

PROPOSITION 3.9. *Let Λ be an artin algebra and f: $A \to B$ a map in* $\mathrm{mod}\,\Lambda$ *with A and B in $\mathscr{C}$. Then* $\mathrm{Im}(f,) \subset \underline{r}^\infty(A,)$ *if and only if* $\mathrm{Im}(\mathrm{DTr}f,) \subset \underline{r}^\infty(\mathrm{DTr}A,)$ *for all choices of* $\mathrm{DTr}f$.

On the basis of this result we get that the following condition (B_3) can replace our condition (B).

(B_3) If f: $A \to B$ is a monomorphism where A and B are indecomposable and in $\mathscr{C}$, and $\mathrm{Im}(f,) \not\subset \underline{r}^\infty(A,)$, then $\mathrm{DTr}f$: $\mathrm{DTr}A \to \mathrm{DTr}B$ is a monomorphism.

Further, we have the following result which we also state without proof.

THEOREM 3.10. *Let Λ be an artin algebra satisfying condition (A), and let C be indecomposable in $\mathscr{C}$. Then $\alpha(X) \leqq 2$ for all indecomposable X in $[C]$ if and only if some indecomposable B of minimal length in $[C]$ has the property that B is periodic or*

$$\mathrm{Hom}(B, \mathrm{TrD}^i B) = \underline{r}^{\infty}(B, \ )(\mathrm{TrD}^i B)$$

for all $i > 0$.

In other words, we have the following condition, which together with (A) is implied by all our conditions (B), (B_1), (B_2) and (B_3).

(B_4) There is some indecomposable module B in $\mathscr{C}$ of minimal length in $[B]$ such that either

$$\mathrm{Hom}(B, \mathrm{TrD}^i B) = \underline{r}^{\infty}(B, \ )((\mathrm{TrD}^i B) \text{ for all } i > 0$$

or B is periodic.

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