

SOME OPTIMAL DIVIDENDS PROBLEMS

BY

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ABSTRACT

We consider a situation originally discussed by De Finetti (1957) in which a surplus process is modified by the introduction of a constant dividend barrier. We extend some known results relating to the distribution of the present value of dividend payments until ruin in the classical risk model and show how a discrete time risk model can be used to provide approximations when analytic results are unavailable. We extend the analysis by allowing the process to continue after ruin.

1. INTRODUCTION

In this paper we study optimal dividend strategies, a problem first discussed by De Finetti (1957). De Finetti considered a discrete time risk model and produced some results in the situation when the aggregate gain of an insurer per unit time is either 1 or -1 . A summary of De Finetti's work can be found in Bühlmann (1970, Section 6.4.5), or for a more recent reference see Gerber and Shiu (2004, Appendix B).

The problem of optimal dividend strategies has also been considered in continuous time. The textbooks by Bühlmann (1970) and Gerber (1979) discuss the problem in the context of the classical risk model. See also Gerber and Shiu (1998, Section 7).

An optimal dividend strategy is almost always taken to be a strategy which maximises the expected discounted future dividends. Such strategies have been considered in relation to a variety of models for the insurance surplus process. These range from De Finetti's (1957) very simple model, through the classical continuous time compound Poisson model to more sophisticated models, *e.g.* Paulsen and Gjessing (1997), who consider a claims process which includes a Brownian motion element and stochastic interest on reserves driven by an independent Brownian motion, and Højgaard (2002), who considers a process with dynamic control of the premium income. Some authors have considered the problem of determining the optimal form of the barrier, *e.g.* De Finetti (1957) and Asmussen and Taksar (1997); others have assumed that the barrier is constant and considered the problem of finding the optimal level for the barrier. Gerber and Shiu (2004, Section 1) is a rich source of references for the literature on this subject.

In this paper we consider both a discrete time risk model and the classical continuous time risk model. We show how our discrete time risk model can be used to tackle problems for which analytical solutions do not exist in the classical continuous time model. In the classical continuous time model, we generalise a result given in Bühlmann (1970) and Gerber (1979). Our result has a counterpart in the Brownian motion risk model considered by Gerber and Shiu (2004). In each of the references mentioned so far, the risk process continues until ruin occurs, an event which is certain as each study is in the framework of a dividend barrier whose level is constant. We consider the notion that the risk process can continue after ruin as a result of a suitable injection of surplus. Siegl and Tichy (1999) also consider the expected present value of dividends for a surplus model which continues after ruin. However, their model is different to ours in several important aspects.

We define the classical surplus process $\{U(t)\}_{t \geq 0}$ as

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i$$

where u is the insurer's initial surplus, c is the rate of premium income per unit time, $N(t)$ is the number of claims up to time t , and X_i is the amount of the i th claim. $\{N(t)\}_{t \geq 0}$ is a Poisson process, with parameter λ , and $\{X_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed random variables, independent of $\{N(t)\}_{t \geq 0}$. We denote by μ_k the k th moment of X_1 . Let F and f denote the distribution function and density function respectively of X_1 , with $F(0) = 0$, and let $c = (1 + \theta)\lambda\mu_1$, where $\theta > 0$ is the premium loading factor. The ultimate ruin probability from initial surplus u is denoted $\psi(u)$ and defined by

$$\psi(u) = \Pr(U(t) < 0 \text{ for some } t > 0).$$

We consider the above surplus process modified by the introduction of a dividend barrier, b . When the surplus reaches the level b , premium income no longer goes into the surplus process but is paid out as dividends to shareholders. Thus, when the modified surplus process attains the level b , it remains there until the next claim occurs. We say that ruin occurs when the modified surplus process falls below 0, and ruin is certain as we are effectively considering a surplus process in the presence of a reflecting upper barrier. We use the notation T_u to denote the time of ruin.

Let the random variable D_u denote the present value at force of interest δ per unit time of dividends payable to shareholders until ruin occurs (given initial surplus u), with $V_n(u, b) = E[D_u^n]$. An integro-differential equation for $V_1(u, b)$ can be found in Bühlmann (1970) and Gerber (1979), together with a solution in the case when the individual claim amount distribution is exponential. A solution for $V_n(u, b)$ in the case of the Brownian motion risk model (as defined in Klugman et al (1998)) is given in Gerber and Shiu (2004). In Section 2 we derive an integro-differential equation for $V_n(u, b)$, and solve this equation when the individual claim amount distribution is exponential. In Section 3 we consider the distribution of D_u when $\delta = 0$.

In Section 4 we consider both the time of ruin and the deficit at ruin and derive results which we then apply in Section 6 where we allow the process to continue after ruin. In Section 5 we consider a discrete time model. Notation for this model is introduced in that section.

2. THE MOMENTS OF D_u

In this section we derive an integro-differential equation for $V_n(u, b)$, and find a boundary condition. We then use this equation to calculate some moments in the case when the individual claim amount distribution is exponential. We start, however, with a lemma which we apply in Theorem 2.1.

Lemma 2.1. *For any positive δ and μ and any positive integer m , define:*

$$I(m, \mu) = \mu \int_0^\infty e^{-\mu t} \left(\frac{1 - e^{-\delta t}}{\delta} \right)^m dt.$$

Then

$$I(m, \mu) = m! \prod_{k=1}^m (\mu + k\delta)^{-1}.$$

Proof. Integrating by parts, it can be seen that

$$I(m, \mu) = \frac{m}{\mu + \delta} I(m - 1, \mu + \delta).$$

Hence

$$I(m, \mu) = \frac{m(m-1) \dots 2}{(\mu + \delta)(\mu + 2\delta) \dots (\mu + (m-1)\delta)} I(1, \mu + (m-1)\delta).$$

It can be checked that

$$I(1, \mu + (m-1)\delta) = (\mu + m\delta)^{-1}$$

and the result follows. ■

Remark 2.1. *We remark that the above result holds when $\delta = 0$, provided we interpret $(1 - e^{-\delta t})/\delta$ as t in this case.*

Theorem 2.1. *For $n = 1, 2, 3, \dots$, $V_n(u, b)$ satisfies the integro-differential equation*

$$\frac{d}{du} V_n(u, b) = \frac{\lambda + n\delta}{c} V_n(u, b) - \frac{\lambda}{c} \int_0^u f(x) V_n(u - x, b) dx \quad (2.1)$$

with boundary condition

$$\frac{d}{du} V_n(u, b) \Big|_{u=b} = nV_{n-1}(b, b). \tag{2.2}$$

Proof. Let $0 \leq u < b$, and let τ denote the time at which the surplus would reach b if there were no claims, so that $u + c\tau = b$. By considering whether or not a claim occurs before time τ , we have

$$V_n(u, b) = e^{-(\lambda + n\delta)\tau} V_n(b, b) + \int_0^\tau \lambda e^{-(\lambda + n\delta)t} \int_0^{u+ct} f(x) V_n(u + ct - x, b) dx dt, \tag{2.3}$$

and substituting $s = u + ct$ we get

$$V_n(u, b) = e^{-(\lambda + n\delta)(b-u)/c} V_n(b, b) + \frac{\lambda}{c} \int_u^b e^{-(\lambda + n\delta)(s-u)/c} \int_0^s f(x) V_n(s - x, b) dx ds.$$

Differentiating with respect to u , we obtain

$$\frac{d}{du} V_n(u, b) = \frac{\lambda + n\delta}{c} V_n(u, b) - \frac{\lambda}{c} \int_0^u f(x) V_n(u - x, b) dx,$$

which is equation (2.1).

To obtain the boundary condition (2.2), we first consider $V_n(b, b)$. By conditioning on the time and the amount of the first claim we have

$$V_n(b, b) = \int_0^\infty \lambda e^{-(\lambda + n\delta)t} (c\bar{s}_\tau)^n dt + \sum_{j=1}^n \int_0^\infty \binom{n}{j} \lambda e^{-(\lambda + n\delta)t} (c\bar{s}_\tau)^{n-j} \int_0^b f(x) V_j(b - x, b) dx dt$$

where, in standard actuarial notation, $\bar{s}_\tau = (e^{\delta t} - 1) / \delta$.

