ON THE LIMIT OF THE MODULUS OF A BOUNDED REGULAR FUNCTION

by N. A. BOWEN
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1. Introduction

M. L. Cartwright has given ((**2**), 180-181) the following theorem, together with a neat proof of it.

Theorem C. *Suppose that* $f(z)$ *is regular and*

$$|f(z)|<1 \qquad (1)$$

in the half-strip S

$$\alpha<x<\beta, \quad y>1 \quad (z = x+iy) \qquad (2)$$

of the complex plane.

Suppose also that for some constant a in $\alpha<a<\beta$

$$|f(a+iy)|\to 1 \qquad (3)$$

as $y\to\infty$. *Then for every* $\delta>0$

$$|f(x+iy)|\to 1 \qquad (4)$$

uniformly as $y\to\infty$ *for* $\alpha+\delta\leqq x\leqq\beta-\delta$.

The object of this note is to give a new proof of Theorem C, with reduced hypotheses. Observing the strong resemblance between Theorem C and Montel's limit theorem, a standard proof of which follows from Vitali's convergence theorem, one wonders whether a result of Vitali type but involving the moduli of a sequence of regular and bounded functions instead of the sequence of functions itself, exists, and, if so, whether Theorem C can be obtained from it. Theorems 1 and 2 below show that this is in fact the case.

2. Theorem 1

Let $f_n(z)$ *be a sequence of functions regular and satisfying*

$$|f_n(z)|<1 \quad (n = 1, 2, 3, \ldots) \qquad (5)$$

for every z in the circle γ,

$$|z|<1. \qquad (6)$$

Let z_n *be a sequence of points in* (6) *such that*

$$z_n\to 0 \text{ as } n\to\infty, \qquad (7)$$

and suppose that

$$|f_n(z_n)|\to 1 \text{ as } n\to\infty. \qquad (8)$$

Then for every $\delta>0$

$$|f_n(z)|\to 1 \qquad (9)$$

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uniformly in γ_δ,

$$|z|\leqq 1-\delta$$

as $n\to\infty$.

Remarks on Theorem 1

(i) It is convenient in (7) to take 0 for the limit point of the z_n, but it will be seen that the method with slight elaboration applies if *any* limit point b satisfying $|b|<1$ be given. Further, it is clear, by conformal transformation or otherwise, that the circle γ may be replaced by any simply connected bounded region, γ_δ being transformed into an interior region.

(ii) It is of course tempting to try to replace the upper bound 1 in (5) by $M(>1)$ as in Vitali's Theorem. This however cannot be done, since it would imply by the method given in Section **4** below, that the upper bound 1 of $|f(z)|$ in Theorem C above could likewise be replaced by $M(>1)$. The function † $F(z) = e^{\sinh z}$ considered in the half strip $-1<x<1$, $y>1$ shows that Theorem C thus modified is false; $|F(z)|$, though bounded, tends to unity as $y\to\infty$ along one line only, viz. $x = 0$.

3. Proof of Theorem 1

Applying Schwarz' Lemma ‡ to the function $f_n(z)-f_n(0)$ in the circle γ ($|z|<1$) and using (5), we get

$$|f_n(z)-f_n(0)|<2\omega \quad (|z|\leqq\omega,\ 0<\omega<1) \qquad \text{.....................(10)}$$

and hence

$$|f_n(0)|>|f_n(z)|-2\omega.$$

Thus, given $\varepsilon>0$, we find that

$$|f_n(0)|>1-\varepsilon \quad (n>n_0(\varepsilon))\text{.................................(11)}$$

by setting $\omega = \frac{1}{3}\varepsilon$, $z = z_n$, and taking n_0 large enough to ensure that $|z_n|\leqq\frac{1}{3}\varepsilon$ and $|f_n(z_n)|>1-\frac{1}{3}\varepsilon$, inequalities that follow from (7) and (8) respectively.

Again, for all n sufficiently large, $f_n(z)$ can have no zero in the circle $\gamma_{\frac{1}{2}\delta}$, viz. $|z|\leqq 1-\frac{1}{2}\delta$, $(0<\delta<1)$. For if we suppose on the contrary that $f_n(z)$ has a zero at $z = c$, where $|c|\leqq 1-\frac{1}{2}\delta$, then, applying the maximum modulus theorem to the function $f_n(z)(1-\bar{c}z)/(z-c)$ which is regular in $|z|<1$, we get, by (5),

$$|f_n(z)|\leqq\left|\frac{z-c}{1-\bar{c}z}\right| \quad (|z|<1)$$

and hence

$$|f_n(0)|\leqq|c|\leqq 1-\tfrac{1}{2}\delta,$$

which contradicts (11) if ε be chosen less than $\frac{1}{2}\delta$, that is, for all sufficiently large n. Thus, for all n sufficiently large, each $f_n(z)$ has no zeros in $\gamma_{\frac{1}{2}\delta}$.

