ON THE RADICAL OF A RING WITH MINIMUM CONDITION

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The purpose of this note is to establish the following characterisation of the radical:

THEOREM. *Let R be a ring with the minimum condition for left ideals. Then the radical of R is the intersection of the maximal nilpotent subrings of R.*

We prove first the following lemmas, assuming throughout that R is a ring with minimum condition:

LEMMA 1. *Suppose R is the direct sum $R_1 \oplus \cdots \oplus R_k$ of the ideals R_i, $i = 1, 2, \cdots, k$. Let N_i be a maximal nilpotent subring of R_i ($i = 1, 2, \cdots, k$), and let $N = N_1 \oplus \cdots \oplus N_k$. Then N is a maximal nilpotent subring of R. Conversely, if N is a maximal nilpotent subring of R, then the R_i-component*

$$N_i = \{x | \, x = n-y \in R_i \text{ for some } n \in N, \ y \in \sum_{j \neq i} R_j\}$$

of N is a maximal nilpotent subring of R_i, and

$$N = N_1 \oplus \cdots \oplus N_k.$$

PROOF. Consider the product $a_1 a_2 \cdots a_t$ of elements $a_i \in R$. Each a_i is uniquely expressible in the form

$$a_i = b_{i1}+b_{i2}+ \cdots +b_{ik}, \quad b_{ij} \in R_j.$$

Then

$$a_1 a_2 \cdots a_t = b_{11} b_{21} \cdots b_{t1}+b_{12} b_{22} \cdots b_{t2}+ \cdots +b_{1k} b_{2k} \cdots b_{tk}$$

since R is the direct sum of the ideals R_i, $i = 1, 2, \cdots, k$. Thus $a_1 a_2 \cdots a_t = 0$ if and only if $b_{1i} b_{2i} \cdots b_{ti} = 0$ for all $i = 1, 2, \cdots, k$. Thus a subring S of R is nilpotent if and only if for all i, the R_i-component

$$S_i = \{x | \, x = s-y \in R_i \text{ for some } s \in S, \ y \in \sum_{j \neq i} R_j\}$$

of S is nilpotent.

(i) Let N_i be a maximal nilpotent subring of R_i ($i = 1, 2, \cdots, k$), and let $N = N_1 \oplus \cdots \oplus N_k$. Then N is a nilpotent subring of R. Suppose S

is a nilpotent subring of R and $S \geqq N$. Then the component $S_i \geqq N_i$ and is nilpotent. Since N_i is maximal nilpotent in R_i, we must have $S_i = N_i$. But

$$S \leqq \sum_{i=1}^{k} S_i = \sum_{i=1}^{k} N_i = N.$$

Therefore $S = N$ and N is maximal nilpotent in R.

(ii) Let N be a maximal nilpotent subring of R. Then the components N_i of N are nilpotent. Suppose $R_i \geqq S_i \geqq N_i$ and S_i is nilpotent, $i = 1, 2, \cdots, k$. Then $S = S_1 \oplus \cdots \oplus S_k$ is a nilpotent subring of R and $S \geqq N$. Therefore $S = N$ which implies $S_i = N_i$. Thus N_i is a maximal nilpotent subring of R_i and $N = N_1 \oplus \cdots \oplus N_k$.

LEMMA 2. *Suppose R is simple, non-null. Then there exist maximal nilpotent subrings U, L of R such that $U \cap L = 0$.*

PROOF. R is isomorphic to the ring of endomorphisms of some finite-dimensional left vector space over some division ring D. From any basis of V, we obtain a faithful representation of R by matrices (d_{ij}) with elements d_{ij} in D. If N is any nilpotent subring of R, we can choose the basis of V such that every element of N is represented by an upper triangular matrix (d_{ij}), $d_{ij} = 0$ for $i \geqq j$. Clearly the subring U of all elements of R which are represented (for some given basis of V) by upper triangular matrices is a maximal nilpotent subring of R. The subring L of elements represented by lower triangular matrices (d_{ij}), $d_{ij} = 0$ for $i \leqq j$, is also a maximal nilpotent subring of R and $U \cap L = 0$.

LEMMA 3. *Suppose R is semi-simple. Then the intersection of the maximal nilpotent subrings of R is 0.*

PROOF. R is the direct sum $S_1 \oplus \cdots \oplus S_k$ of simple non-null ideals S_i. For each i, there exist maximal nilpotent subrings U_i, L_i of S_i such that $U_i \cap L_i = 0$. Put $U = U_1 \oplus \cdots \oplus U_k$ and $L = L_1 \oplus \cdots \oplus L_k$. Then U, L are maximal nilpotent subrings of R and $U \cap L = 0$.

LEMMA 4. *Let N be the radical of R and let K be a subring of R. Then K is a maximal nilpotent subring of R if and only if $K \geqq N$ and K/N is a maximal nilpotent subring of R/N.*

PROOF. If K is nilpotent, then so is $(K+N)/N$. But $(K+N)/N$ and N both nilpotent implies that $K+N$ is nilpotent. Thus if K is maximal nilpotent, then $K = K+N$ and therefore $K \geqq N$. Suppose $K \geqq N$. Then K is nilpotent if and only if K/N is nilpotent. Thus $K(\geqq N)$ is maximal nilpotent in R if and only if K/N is maximal nilpotent in R/N.

PROOF OF THEOREM. Let N be the radical of R, and let M_α be the maximal nilpotent subrings of R. Then $M_\alpha \geqq N$ for all α, and

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$$\left(\bigcap_{\alpha} M_\alpha\right)/N = \bigcap_{\alpha} (M_\alpha/N).$$

But the M_α/N are all the maximal nilpotent subrings of the semi-simple ring R/N. Therefore

$$\bigcap_{\alpha} (M_\alpha/N) = 0$$

and therefore

$$\bigcap_{\alpha} M_\alpha = N.$$

Reference

[1] Artin, E., Nesbitt, C. J. and Thrall, R. M., Rings with minimum condition (University of Michigan Press, Ann Arbor, 1944).

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