Applying Lemma 2.1 we have

$$\begin{aligned} V_n(b, b) &= c^n I(n, \lambda) + \sum_{j=1}^n c^{n-j} \binom{n}{j} \frac{\lambda}{\lambda + j\delta} I(n - j, \lambda + j\delta) \int_0^b f(x) V_j(b - x, b) dx \\ &= c^n n! \prod_{j=1}^n (\lambda + j\delta)^{-1} \\ &\quad + \sum_{j=1}^n c^{n-j} \binom{n}{j} (n - j)! \lambda \left[\prod_{m=j}^n (\lambda + m\delta)^{-1} \right] \int_0^b f(x) V_j(b - x, b) dx. \end{aligned}$$

Also, by rearranging (2.1) we have

$$V_n(b, b) = \frac{c}{\lambda + n\delta} \frac{d}{du} V_n(u, b) \Big|_{u=b} + \frac{\lambda}{\lambda + n\delta} \int_0^b f(x) V_n(b - x, b) dx.$$

Hence, by equating the above two expressions for $V_n(b, b)$ we obtain

$$\begin{aligned} & \frac{d}{du} V_n(u, b) \Big|_{u=b} \\ &= c^{n-1} n! \prod_{j=1}^{n-1} (\lambda + j\delta)^{-1} \\ &+ \sum_{j=1}^{n-1} c^{n-1-j} \binom{n}{j} (n-j)! \lambda \left[\prod_{m=j}^{n-1} (\lambda + m\delta)^{-1} \right] \int_0^b f(x) V_j(b - x, b) dx. \end{aligned} \tag{2.4}$$

We can now prove (2.2) by induction. We note that $V_0(b, b) = E[D_u^0] = 1$. Equation (2.2) is already known to be true for $n = 1$. See, for example, Gerber and Shiu (1998, Section 7). Now let us assume that it is true for $n = 1, 2, \dots, m$. Then by (2.1) and (2.2)

$$V_m(b, b) = \frac{cm}{\lambda + m\delta} V_{m-1}(b, b) + \frac{\lambda}{\lambda + m\delta} \int_0^b f(x) V_m(b - x, b) dx$$

or

$$\frac{\lambda}{\lambda + m\delta} \int_0^b f(x) V_m(b - x, b) dx = V_m(b, b) - \frac{cm}{\lambda + m\delta} V_{m-1}(b, b).$$

Next, by (2.4),

$$\begin{aligned} & \frac{d}{du} V_{m+1}(u, b) \Big|_{u=b} \\ &= c^m (m+1)! \prod_{j=1}^m (\lambda + j\delta)^{-1} \\ &+ \sum_{j=1}^m c^{m-j} \binom{m+1}{j} (m+1-j)! \left[\prod_{r=j}^m (\lambda + r\delta)^{-1} \right] [(\lambda + j\delta) V_j(b, b) - cj V_{j-1}(b, b)] \\ &= c^m (m+1)! \prod_{j=1}^m (\lambda + j\delta)^{-1} + (m+1) V_m(b, b) + \\ &+ \sum_{j=1}^{m-1} c^{m-j} \binom{m+1}{j} (m+1-j)! \left[\prod_{r=j+1}^m (\lambda + r\delta)^{-1} \right] V_j(b, b) \end{aligned}$$

$$\begin{aligned}
 & - \sum_{j=2}^m c^{m-j} \binom{m+1}{j} (m+1-j)! \left[\prod_{r=j}^m (\lambda+r\delta)^{-1} \right] c_j V_{j-1}(b, b) \\
 & - c^m (m+1)! \left[\prod_{r=1}^m (\lambda+r\delta)^{-1} \right] V_0(b, b) \\
 = & (m+1) V_m(b, b) + \sum_{j=1}^{m-1} c^{m-j} \binom{m+1}{j} (m+1-j)! \left[\prod_{r=j+1}^m (\lambda+r\delta)^{-1} \right] V_j(b, b) \\
 & - \sum_{s=1}^{m-1} c^{m-s} \binom{m+1}{s+1} (m-s)! \left[\prod_{r=s+1}^m (\lambda+r\delta)^{-1} \right] (s+1) V_s(b, b) \\
 = & (m+1) V_m(b, b)
 \end{aligned}$$

which completes the proof. ■

We remark that Gerber and Shiu (2004, formula (4.11)) have shown that (2.2) also holds for the Brownian motion risk model.

Example 2.1. Let $F(x) = 1 - \exp\{-\alpha x\}$, $x \geq 0$, with $\alpha > 0$. Then by a standard technique (see, for example, Gerber (1979, p. 116)) we get

$$\frac{d^2}{du^2} V_n(u, b) + \left(\alpha - \frac{\lambda + n\delta}{c} \right) \frac{d}{du} V_n(u, b) - \frac{\alpha n\delta}{c} V_n(u, b) = 0.$$

For this individual claim amount distribution, Lundberg’s fundamental equation with force of interest $n\delta$ (see Gerber and Shiu (1998, Section 2)) is

$$s^2 + \left(\alpha - \frac{\lambda + n\delta}{c} \right) s - \frac{\alpha n\delta}{c} = 0 \tag{2.5}$$

which gives

$$V_n(u, b) = k_{1,n} \exp\{r_{1,n}u\} + k_{2,n} \exp\{r_{2,n}u\} \tag{2.6}$$

where $r_{1,n}$ and $r_{2,n}$ are the roots of (2.5). To solve for $k_{1,n}$ and $k_{2,n}$ (which are both functions of δ), we use equation (2.1) and insert the functional form for V_n given by (2.6) and the form of the density f . Integrating out we find that

$$\frac{k_{1,n}}{k_{2,n}} = - \frac{\alpha + r_{1,n}}{\alpha + r_{2,n}}$$

so that

$$V_n(u, b) = \frac{k_{1,n}}{\alpha + r_{1,n}} \left((\alpha + r_{1,n}) \exp\{r_{1,n}u\} - (\alpha + r_{2,n}) \exp\{r_{2,n}u\} \right)$$

and

$$\begin{aligned} \left. \frac{d}{du} V_n(u, b) \right|_{u=b} &= nV_{n-1}(b, b) \\ &= \frac{k_{1,n}}{a + r_{1,n}} ((\alpha + r_{1,n})r_{1,n} \exp\{r_{1,n}b\} - (\alpha + r_{2,n})r_{2,n} \exp\{r_{2,n}b\}). \end{aligned}$$

Hence

$$V_n(u, b) = nV_{n-1}(b, b) \frac{(\alpha + r_{1,n}) \exp\{r_{1,n}u\} - (\alpha + r_{2,n}) \exp\{r_{2,n}u\}}{(\alpha + r_{1,n})r_{1,n} \exp\{r_{1,n}b\} - (\alpha + r_{2,n})r_{2,n} \exp\{r_{2,n}b\}}. \tag{2.7}$$

Equation (2.7) is well known when $n = 1$. See, for example, Bühlmann (1970). Table 2.1 shows some values of the mean, standard deviation and coefficient of skewness of D_{20} when $\alpha = 1$, $\delta = 0.1$, $\lambda = 100$ and $c = 110$, with b varying. We comment on these values in the next section.

3. THE DISTRIBUTION OF D_u WHEN $\delta = 0$

In this section we consider the special case when $\delta = 0$. Gerber and Shiu (2004) consider this case for the Brownian motion risk model and show that the distribution of D_u is a mixture of a degenerate distribution at 0 and an exponential distribution. As shown below, this is also the case for the classical risk model.

