BULL. AUSTRAL. MATH. SOC. 42A16, 46E99
VOL. 27 (1983), 107-113.

FUNCTION SPACES ON THE UNIT CIRCLE

RICHARD J. LOY

In this note we give some negative results concerning the question of whether certain integrable functions on the unit circle with mean value zero are expressible as finite sums of differences $g - g_\alpha$ of integrable functions g, where g_α denotes the translate of g by α.

Let E denote a Fréchet space of functions as the unit circle $\mathbb{T}$. Suppose $E \subset L^1(\mathbf{T})$ and is translation invariant: if $f \in E$ and $\alpha \in \mathbb{T}$ then $f_\alpha : t \mapsto f(t-\alpha)$ also lies in E. Let $\hat{} : L^1(\mathbb{T}) \to c_0(\mathbb{Z})$ denote the Fourier transform and suppose $\hat{}$ is continuous on E. Set $E_0 = \{f \in E : \hat{f}(0) = 0\}$. We will always be working on $\mathbb{T}$ and its dual $\mathbb{Z}$, and so henceforth write L^p for $L^p(\mathbf{T})$, l^p for $l^p(\mathbb{Z})$ and so on.

In the investigation of translation invariant linear functionals on E one is led to consider the subspaces of E_0 defined by

$$\Delta_m(E) = \left\{f \in E_0 : f = \sum_{i=1}^{m} g_i - (g_i)_{\alpha_i} \text{ for some } g_1, \ldots, g_m \in E_0, \alpha_1, \ldots, \alpha_m \in \mathbb{T}\right\},$$

$$\Delta(E) = \bigcup_{m \geq 1} \Delta_m(E) .$$

Indeed, a linear functional ϕ on E is translation invariant if and only

Received 21 September 1982.

if $\ker \phi \supset \Delta(E)$, and up to a scalar multiple there is only one such (namely $f \mapsto \hat{f}(0)$) precisely when $\Delta(E) = E_0$. Further, every translation invariant linear functional on E is continuous if and only if $\Delta(E)$ is closed and of finite codimension in E_0 .

The role of the individual $\Delta_m(E)$ becomes apparent from the following.

THEOREM 1. *Suppose* E *is separable and the map* $E \times \mathbb{T} \to E : (f, \alpha) \mapsto f_\alpha$ *is continuous. Then* $\Delta(E)$ *is closed and of finite codimension in* E_0 *if and only if* $\Delta_m(E)$ *has finite codimension in* E_0 *for some* m .

Proof. The hypotheses on E ensure that each $\Delta_m(E)$, and $\Delta(E)$, are analytic subspaces of E_0 and so they are necessarily closed if of finite codimension. Thus $\operatorname{codim} \Delta_m(E) < \infty$ for some m implies $\Delta(E)$ closed and of finite codimension. Conversely, if $\operatorname{codim} \Delta(E) < \infty$ it is closed, and since $\Delta(E) = \bigcup_{m \geq 1} \Delta_m(E)$ some $\Delta_k(E)$ is nonmeagre in $\Delta(E)$. But then $\Delta(E) = \Delta_k(E) - \Delta_k(E)$ by the Pettis lemma. Since $\Delta_{2k}(E) = \Delta_k(E) - \Delta_k(E)$ we thus have $\operatorname{codim} \Delta_{2k}(E) < \infty$.

Most of what is known for specific E is detailed in greater generality in the survey paper [8] and a brief resumé suffices here: $\Delta_2(L^2) \subsetneqq \Delta_3(L^2) = L_0^2$, $\Delta_2(A) = A_0$, $\Delta(L^1) \neq L_0^1$, $\Delta_1(C^\infty) = C_0^\infty$. The C^∞ result is proven using distributions, and consideration of the orders of the distributions involved (see [7]) enables the further conclusion $\Delta_1(C) \supset C_0^2$. However, the sharper result $\Delta_1(C^\varepsilon) \supset C_0^{1+\delta}$, for any $\delta > \varepsilon > 0$, is given in [2] (as is another proof of the C^∞ result). Finally, $\Delta(L^\infty) \neq L_0^\infty$ is clear from the results of [10].

In [5] there is an inconclusive discussion of whether for each $f \in C_0$ there is some irrational $\alpha \in \mathbb{T}$ such that

https://doi.org/10.1017/S0004972700011527 Published online by Cambridge University Press

$$(*) \qquad \sup_n \left| \sum_{r=1}^{n} f(r\alpha) \right| < \infty ,$$

this being equivalent to $f = g - g_\alpha$ for some $g \in C$ by Theorem 14.11 of [3]. Here we show $\Delta_1(C) \neq C_0$ so (*) fails. Indeed for 'most' $f \in C_0$, (*) fails for 'most' α.

