Canad. Math. Bull. Vol. **18** (5), 1975

ON THE EDELSTEIN CONTRACTIVE MAPPING THEOREM

BY
LUDVIK JANOS

ABSTRACT. Let X be a metrizable topological space and $f:X\to X$ a continuous selfmapping such that for every $x\in X$ the sequence of iterates $\{f^n(x)\}$ converges. It is proved that under these conditions the following two statements are equivalent:

1. There is a metrization of X relative to which f is contractive in the sense of Edelstein.

2. For any nonempty f-invariant compact subset Y of X the intersection of all iterates $f^n(Y)$ is a one-point set. The relation between this type of contractivity and the Banach contraction principle is also discussed.

1. **Introduction.** A mapping f from a metric space (X, d) into itself is said to be *contractive* (in sense of Edelstein) if $x\neq y$ implies $d(f(x), f(y))<d(x, y)$. The theorem of Edelstein [3] states that *a contractive mapping f has a unique fixed point if for some $x\in X$ the sequence $\{f^n(x)\}$ of iterates has a convergent subsequence.* Recently Bryant and Guseman [1] partially answered the question posed by Nadler [7] to characterize those metric spaces for which a contractive selfmapping f, which has a fixed point, has the property that the sequence $\{f^n(x)\}$ converges for every $x\in X$. We look at this problem from another angle, focusing our attention rather on the mapping f itself than on the space (X, d). Our main objective is to give a purely topological formulation of the Edelstein theorem under the additional assumption that the sequence $\{f^n(x)\}$ converges for every $x\in X$. It turns out that this is the case if and only if the function f shrinks nonempty compact subsets of X to a point.

THEOREM 1.1. *Let X be a metrizable topological space, $f:X\to X$ continuous and such that the sequence $\{f^n(x)\}$ converges for every $x\in X$. Then the following two statements are equivalent:*

(i) *There is a metric d on X compatible with the topology of X and such that f is contractive relative to d.*

(ii) *For every nonempty compact f-invariant subset Y of X the intersection of all iterates $f^n(Y)$ is a one point set.*

REMARK 1.1. It is seen that in our case the condition (ii) implies that f has a unique fixed point $a\in X$. Indeed, denoting by a the limit of $\{f^n(x)\}$ we infer, due

Received by the editors October 23, 1974 and, in revised form, January 29, 1975.

https://doi.org/10.4153/CMB-1975-118-8 Published online by Cambridge University Press

to continuity of f that $f(a)=a$. If there were two distinct fixed points a, b then the f-invariant compact set $\{a, b\}$ would not shrink to a point as assumed in (ii).

2. **Proof of the theorem.**

(1) Proof of (i)⇒(ii). If $Y\subset X$ is a nonempty compact subset such that $f(Y)\subset Y$, our task is to show that $A=\bigcap_1^\infty f^n(Y)$ is a one-point set. From compactness of Y it follows that A is nonempty, compact, and that f maps A onto itself. Let $d[A]$ denotes the diameter of A relative to the metric d for which f is contractive. If A were not a singleton then $d[A]$ would be >0 which in turn would imply that the diameter $d[f(A)]$ of $f(A)$ is strictly less than $d[A]$ which would contradict the fact that f maps A onto itself.

(2) Proof of (ii)⇒(i).

LEMMA 2.1. *Let (X, d) be a compact metric space and $f:X\to X$ continuous and such that $\bigcap_1^\infty f^n(X)$ be a one-point set. Then the function d^* defined by*

$$d^*(x, y) = \sup_{n\geq 0} d(f^n(x), f^n(y))$$

(*where we put $f^0(x)=x$ and the supremum is taken over all non-negative integers*) *is a metric on X, is topologically equivalent to d and has the property that f is nonexpansive relative to d^*, i.e. $d^*(f(x), f(y))\leq d^*(x, y)$ for all $x, y \in X$.*

Proof. For the proof see [4, Theorem 1, p. 287].