Consider first the case when $0 \leq u < b$. Then the probability that there is a first dividend stream is the probability that the surplus reaches b without ruin occurring first, the probability of which is $\chi(u, b)$ where

$$\chi(u, b) = \frac{1 - \psi(u)}{1 - \psi(b)}.$$

TABLE 2.1

MEAN, STANDARD DEVIATION AND COEFFICIENT OF SKEWNESS OF D_{20} .

b	Mean	St. Dev.	Skewness
20	46.496	35.705	0.8737
30	65.011	43.875	0.1472
40	72.355	42.811	-0.2733
50	71.324	39.706	-0.4133
60	66.896	36.866	-0.3978
70	61.620	34.386	-0.3246
80	56.404	32.129	-0.2361
90	51.520	30.023	-0.1464
100	47.025	28.042	-0.0596

See Dickson and Gray (1984). Given that there is a first dividend stream, the probability that there is a second stream of dividend payments is $p(b)$ where

$$p(b) = \int_0^b f(x)\chi(b - x, b)dx \stackrel{\text{def}}{=} 1 - q(b)$$

since a second stream can occur only if the amount of the claim that takes the surplus process away from the dividend barrier is no more than b . Hence, if N denotes the number of streams of dividend payments, N has a zero-modified geometric distribution with $\Pr(N = 0) = 1 - \chi(u, b)$ and for $r = 1, 2, 3, \dots$,

$$\Pr(N = r) = \chi(u, b)p(b)^{r-1}q(b).$$

Thus

$$M_N(t) = 1 - \chi(u, b) + \frac{\chi(u, b)q(b)e^t}{1 - p(b)e^t}.$$

Now let Δ_i denote the amount of the i th dividend stream, so that $D_u = \sum_{i=1}^N \Delta_i$. Clearly $\{\Delta_i\}_{i=1}^\infty$ is a sequence of independent and identically distributed random variables. We note that the time until a claim from the time the surplus process reaches b is exponentially distributed with mean $1/\lambda$ due to the memoryless property of the exponential distribution. Hence, the distribution of Δ_1 is exponential with mean c/λ , and so

$$M(t) = E[\exp\{t\Delta_1\}] = \frac{\lambda}{\lambda - ct}$$

giving

$$\begin{aligned} M_{D_u}(t) &= M_N[\log M(t)] = 1 - \chi(u, b) + \frac{\chi(u, b)q(b)M(t)}{1 - p(b)M(t)} \\ &= 1 - \chi(u, b) + \chi(u, b)\frac{q(b)\lambda}{q(b)\lambda - ct}. \end{aligned}$$

Thus, the distribution of D_u is a mixture of a degenerate distribution at 0 and an exponential distribution with mean $c/(\lambda q(b))$.

Applying the above argument when $u = b$, we see that the distribution of D_b is exponential with mean $c/(\lambda q(b))$.

We remark that when $\delta = 0$, the coefficient of skewness of D_u is always positive. However, it can be seen from Table 2.1 that this is not the case when $\delta > 0$.

4. TIME AND SEVERITY OF RUIN

In this section we derive results relating to the time and severity of ruin that we will apply in Section 6. Let Y_u denote the deficit at ruin given initial surplus u .

We will consider $E[e^{-\delta T_u} Y_u^n]$, of which $E[e^{-\delta T_u}]$ is an important special case. To do this, we adopt the approach of Gerber and Shiu (1998) and consider a function $\phi_n(u, b)$ defined by $\phi_n(u, b) = E[e^{-\delta T_u} Y_u^n]$. This is just the function ϕ defined in equation (2.10) of Gerber and Shiu (1998), modified to our model. In particular, $P(T_u < \infty) = 1$ in our model. Properties of a more general version of the function $\phi_n(u, b)$ are discussed in detail by Lin et al (2003).

Let τ be as previously defined. Then by conditioning on the time and the amount of the first claim,

$$\begin{aligned} \phi_n(u, b) &= \int_0^\tau \lambda e^{-(\lambda+\delta)t} \int_{u+ct}^\infty (y-u-ct)^n f(y) dy dt \\ &\quad + \int_\tau^\infty \lambda e^{-(\lambda+\delta)t} \int_b^\infty (y-b)^n f(y) dy dt \\ &\quad + \int_0^\tau \lambda e^{-(\lambda+\delta)t} \int_0^{u+ct} f(y) \phi_n(u+ct-y, b) dy dt \\ &\quad + \int_\tau^\infty \lambda e^{-(\lambda+\delta)t} \int_0^b f(y) \phi_n(b-y, b) dy dt \\ &= \frac{1}{c} \int_u^b \lambda e^{-(\lambda+\delta)(s-u)/c} \int_s^\infty (y-s)^n f(y) dy ds \\ &\quad + \frac{1}{c} \int_b^\infty \lambda e^{-(\lambda+\delta)(s-u)/c} \int_b^\infty (y-b)^n f(y) dy ds \\ &\quad + \frac{1}{c} \int_u^b \lambda e^{-(\lambda+\delta)(s-u)/c} \int_0^s f(y) \phi_n(s-y, b) dy ds \\ &\quad + \frac{1}{c} \int_b^\infty \lambda e^{-(\lambda+\delta)(s-u)/c} \int_0^b f(y) \phi_n(b-y, b) dy ds \end{aligned}$$

giving

$$\begin{aligned} ce^{-(\lambda+\delta)uc} \phi_n(u, b) &= \int_u^b \lambda e^{-(\lambda+\delta)s/c} \int_s^\infty (y-s)^n f(y) dy ds \\ &\quad + \int_b^\infty \lambda e^{-(\lambda+\delta)s/c} \int_b^\infty (y-b)^n f(y) dy ds \\ &\quad + \int_u^b \lambda e^{-(\lambda+\delta)s/c} \int_0^s f(y) \phi_n(s-y, b) dy ds \\ &\quad + \int_b^\infty \lambda e^{-(\lambda+\delta)s/c} \int_0^b f(y) \phi_n(b-y, b) dy ds. \end{aligned}$$

Hence

$$\begin{aligned} &ce^{-(\lambda+\delta)uc} \left(\frac{-(\delta+\lambda)}{c} \phi_n(u, b) + \frac{d}{du} \phi_n(u, b) \right) \\ &= -\lambda e^{-(\lambda+\delta)uc} \int_u^\infty (y-u)^n f(y) dy - \lambda e^{-(\lambda+\delta)uc} \int_0^u f(y) \phi_n(u-y, b) dy \end{aligned}$$

or

$$\begin{aligned} \frac{d}{du} \phi_n(u, b) &= \frac{\delta + \lambda}{c} \phi_n(u, b) - \frac{\lambda}{c} \int_u^\infty (y - u)^n f(y) dy \\ &\quad - \frac{\lambda}{c} \int_0^u f(y) \phi_n(u - y, b) dy \end{aligned} \tag{4.1}$$

which can be written as

$$\begin{aligned} \phi_n(u, b) &= \frac{c}{\lambda + \delta} \frac{d}{du} \phi_n(u, b) + \frac{\lambda}{\lambda + \delta} \left[\int_u^\infty (y - u)^n f(y) dy \right. \\ &\quad \left. + \int_0^u f(y) \phi_n(u - y, b) dy \right]. \end{aligned}$$

Also

$$\begin{aligned} \phi_n(b, b) &= \int_0^\infty \lambda e^{-(\lambda + \delta)t} dt \left[\int_b^\infty (y - b)^n f(y) dy + \int_0^b f(y) \phi_n(b - y, b) dy \right] \\ &= \frac{\lambda}{\lambda + \delta} \left[\int_b^\infty (y - b)^n f(y) dy + \int_0^b f(y) \phi_n(b - y, b) dy \right] \end{aligned}$$

so that

$$\left. \frac{d}{du} \phi_n(u, b) \right|_{u=b} = 0. \tag{4.2}$$

Example 4.1. Let us again consider the case when $F(x) = 1 - \exp\{-\alpha x\}$, $x \geq 0$, with $\alpha > 0$. It is straightforward to show that

$$\frac{d^2}{du^2} \phi_n(u, b) + \left(\alpha - \frac{\delta + \lambda}{c} \right) \frac{d}{du} \phi_n(u, b) - \frac{\alpha \delta}{c} \phi_n(u, b) = 0$$

giving

$$\phi_n(u, b) = h_1 e^{r_1 u} + h_2 e^{r_2 u}$$

where r_1 and r_2 are what were called $r_{1,1}$ and $r_{2,1}$ in Example 2.1. Inserting for ϕ_n and f in (4.1) we find that

$$\frac{n!}{\alpha^n} = \frac{\alpha h_1}{\alpha + r_1} + \frac{\alpha h_2}{\alpha + r_2}, \tag{4.3}$$

where $h_i = h_i(\delta)$ for $i = 1, 2$. Also

$$\left. \frac{d}{du} \phi_n(u, b) \right|_{u=b} = h_1 r_1 e^{r_1 b} + h_2 r_2 e^{r_2 b}$$

gives

$$\frac{h_1}{h_2} = \frac{-r_2}{r_1} e^{(r_2-r_1)b}.$$

From (4.3) it follows that

$$E[e^{-\delta T_u} Y_u^n] = E[e^{-\delta T_u}] E[Y_u^n] = E[e^{-\delta T_u}] \frac{n!}{\alpha^n},$$

a result we could have anticipated as the distribution of the deficit at ruin is exponential with mean $1/\alpha$, independent of the time of ruin. A little algebra shows that

$$E[e^{-\delta T_u}] = \frac{\lambda}{c} \frac{r_1 e^{r_1 b + r_2 u} - r_2 e^{r_2 b + r_1 u}}{(\alpha + r_1) r_1 e^{r_1 b} - (\alpha + r_2) r_2 e^{r_2 b}}, \quad (4.4)$$

where we have used the fact that $(\alpha + r_1)(\alpha + r_2) = \lambda \alpha c$. An alternative method of deriving this result is given by Lin et al (2003).