† Given in (3), 401, where, incidentally, it is shown that $M>1$ may be taken if the line $x=a$ of Theorem C be replaced by *two* lines $x=a$, $x=b$ and, in some cases, $f(z)\neq 0$ in the half-strip is assumed.

‡ (4), 168.

We may therefore apply Carathéodory's inequality† to $\log \phi(z)$ in $\gamma_{\frac{1}{2}\delta}$, where $\phi(z) = \{f_n(z)\}/\{f_n(0)\}$. This gives ‡

$$\log \{m(f_{n,r})/|f_n(0)|\} \geqq -\frac{2r}{w'-r} \log \{M(f_{n,r})/|f_n(0)|\}, \quad (0 \leqq r < w' = 1-\tfrac{1}{2}\delta).$$

Whence, over $0 \leqq r \leqq 1-\delta$, using (5) and (11),

$$\log m(f_{n,r}) \geqq -\frac{2r}{w'-r} \log M(f_{n,r}) + \left(\frac{w'+r}{w'-r}\right) \log |f_n(0)|$$

$$\geqq -8\delta^{-1}\varepsilon \quad (0<\varepsilon<\tfrac{1}{2}\delta,\ n>n_0(\varepsilon)),$$

and, remembering that $\delta(>0)$ is fixed, we have *a fortiori*

$$\log |f_n(z)| > -\varepsilon' \quad (n > N_0(\varepsilon')), \qquad \text{.........................(12)}$$

where $\varepsilon'(>0)$ is arbitrary, uniformly over the circle γ_δ,

$$|z| \leqq 1-\delta. \qquad \text{.........................(13)}$$

The inequalities (12) and (5) give the result (9) uniformly over the circle γ_δ defined by (13).

4. Theorem 2

The conclusion of Theorem C remains valid if the hypothesis (3) *is replaced by*

$$|f(x_n+iy_n)| \to 1 \text{ as } n \to \infty \qquad \text{.........................(14)}$$

where the points $z_n = x_n + iy_n$ *include a sequence U having the following properties,*

$$\alpha+\delta' \leqq x_n \leqq \beta-\delta', \quad (\delta' \geqq \delta) \qquad \text{.........................(15)}$$

the x_n *have only one limit point* x_0 *say,*

$$2 < y_n \uparrow \infty, \qquad \text{.........................(16)}$$

and

$$\overline{\lim_{n\to\infty}} (y_{n+1}-y_n) < \infty. \qquad \text{.........................(17)}$$

Proof of Theorem 2

Let $z_n \in U$. By (16) and (17)

$$0 < y_{n+1}-y_n < \tfrac{1}{2}\lambda, \quad (n>n_0) \qquad \text{.........................(18)}$$

where λ is a constant in (say) $2<\lambda<\infty$.

For all such n consider the sequence

$$f_n(z) = f\{z+i(y_n-1)\} \qquad \text{.........................(19)}$$

in the rectangle R:

$$\alpha < x < \beta, \quad 0 < y < \lambda.$$

Clearly

$$|f_n(z)| < 1$$

† (1), 3; (4) 174-5.
‡ $m(f_{n,r})$, $M(f_{n,r})$ denote respectively min $|f(z)|$, max $|f(z)|$ on $|z|=r$.

in R, by (1), and

$$|f_n(x_n+i)| = |f(x_n+iy_n)| \to 1 \text{ as } n \to \infty.$$

Also, we have assumed that the x_n concerned have only one limit point x_0 say, which by (15) satisfies the inequality

$$\alpha+\delta \leqq \alpha+\delta' \leqq x_0 \leqq \beta-\delta' \leqq \beta-\delta.$$

It follows by Theorem 1, applied for a rectangular region (see remarks on Theorem 1 in Section 2), that the sequence $|f_n(z)|$ converges uniformly to 1 in the rectangle R_δ:

$$\alpha+\delta \leqq x \leqq \beta-\delta, \quad \delta \leqq y \leqq \lambda-\delta$$

and hence, by (18) and (19), remembering also that $\delta > 0$ is small, we have

$$|f(z)| \to 1$$

uniformly as $y \to \infty$ in $\alpha+\delta \leqq x \leqq \beta-\delta$.

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The University
Leicester