THEOREM 2. *(i)* $C_0 \not\subset \Delta_1(L^1)$ *so in particular* $C_0 \neq \Delta_1(C)$.

(ii) $L_0^p \not\subset \Delta_m(L^1)$ *if* $1 \le m < p(p-1)^{-1}$ *for* $1 < p < \infty$.

Proof. We use the same consequence of diophantine approximation theory as has been utilized in [9], [4], [6]: if $f \in \Delta_m(L^1)$ then

$$\liminf_{k\to\infty} k^{1/m}|\hat{f}(k)| = 0 .$$

Thus for *(i)* it suffices to note that the Hardy-Littlewood function

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n} e^{in\log n} \cdot e^{inx}$$

is in C_0 yet $|\hat{f}(k)| = k^{-1}$ for $k \ge 1$.

For *(ii)* suppose $1 \le m < p(p-1)^{-1}$ for some $1 < p < \infty$. Then $p - 2 - pm^{-1} < -1$ and so $\sum n^{p-2}n^{-p/m} < \infty$, so that by [1], §7.3,

$$f(x) = \sum_{n=1}^{\infty} n^{-1/m} \sin nx$$

defines $f \in L_0^p$ with $|\hat{f}(k)| = \frac{1}{2}k^{-1/m}$ for $k \ge 1$.

We remark that $\Delta(L^1) \neq L_0^1$ is proved similarly to *(ii)* by using the function

$$g(x) = \sum_{n=2}^{\infty} \frac{\cos nx}{\log n} .$$

Alternatively this result follows from *(ii)* by the argument of Theorem 1 and the observation $L^p \subset L^1$ if $p \geq 1$.

Note also that if $1 \leq p \leq 2$ and $f \in L^p$ then $f \in F_q$ (that is, $\hat{f} \in l^q$) for $q = p(p-1)^{-1}$ by the Hausdorff-Young inequality. Thus if $\hat{f}(0) = 0$ we have $f \in \Delta_m(F_q)$ for $m > q$ by [9]. But whether or not $f \in \Delta_m(L^p)$ for $m > q$ remains open.

THEOREM 3. *The set*

$$\left\{f \in C_0 : \sup_n \left|\sum_{r=1}^{n} f(r\alpha)\right| = \infty \text{ for } \alpha \text{ in a nonmeagre set of full measure}\right\}$$

is nonmeagre in C_0.

Proof. For $k = 1, 2, \ldots$ define

$$M_k = \left\{f \in C_0 : \sup_n \left|\sum_{r=1}^{n} f(r\alpha)\right| > k \text{ for } \alpha \text{ in a set of measure greater than } 1-k^{-1}\right\}.$$

Then each M_k is open in C_0. For suppose $f \in M_k$, so the inequality will hold on a compact set S of measure greater than $1 - k^{-1}$. If $\alpha \in S$ there is $\delta(\alpha) > 0$ and an integer $n(\alpha) \geq 1$ such that

$$\left|\sum_{r=1}^{n(\alpha)} f(r\alpha)\right| > k + \delta(\alpha) .$$

Since f is continuous there is thus an open neighbourhood $U(\alpha)$ of α such that

$$\left|\sum_{r=1}^{n(\alpha)} f(r\beta)\right| > k + \delta(\alpha)$$

for $\beta \in U(\alpha)$. Let $U(\alpha_1), \ldots, U(\alpha_p)$ be a finite cover of S by such neighbourhoods and set $\varepsilon = \min\{\delta(\alpha_1), \ldots, \delta(\alpha_p)\}$, $m = \max\{n(\alpha_1), \ldots, n(\alpha_p)\}$. Now take $g \in C_0$ with $\|f-g\| < \varepsilon m^{-1}$. Then

https://doi.org/10.1017/S0004972700011527 Published online by Cambridge University Press

if $\alpha \in S$, say $\alpha \in U(\alpha_j)$,

$$\left|\sum_{r=1}^{n(\alpha_j)} g(r\alpha)\right| \geq \left|\sum_{r=1}^{n(\alpha_j)} f(r\alpha)\right| - \varepsilon n(\alpha_j)m^{-1}$$
$$> k + \delta(\alpha_j) - \varepsilon$$
$$\geq k .$$

Thus $\{g \in C_0 : \|f-g\| < \varepsilon m^{-1}\} \subset M_k$.