5. DISCRETE TIME MODELLING

In this section we consider the discrete time model described by Dickson and Waters (1991). In this model, the surplus process is $\{U(n)\}_{n=0}^{\infty}$, where $U(0) = u$ (an integer) and $U(n) = U(n-1) + 1 - Z_n$, where $\{Z_n\}_{n=1}^{\infty}$ is a sequence of independent and identically distributed random variables. The premium income per unit time is 1, and Z_n denotes the aggregate amount of claims in the n th time period, with $E[Z_1] < 1$.

The distribution of Z_1 is compound Poisson with Poisson parameter $1/[(1 + \theta)\beta]$, where we choose β to be a positive integer, and individual claims are distributed on the non-negative integers, with a mean of β . A consequence of this is that the premium contains a loading of θ . Dickson and Waters (1991, Section 1) explain how this model can be used to approximate the classical risk model of Section 1 using a process of rescaling of both time and monetary units, and by discretising the individual claim amount distribution F .

Let b again be a dividend barrier, and let b be an integer. A dividend of 1 is payable at time n only if the surplus was at level b at time $n-1$ and there are no claims at time n , for $n = 1, 2, 3, \dots$. With a slight abuse of notation, we again let D_u denote the present value of dividends payable until ruin at force of interest $\delta \geq 0$ per unit time, and again denote $E[D_u^n]$ by $V_n(u, b)$. We are thus generalising De Finetti's original model by allowing the increment of the (unmodified) surplus process to be one of $1, 0, -1, -2, \dots$, compared with 1 or -1 in De Finetti's model. We define ruin to be the event that the surplus goes to zero or below at some time $n > 0$, but we allow the surplus at time 0 to be 0.

Conditioning on the aggregate claim amount in the first time period, and letting $g_j = Pr(Z_1 = j)$, we find that for $u = 0, 1, 2, \dots, b-1$

$$V_n(u, b) = e^{-n\delta} \sum_{j=0}^u g_j V_n(u + 1 - j, b) \tag{5.1}$$

and

$$V_n(b, b) = e^{-n\delta} \left(g_0 \sum_{j=0}^n \binom{n}{j} V_j(b, b) + \sum_{j=1}^b g_j V_n(b + 1 - j, b) \right). \tag{5.2}$$

Thus, we have $b + 1$ linear equations for the unknowns $V_n(u, b)$, $u = 0, 1, 2, \dots, b$, and we can solve these by standard methods. Claramunt, Mármol and Alegre (2002) investigate the expected present value of dividends in a discrete time model and formulae (5.1) and (5.2) above with $n = 1$ correspond to their formula (4).

The same approach can be applied to finding moments of the discounted time of ruin and severity of ruin. We again let T_u denote the time of ruin from initial surplus u , and note that $\Pr(T_u < \infty) = 1$. Defining $\phi_0(u, b) = E[e^{-\delta T_u}]$ and letting

$$\tilde{G}(k) = 1 - G(k) = \sum_{j=k+1}^{\infty} g_j$$

we have

$$\phi_0(u, b) = e^{-\delta} \left(\sum_{j=0}^u g_j \phi_0(u + 1 - j, b) + \tilde{G}(u) \right) \tag{5.3}$$

for $u = 0, 1, 2, \dots, b - 1$, and

$$\phi_0(b, b) = e^{-\delta} \left(g_0 \phi_0(b, b) + \sum_{j=1}^b g_j \phi_0(b + 1 - j, b) + \tilde{G}(b) \right). \tag{5.4}$$

Let Y_u again denote the deficit at ruin, and let $\phi_1(u, b) = E[e^{-\delta T_u} Y_u]$. Then

$$\begin{aligned} \phi_1(u, b) &= e^{-\delta} \left(\sum_{j=0}^u g_j \phi_1(u + 1 - j, b) + \sum_{j=u+1}^{\infty} g_j (j - u - 1) \right) \\ &= e^{-\delta} \left(\sum_{j=0}^u g_j \phi_1(u + 1 - j, b) + E[Z_1] - \sum_{j=1}^u j g_j - (u + 1) \tilde{G}(u) \right) \end{aligned}$$

for $u = 0, 1, 2, \dots, b - 1$, and

$$\begin{aligned} \phi_1(b, b) &= e^{-\delta} \left(g_0 \phi_1(b, b) + \sum_{j=1}^b g_j \phi_1(b + 1 - j, b) + \sum_{j=b+1}^{\infty} g_j (j - b - 1) \right) \\ &= e^{-\delta} \left(g_0 \phi_1(b, b) + \sum_{j=1}^b g_j \phi_1(b + 1 - j, b) + E[Z_1] - \sum_{j=1}^b j g_j - (b + 1) \tilde{G}(b) \right). \end{aligned}$$

Calculation of the functions ϕ_0 and ϕ_1 involves solving $b + 1$ linear equations in the same number of unknowns. This can be done by standard methods.

Other quantities can similarly be calculated for this model. For example, for $u = 0, 1, 2, \dots, b - 1$,

$$E[T_u] = \sum_{j=0}^u g_j (1 + E[T_{u+1-j}]) + \bar{G}(u) = 1 + \sum_{j=0}^u g_j E[T_{u+1-j}]$$

and

$$\begin{aligned} E[T_b] &= g_0(1 + E[T_b]) + \sum_{j=1}^b g_j(1 + E[T_{b+1-j}]) + \bar{G}(b) \\ &= 1 + g_0 E[T_b] + \sum_{j=1}^b g_j E[T_{b+1-j}]. \end{aligned}$$

Two quantities we will need in Section 6 are $\xi(u, b) \stackrel{\text{def}}{=} E[D_u e^{-\delta T_u}]$ and $\gamma(u, b) \stackrel{\text{def}}{=} E[D_u e^{-\delta T_u} Y_u]$. We calculate these as follows.

For $u = 0, 1, 2, \dots, b - 1$,

$$\xi(u, b) = e^{-\delta} \sum_{j=0}^u g_j \xi(u + 1 - j, b)$$

noting that no dividends are payable if ruin occurs at time 1, and

$$\xi(b, b) = e^{-\delta} \left(\sum_{j=1}^b g_j \xi(b + 1 - j, b) + g_0 [\xi(b, b) + \phi_0(b, b)] \right).$$

Similarly, for $u = 0, 1, 2, \dots, b - 1$,

$$\gamma(u, b) = e^{-\delta} \sum_{j=0}^u g_j \gamma(u + 1 - j, b)$$

and

$$\gamma(b, b) = e^{-\delta} \left(\sum_{j=1}^b g_j \gamma(b + 1 - j, b) + g_0 [\gamma(b, b) + \phi_1(b, b)] \right).$$

Example 5.1. *In this example we compare approximations with exact values based on an individual claim amount distribution in the classical risk model that is exponential with mean 1. To perform calculations in our discrete model, we discretised this exponential distribution according to the method of De Vylder and Goovaerts (1988). This discretisation method is mean preserving which facilitates calculation of the function ϕ_1 . In the classical risk model, let us set $\lambda = 100$,*

TABLE 5.1
 EXACT AND APPROXIMATE VALUES OF $V_1(u, 100)$, $V_2(u, 100)$, $E[T_u]$ AND $E[\exp\{-\delta T_u\}]$

u	$V_1(u, 100)$		$V_2(u, 100)$		$E[T_u]$		$E[\exp\{-\delta T_u\}]$	
	Exact	Approx.	Exact	Approx.	Exact	Approx.	Exact	Approx.
0	4.6812	4.6810	278.90	278.88	976.07	976.45	0.9009	0.9008
10	33.353	33.352	2,030.8	2,030.6	6,803.9	6,806.5	0.3343	0.3343
20	47.025	47.023	2,997.7	2,997.4	9,151.2	9,154.8	0.1242	0.1242
30	55.423	55.421	3,760.6	3,760.3	10,096	10,100	0.0462	0.0462
40	62.185	62.182	4,533.0	4,532.6	10,477	10,481	0.0173	0.0173
50	68.689	68.686	5,403.6	5,403.2	10,629	10,633	0.0066	0.0066
60	75.482	75.479	6,421.0	6,420.4	10,690	10,694	0.0027	0.0027
70	82.802	82.799	7,622.7	7,622.1	10,714	10,718	0.0012	0.0012
80	90.779	90.775	9,047.0	9,046.3	10,723	10,727	0.0007	0.0007
90	99.505	99.500	10,737	10,736	10,726	10,730	0.0006	0.0006
100	109.06	109.06	12,741	12,740	10,726	10,731	0.0005	0.0005

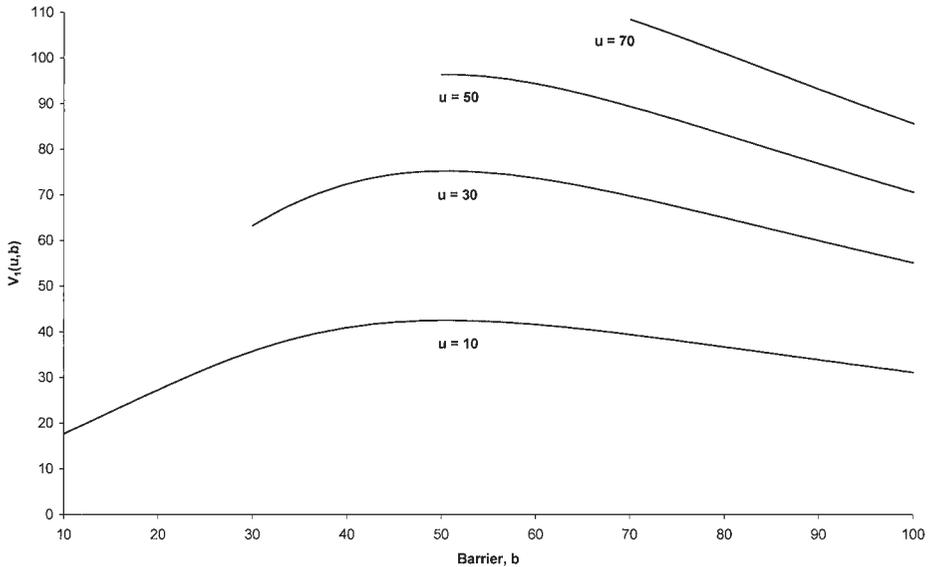
$\delta = 0.1$ and $\theta = 0.1$. In our discrete model, let $\beta = 100$. Table 5.1 shows exact and approximate values of $V_1(u, 100)$, $V_2(u, 100)$, $E[T_u]$ and $E[\exp\{-\delta T_u\}]$ for a range of values for u . (A formula for $E[T_u]$ is given by Gerber (1979, p. 150).) We see from this table that the approximate values are very close to the true values. We have observed this to be the case for other values of b but we also found that for a given value of β , accuracy improves as b increases. Also, we note that in Dickson and Waters (1991) a smaller value of β was adequate to provide good approximations – in that case to non-ruin probabilities. We believe that a small value such as $\beta = 20$ will not be particularly appropriate for our current purpose. In all subsequent calculations in this paper, a scaling factor of $\beta = 100$ has been used in our discrete model, and the above mentioned discretisation procedure has been applied.

We conclude this section by noting that the methodology of Section 3 can be applied to show that when $\delta = 0$, the distribution of D_u is a mixture of a degenerate distribution at 0 and a geometric distribution on the non-negative integers for $0 \leq u < b$, and the distribution of D_b is geometric.

6. OPTIMAL DIVIDENDS

One approach to selecting the level of the dividend barrier is to set it at b^* where b^* is the value of b which maximises $V_1(u, b)$. See Bühlmann (1970) or Gerber (1979). In the case when F is an exponential distribution with mean $1/\alpha$, the optimal level is

$$b^* = \frac{1}{r_1 - r_2} \log \frac{r_2^2(\alpha + r_2)}{r_1^2(\alpha + r_1)}$$

FIGURE 1: $V_1(u, b)$, Pareto claims

where r_1 and r_2 are as in Example 4.1. If this quantity is negative, the optimal level is $b^* = u$. See Gerber and Shiu (1998). Generally, it is difficult to find the optimal level analytically, but it is not difficult to find it numerically if we can find an expression for $V_1(u, b)$. In other cases, we can use our discrete time model to find an approximation to the value of b^* . As an example of this, Figure 1 shows $V_1(u, b)$ as a function of b when $\lambda = 100$, $c = 110$ and $\delta = 0.1$ for $u = 10, 30, 50$ and 70 when the individual claim amount distribution is Pareto(4,3). From this we observe the features that are known in the case of exponential claims, namely that there is an optimal level here for b independent of u (51 to the nearest integer), and that if u exceeds this optimal level, the maximum value of $V_1(u, b)$ occurs when $b = u$.

Under the criterion of maximising the expected present value of dividend payments until ruin, the optimal strategy is to set a constant dividend barrier. See, for example, Borch (1990). Asmussen and Taksar (1997) have shown that this result also holds, under certain conditions, when the surplus is driven by a Brownian motion. In what follows, we investigate the expected present value of dividend payments under a constant dividend barrier. We make no claims about the optimality of a constant dividend barrier, but view our study as a natural extension of existing work.

6.1. Modification 1

As the shareholders benefit from the dividend income until ruin, it is reasonable to expect that the shareholders provide the initial surplus u and take care

of the deficit at ruin. Under this scenario the expected present value of (net) income to shareholders is

$$L(u, b) \stackrel{\text{def}}{=} V_1(u, b) - u - E[e^{-\delta T_u} Y_u].$$

Consistent with the approach of the previous subsection, a reasonable objective is now to find the value of b that maximises $L(u, b)$ for a fixed value of u .

Example 6.1. Again let $F(x) = 1 - \exp\{-\alpha x\}$, $x \geq 0$, with $\alpha > 0$. Then from (2.7) and (4.4),

$$\begin{aligned} \frac{d}{db} L(u, b) &= \frac{d}{db} V_1(u, b) - \frac{d}{db} E[e^{-\delta T_u} Y_u] \\ &= - \frac{(\alpha + r_1) e^{r_1 u} - (\alpha + r_2) e^{r_2 u}}{[(\alpha + r_1) r_1 e^{r_1 b} - (\alpha + r_2) r_2 e^{r_2 b}]^2} [(\alpha + r_1) r_1^2 e^{r_1 b} - (\alpha + r_2) r_2^2 e^{r_2 b}] \\ &\quad - \frac{\lambda (r_1 - r_2) r_1 r_2 e^{(r_1 + r_2) b} [(\alpha + r_1) e^{r_1 u} - (\alpha + r_2) e^{r_2 u}]}{ca [(\alpha + r_1) r_1 e^{r_1 b} - r_2 (\alpha + r_2) e^{r_2 b}]^2} \end{aligned}$$

and this partial derivative is zero when

$$(\alpha + r_1) r_1^2 e^{r_1 b} - (\alpha + r_2) r_2^2 e^{r_2 b} = \frac{\lambda \delta}{c^2} (r_1 - r_2) e^{(r_1 + r_2) b},$$