Each M_k is dense in C_0 . To see this, let h denote the Hardy-Littlewood function defined above, normalized so $\|h\| = 1$. Let $\mathbb{J}$ denote the set of irrationals in $\mathbb{T}$. Take $f \in C_0$, $\varepsilon > 0$ and define the Borel set

$$\mathbb{J}_1 = \left\{\alpha \in \mathbb{J} : \sup_n \left|\sum_{r=1}^{n} f(r\alpha)\right| \leq k\right\} .$$

If $\mathbb{J}_1$ has measure less than k^{-1} then $f \in M_k$. Otherwise take $0 < \delta_1 < \varepsilon$ so that by *(i)* of Theorem 2,

$$\sup_n \left|\sum_{r=1}^{n} (f+\delta_1 h)(r\alpha)\right| = \infty$$

for $\alpha \in \mathbb{J}_1$, and so the Borel set

$$\mathbb{J}_2 = \left\{\alpha \in \mathbb{J} : \sup_n \left|\sum_{r=1}^{n} (f+\delta_1 h)(\alpha)\right| \leq k\right\}$$

is disjoint from $\mathbb{J}_1$. We may continue in this manner to obtain a sequence of distinct numbers $\{\delta_i\}$ with $0 < \delta_i < \varepsilon$ and disjoint Borel sets $\{\mathbb{J}_i\}$. Since the $\{\mathbb{J}_i\}$ are disjoint there is some $\mathbb{J}_j$ with measure less than k^{-1} . But then $f + \delta_j h \in M_k$ and has distance less than ε from f .

Let $M = \bigcap M_k$, nonmeagre in C_0 by the above. If $f \in M$ then certainly

$$V = \left\{\alpha \in \mathbb{T} : \sup_n \left|\sum_{r=1}^{n} f(r\alpha)\right| = \infty\right\}$$

has full measure. Finally, the function $\alpha \mapsto \sup_n \left|\sum_{r=1}^{n} f(r\alpha)\right|$ is lower semicontinuous and so is continuous at the points of a nonmeagre set W. It cannot be finite at any point of continuity since V has full measure. Thus $W \subset V$ and the result follows.

References

[1] R.E. Edwards, *Fourier series: a modern introduction*, Volume 1 (Holt, Rinehart and Winston, New York, 1967). See also: Second edition (Graduate Texts in Mathematics, **64**. Springer-Verlag, New York, Heidelberg, Berlin, 1979).

[2] Michael Robert Hermann, "Sur la conjugaison différentiable des diffeomorphismes du cercle a des rotations", *Inst. Hautes Études Sci. Publ. Math.* **49** (1979).

[3] Walter Helbig Gottschalk and Gustav Arnold Hedlund, *Topological dynamics* (American Mathematical Society Colloquium Publications, **36**. American Mathematical Society, Providence, Rhode Island, 1955).

[4] C.J. Lester, "Continuity of operators on $L_2(G)$ and $L_1(G)$ commuting with translations", *J. London Math. Soc.* (2) **11** (1975), 144-146.

[5] Richard J. Loy, "On the uniqueness of Riemann integration", *Automatic continuity and radical Banach algebras* (Lecture Notes in Mathematics. Springer-Verlag, Berlin, Heidelberg, New York, to appear).

[6] Peter Ludvik, "Discontinuous translation invariant linear functionals on $L^1(G)$ ", *Studia Math.* **56** (1976), 21-30.

[7] Gary H. Meisters, "Periodic distributions and non-Liouville numbers", *J. Funct. Anal.* **26** (1977), 68-88.

[8] Gary Hosler Meisters, "Some problems and results on translation-invariant forms", *Automatic continuity and radical Banach algebras* (Lecture Notes in Mathematics. Springer-Verlag, Berlin, Heidelberg, New York, to appear).

[9] Gary H. Meisters and Wolfgang M. Schmidt, "Translation-invariant linear forms on $L^2(G)$ for compact abelian groups G", *J. Funct. Anal.* 11 (1972), 407-424.

[10] Joseph Max Rosenblatt, "Invariant means for the bounded measurable functions on a non-discrete locally compact group", *Math. Ann.* 220 (1976), 219-228.

Department of Mathematics,
Faculty of Science,
Australian National University,
PO Box 4,
Canberra, ACT 2600,
Australia.