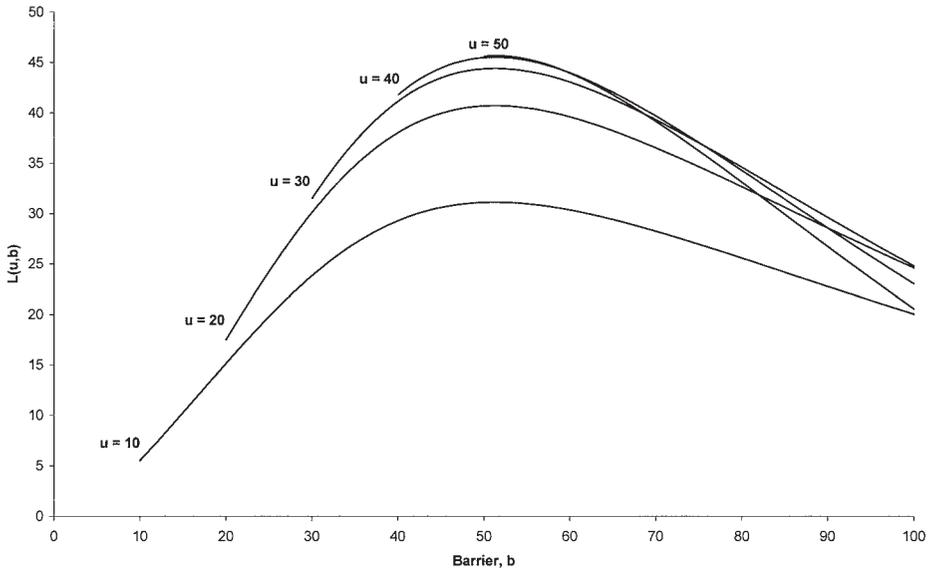


FIGURE 2: $L(u, b)$, Pareto claims

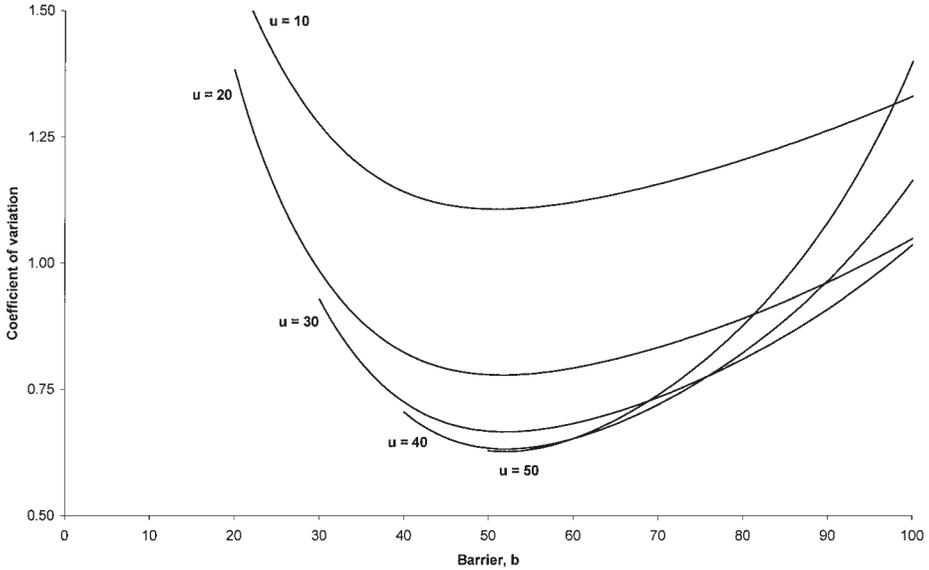


FIGURE 3: Coefficient of variation under Modification 1, exponential claims

using $r_1 r_2 = -\alpha \delta / c$. For the same numerical inputs as in Example 2.1 we find that the optimal value of b is 43.049.

Figure 2 shows values of $L(u, b)$ for $u = 10, 20, \dots, 50$ when individual claims are distributed as Pareto(4,3), $c = 110$, $\lambda = 100$ and $\delta = 0.1$. These plots are based on calculated values of $L(u, b)$ for integer values of b using the approach of the previous section, and they suggest that an optimal barrier level is around 51.

Figure 3 shows the coefficient of variation of the present value of (net) income to shareholders in the case of exponential claims, as approximated by the model of the previous section. To calculate this we require $E[(D_u - e^{-\delta T_u} Y_u)^2]$ which can be calculated as

$$V_2(u, b) - 2\gamma(u, b) + E[e^{-2\delta T_u} Y_u^2]$$

where the final term can be calculated using the approach in Section 5 to the calculation of $\phi_0(u, b)$. We note from Figure 3 that the value of b which minimises the coefficient of variation varies with u , but is around 51, compared with the value 43.049 being optimal in the sense of maximising $L(u, b)$. As each curve in Figure 3 is relatively flat around its minimum, we conclude that choosing b to minimise $L(u, b)$ is a reasonable strategy. We observed similar features in the case of Pareto(4,3) claims.

An alternative objective is to find the optimal level of investment, assuming there is no restriction on the amount the shareholders can input. As

$$\frac{d}{du} L(u, b) = \frac{d}{du} V_1(u, b) - 1 - \frac{d}{du} E[e^{-\delta T_u} Y_u]$$

we note from (2.2) and (4.2) that $\frac{d}{du}L(u, b)$ is zero when $u = b$. However, we are unable to determine whether the derivative is zero elsewhere, so we cannot say whether the maximum of $L(u, b)$ occurs when $u = b$.

In the case of the numerical illustration in Example 6.1, a numerical search shows that the optimal investment level is 43.049, with the barrier at the same level.

As a matter of mathematical interest we note that for the function ϕ_n defined in Section 4,

$$\left. \frac{d}{du} (V_1(u, b) - u - \phi_n(u, b)) \right|_{u=b} = 0.$$

6.2. Modification 2

We again assume that the shareholders input u , but now suppose that when ruin occurs, the shareholders immediately pay the amount of the deficit at ruin, so that the surplus at the time of ruin is then 0. The insurance operation can then continue from this surplus level, and the operation from the time of ruin is independent of the past, so that each time ruin occurs, the surplus can be restored to 0 and this time point is a renewal point of the new process. The surplus is now moving indefinitely between 0 and b . It can remain at b for a period, but immediately moves away from 0. Now let $\tilde{V}(u, b)$ denote the expected present value of dividends only. Then

$$\tilde{V}(u, b) = V_1(u, b) + E[e^{-\delta T_u}] \tilde{V}(0, b).$$

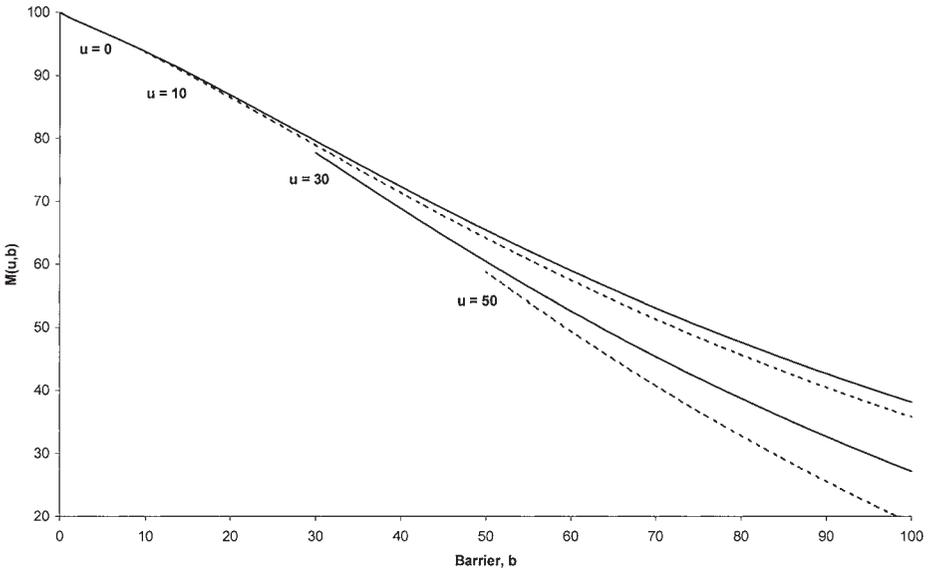


FIGURE 4: $M(u, b)$, exponential claims

Hence

$$\tilde{V}(0, b) = \frac{V_1(0, b)}{1 - E[e^{-\delta T_0}]}.$$

Now let $\tilde{W}(u, b)$ denote the expected present value of payments to be made by the shareholders when the surplus falls below 0. Then

$$\tilde{W}(u, b) = E[e^{-\delta T_u} Y_u] + E[e^{-\delta T_u}] \tilde{W}(0, b)$$

which gives

$$\tilde{W}(0, b) = \frac{E[e^{-\delta T_0} Y_0]}{1 - E[e^{-\delta T_0}]}.$$

Defining $M(u, b)$ to be the expected present value of net income to the shareholders, our strategy is to find the value of b which maximises

$$\begin{aligned} M(u, b) &= \tilde{V}(u, b) - \tilde{W}(u, b) - u \\ &= V_1(u, b) + \frac{E[e^{-\delta T_u}] V_1(0, b)}{1 - E[e^{-\delta T_0}]} \\ &\quad - E[e^{-\delta T_u} Y_u] - \frac{E[e^{-\delta T_u}] E[e^{-\delta T_0} Y_0]}{1 - E[e^{-\delta T_0}]} - u \\ &= L(u, b) + \frac{E[e^{-\delta T_u}] L(0, b)}{1 - E[e^{-\delta T_0}]} \end{aligned}$$

Figure 4 shows $M(u, b)$ as a function of b for $u = 0, 10, 30$ and 50 when $\lambda = 100$, $c = 110$, $\delta = 0.1$ and the individual claim amount distribution is exponential with mean 1. From this figure we deduce that for this set of parameters, the optimal strategy is to set $u = b = 0$ and in the Appendix we prove that this is generally the case.

The explanation for this optimal strategy is that we are dealing only with expected values. When $u = b = 0$, the shareholders are acting as the insurer: they receive the premium income and pay each claim in full when it occurs. The positive loading factor in the premium ensures that the shareholders' expected profit is positive. However, the probability that at some future time the shareholders' outgo will exceed their income is $\psi(0) = 1/(1 + \theta)$.

6.3. Modification 3

The deficiency of the optimal strategy under Modification 2 suggests that the shareholders should seek a solution under which they make no payments after

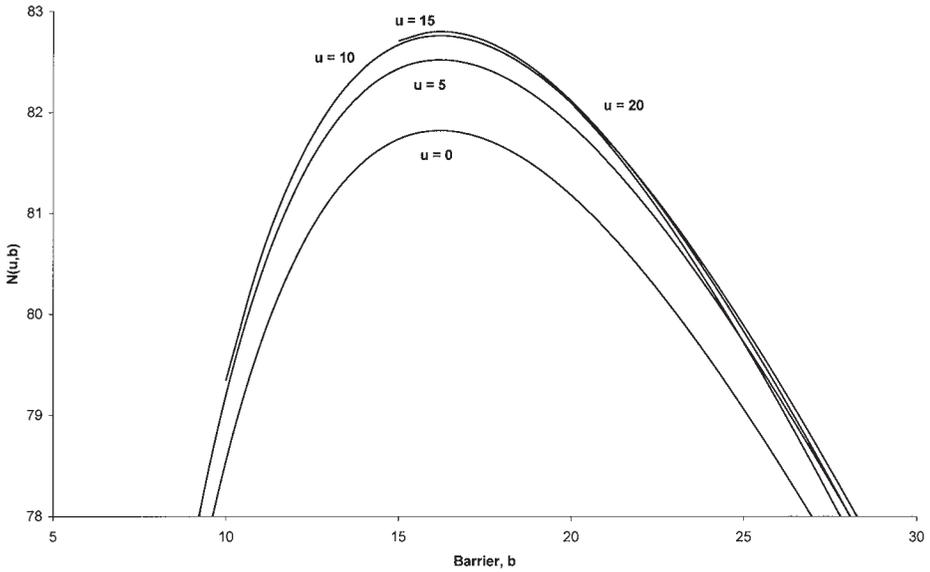


FIGURE 5: $N(u, b)$, exponential claims

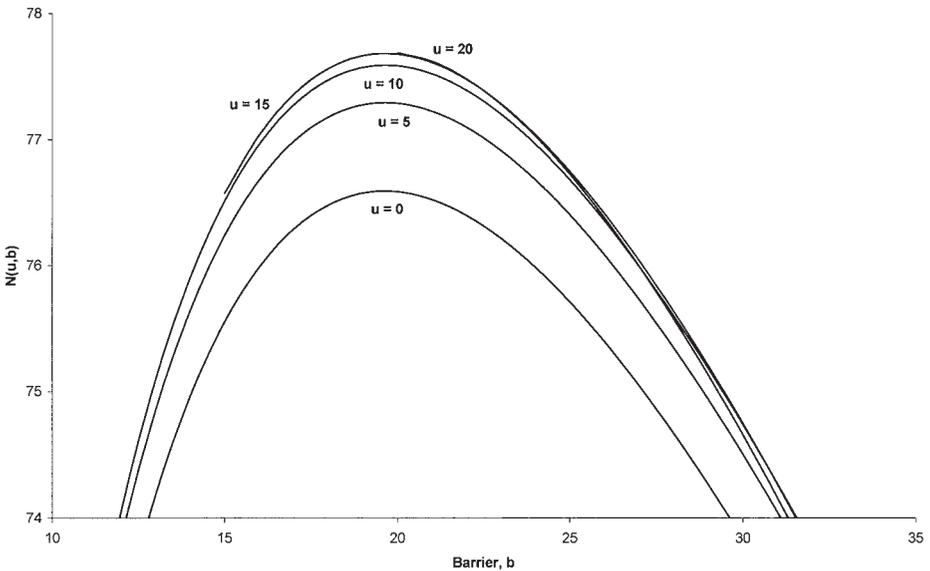


FIGURE 6: $N(u, b)$, Pareto claims

time 0, but can still receive dividend income after ruin occurs. Let us suppose that the shareholders purchase a reinsurance policy which provides them with the amount of the deficit each time that ruin occurs. Such a policy is discussed in the context of the (unrestricted) classical surplus process by Pafumi (1998)

in his discussion of Gerber and Shiu (1998). In our examples, we assume that the reinsurance premium is calculated by the expected value principle. Let θ_R denote the loading factor used by the reinsurer. Then by previous arguments, the reinsurance premium is

$$RP = (1 + \theta_R) \left(E[e^{-\delta T_u} Y_u] + E[e^{-\delta T_u}] \frac{E[e^{-\delta T_0} Y_0]}{1 - E[e^{-\delta T_0}]} \right).$$

Our strategy is now to find the value of b that maximises

$$N(u, b) \stackrel{\text{def}}{=} \tilde{V}(u, b) - u - RP.$$

Let us assume that claims are exponentially distributed with parameter α . Inserting expressions for $\tilde{V}(u, b)$ and RP into the formula for $N(u, b)$, it can be shown after some algebra that

$$\frac{d}{db} N(u, b) = k_1(u)k_2(b)$$

where $k_1(u)$ is a function of u , but not b , and $k_2(b)$ is a function of b , but not u . Further $\frac{d}{db} N(u, b) = 0$ if and only if

$$r_1(\alpha + r_1)e^{r_1 b} - r_2(\alpha + r_2)e^{r_2 b} = (1 + \theta_R) \frac{\lambda}{c} (r_1 - r_2) e^{b(r_1 + r_2)}.$$

Figure 5 shows $N(u, b)$ as a function of b for $u = 0, 5, 10, 15$ and 20 for the following parameter values: $\alpha = 1$, $\lambda = 100$, $c = 110$, $\delta = 0.1$ and $\theta_R = 0.25$. In this case, the optimal value of b is 16.195 . Further, the optimal strategy in this case is to invest this amount and set the barrier at this level, giving $N(16.195, 16.195) = 82.80$. Note that

$$82.80 > 16.195 + 31.85 = u + RP.$$

Figure 6 shows $N(u, b)$ for Pareto(4,3) claim sizes, with all other parameters as in Figure 5. This appears to show the same feature as Figure 5, namely that the optimal value of b is independent of u , but we have not been able to prove that this is true. From Figure 6, the optimal value of b is 20 (to the nearest integer), and $N(20, 20) = 77.68$. Note that in this case $RP = 43.96$ and

$$82.80 - 16.195 - 31.85 > 77.68 - 20 - 43.96.$$

Considering the greater variance associated with Pareto claims, it is somewhat surprising that a difference of only 3.8 in initial surplus is required compared with the case of exponential claims.

Figure 7 shows the coefficient of variation in the case of Pareto(4,3) claims. We observe that this function has an unusual shape. However, the function does not vary greatly around $b = 20$.

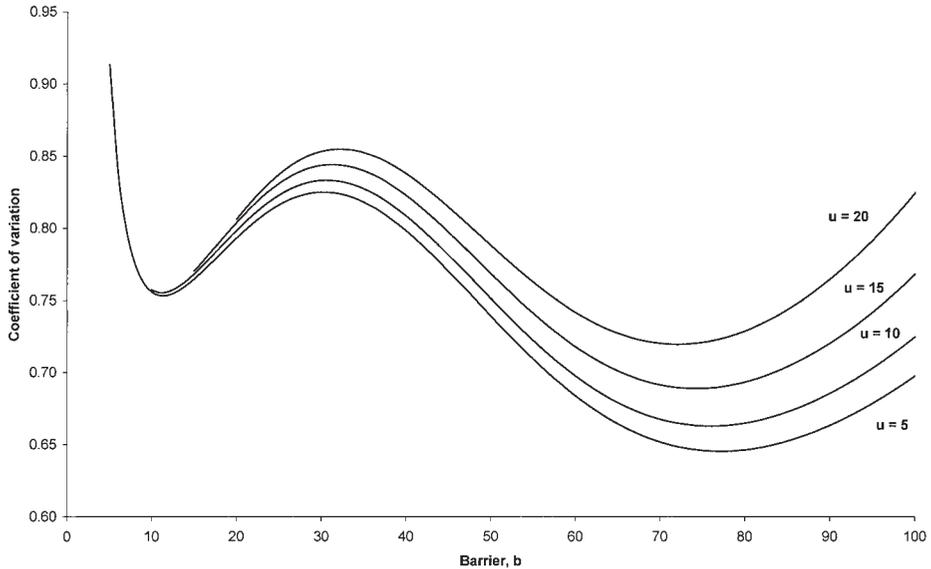


FIGURE 7: Coefficient of variation under Modification 3, Pareto claims

We also calculated values of $N(u, b)$ for different values of θ_R . Our calculations were for integer values of u and b , and we observed that as θ_R increased, the largest value of $N(u, b)$ over all u and b decreased, with this largest value almost always occurring with $u = b$. Further, for increasing θ_R , the value of b maximising $N(b, b)$ increased.

REFERENCES

- ASMUSSEN, S. and TAKSAR, M. (1997) Controlled diffusion models for optimal dividend pay-out. *Insurance: Mathematics & Economics* **20**, 1-15.
- BORCH, K. (1990) *Economics of Insurance*. North Holland, Amsterdam.
- BÜHLMANN, H. (1970) *Mathematical Methods in Risk Theory*. Springer-Verlag, Berlin.
- CLARAMUNT, M.M., MÁRMOL, M. and ALEGRE, A. (2002) *Expected present value of dividends with a constant barrier in the discrete time model*. Proceedings of the 6th International Congress on Insurance: Mathematics & Economics, Lisbon.
- DE FINETTI, B. (1957) Su un' impostazione alternativa della teoria collettiva del rischio. *Transactions of the XVth International Congress of Actuaries* **2**, 433-443.
- DE VYLDER, F. and GOOVAERTS, M.J. (1988) Recursive calculation of finite time survival probabilities. *Insurance: Mathematics & Economics* **7**, 1-8.
- DICKSON, D.C.M. and GRAY, J.R. (1984) Approximations to ruin probability in the presence of an upper absorbing barrier. *Scandinavian Actuarial Journal*, 105-115.
- DICKSON, D.C.M. and WATERS, H.R. (1991) Recursive calculation of survival probabilities. *ASTIN Bulletin* **21**, 199-221.
- GERBER, H.U. (1979) *An Introduction to Mathematical Risk Theory*. S.S. Huebner Foundation, Philadelphia, PA.
- GERBER, H.U. and SHIU, E.S.W. (1998) On the time value of ruin. *North American Actuarial Journal* **2**, 48-78.

- GERBER, H.U. and SHIU, E.S.W. (2004) Optimal dividends: analysis with Brownian motion. *North American Actuarial Journal* **8**(1), 1-20.
- HØJGAARD, B. (2002) Optimal dynamic premium control in non-life insurance. Maximising dividend pay-outs. *Scandinavian Actuarial Journal*, 225-245.
- KLUGMAN, S.A., PANJER, H.H. and WILLMOT, G.E. (1998) *Loss models – from data to decisions*. John Wiley, New York.
- LIN, X.S., WILLMOT, G.E. and DREKIC, S. (2003) The classical Poisson risk model with a constant dividend barrier: analysis of the Gerber-Shiu discounted penalty function. *Insurance: Mathematics & Economics* **33**, 551-566.
- PAFUMI, G. (1998) Discussion of 'On the time value of ruin'. *North American Actuarial Journal* **2**, 75-76.
- PAULSEN, J. and GJESSING, H.K. (1997) Optimal choice of dividend barriers for a risk process with stochastic return on investments. *Insurance: Mathematics & Economics* **20**, 215-223.
- SIEGL, T. and TICHY, R.E. (1999) A process with stochastic claim frequency and a linear dividend barrier. *Insurance: Mathematics & Economics* **24**, 51-65.

APPENDIX

In this Appendix we consider Modification 2, as set out in Section 6.2. We prove that $u = b = 0$ is optimal for this model in the sense that this maximises $M(u, b)$, the expected present value of the net income to the shareholders.

For a given initial surplus u and a dividend barrier at $b (\geq u)$, let $D_t^*(u, b)$ be a random variable denoting the total net income to the shareholders in $(0, t]$. Then, since the shareholders have to provide the initial surplus, the total net income to the shareholders in $[0, t]$ is $D_t^*(u, b) - u$. Hence:

$$M(u, b) = E \left[\int_{0+}^{\infty} e^{-\delta s} dD_s^*(u, b) \right] - u.$$

For any u and b such that $0 \leq u \leq b$ and any $t \geq 0$, let:

$$C_t^*(u, b) = D_t^*(0, 0) - D_t^*(u, b) + u.$$

Then:

$$M(0, 0) - M(u, b) = E \left[\int_{0-}^{\infty} e^{-\delta s} dC_s^*(u, b) \right].$$

We will show that this difference is non-negative by showing in the following Result that the integral is non-negative with probability one.

Result: For any $u (\geq 0)$, $b (\geq u)$, $\delta (\geq 0)$ and $t (\geq 0)$ we have:

$$C_t^*(u, b) \geq 0 \quad \text{w.p. 1} \tag{A1}$$

and

$$\int_{0-}^t e^{-\delta s} dC_s^*(u, b) \geq 0 \quad \text{w.p. 1} \tag{A2}$$

Proof. Let $U_t^*(u, b)$ be the level of the modified surplus at time $t +$, so that:

$$U_t^*(u, b) = u + ct - \sum_{i=1}^{N(t)} X_i - D_t^*(u, b). \tag{A3}$$

By considering the sample paths of the modified surplus processes, it is clear that for $b \geq u \geq 0$:

$$U_t^*(u, b) \geq U_t^*(u, u) \geq U_t^*(0, 0) \equiv 0$$

and hence from (A3) that:

$$D_i^*(0, 0) \geq D_i^*(u, b) - u.$$

This proves (A1).

Let t_1, t_2, \dots denote the (random) times of the claims for the surplus process. To prove (A2), note that the income stream contributing to $D_i^*(u, b)$ consists of continuous payments at constant rate c during some time intervals, i.e. when dividends are being paid, and some negative lump sums at some of the claim instants, i.e. payments to restore the surplus to zero following a claim which causes ruin. The income stream contributing to $D_i^*(0, 0)$ consists of continuous payments at constant rate c in each interval (t_{i-1}, t_i) and negative lump sums at each claim instant equal to the corresponding claim amount. Hence, the income stream contributing to $C_i^*(u, b)$ consists of a non-negative payment of u at time 0, non-positive discrete cash flows at times t_1, t_2, \dots and, in each time interval (t_{i-1}, t_i) , at most one interval of continuous cash flow at constant rate c .

Note that the only cash flows contributing to $C_i^*(u, b)$ between t_{i-1} and t_i are non-negative, so that if:

$$\int_{0-}^{t_{i-1}} e^{-\delta s} dC_s^*(u, b) \geq 0$$

then:

$$\int_{0-}^t e^{-\delta s} dC_s^*(u, b) \geq 0$$

for $t \in [t_{i-1}, t_i)$. Hence, it is sufficient to prove (A2) for $t = t_1, t_2, \dots$. Consider the interval $[0, t_1]$. Since the only non-positive cash flow in this interval is at time t_1 , the accumulated amount of the cash flows must exceed $C_{t_1}^*(u, b)$, so that:

$$\int_{0-}^{t_1} e^{\delta(t_1-s)} dC_s^*(u, b) \geq C_{t_1}^*(u, b) \geq 0.$$

Hence, (A2) is true for $t = t_1$.

Suppose (A2) holds for $t = t_1, \dots, t_n$ for some positive integer n . Then, by hypothesis:

$$e^{\delta t_n} \int_{0-}^{t_n} e^{-\delta s} dC_s^*(u, b) \geq \int_{0-}^{t_n} e^{-\delta s} dC_s^*(u, b) \geq 0.$$

Since the accumulated cash flow to time t_n is positive, the accumulated cash flow to any time t in (t_n, t_{n+1}) is also positive (since cash flows are non-negative in (t, t_{n+1})). Further, as $C_{t_{n+1}}^*(u, b) \geq 0$,

$$e^{\delta t_n} \int_{0-}^{t_{n+1}} e^{-\delta s} dC_s^*(u, b) \geq 0$$

and hence

$$\int_{0-}^{t_{n+1}} e^{-\delta s} dC_s^*(u, b) \geq 0.$$

Formula (A2) then follows by induction. ■

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