Now we show that this lemma applies also to our space X which need not be compact. We apply it so to speak locally, to f-invariant compact subsets. Choosing a metric d on X compatible with the topology of X and defining the function $d^*(x, y)=\sup_{n\geq 0} d(f^n(x), f^n(y))$ as in Lemma 2.1 we have to show that d^* is a metric on X and that d^* and d are equivalent. From the fact that $f^n(x)\to a$ for all $x \in X$ it follows easily that if $Z\subset X$ is an arbitrary nonempty compact subset of X then the set $Z(f)$ defined by

$$Z(f) = \{y \mid y = f^n(z) \text{ for some } z\in Z \text{ and some } n \geq 0\} \cup \{a\}$$

where a is the unique fixed point of f (see Remark 1.1) is compact and f-invariant. This implies that d^* is a well defined metric on X. To prove that d^* is equivalent to d we first observe that $d^*\geq d$, so all we need is to show that any sequence $\{x_n\}$ converging relative to d converges also relative to d^*. Assume this is not so. Then there is some sequence $\{x_n\}\to x$ converging to some $x \in X$ which is divergent relative to d^*. But choosing $Z=\{x_n \mid n\geq 1\} \cup \{x\}$ this would contradict the Lemma 2.1. applied to the compact space $Z(f)$ defined above.

Having proved the equivalence of d^* to d with d^* satisfying

$$d^*(f(x), f(y)) \leq d^*(x, y) \quad \text{for all} \quad x, y\in X$$

we proceed with construction of still another metric d^{**} on X relative to which f is contractive. We define

$$d^{**}(x, y) = \sum_{n=0}^{\infty} \frac{1}{2^n} d^*(f^n(x), f^n(y))$$

and we see at once that d^{**} is a metric equivalent to d^* and hence to d since we have $d^* \leq d^{**} \leq 2d^*$. We verify also readily that f is nonexpansive also relative to d^{**}. To show that f is contractive relative to d^{**} assume for some $x \neq y$ that $d^{**}(f(x), f(y)) = d^{**}(x, y)$. This would imply that $d^*(f^n(x), f^n(y)) = d^*(f^{n+1}(x$ $f^{n+1}(y))$ for all $n > 0$ which would contradict the fact that both sequences converge to the same point $a \in X$. Thus f is contractive relative to d^{**} which completes the proof of our theorem.

3. **Relation to Banach contraction principle.** Let (X, d) be a metric space and let $f: X \to X$ be a contraction in the sense of Banach, i.e. there is a constant $c \in (0, 1)$ such that for all $x, y \in X$ we have $d(f(x), f(y)) \leq cd(x, y)$. If we assume that f has a fixed point $a \in X$ (we do not assume completeness of (X, d)) then it follows that a is a unique fixed point and that the sequence $\{f^n(x)\}$ converges to a for all x. Thus this observation may raise a conjecture whether the contractivity of f in sense of Edelstein together with the assumption that the sequence $\{f^n(x)\}$ converges for every x implies the contractivity in sense of Banach relative to a suitable equivalent remetrization of X. An example of the operator of integration $T: C \to C$ mapping the Frechét space C of all continuous real valued functions defined on the real line R into itself disproves this conjecture (see [5]). There exists a metric on C relative to which T is contractive but there is not a metric on C relative to which T would be a contraction in the sense of Banach. However if one generalizes the idea of contraction using pseudo-metrics on X instead of metrics one arrives at a similar conjecture concerning a converse of the following

THEOREM 3.1. *Let X be a metrizable space and let $\{d_n \mid n \geq 1\}$ be a countable family of pseudometrics on X generating the topology of X relative to which X is complete* (*see* [2, p. 308]). *Let $f: X \to X$ be continuous and such that for some $c \in (0, 1)$ we have*

$$d_n(f(x), f(y)) \leq cd_n(x, y) \quad \text{for all} \quad x, y \in X$$

and all $n \geq 1$. Then there is a unique fixed point $a \in X$ of f towards which every sequence $\{f^n(x)\}$ converges and there is a metric on X inducing the topology of X relative to which f is contractive.

Proof. The fact that f has a unique fixed point and that $\{f^n(x)\}$ converges for every $x \in X$ follows from the generalized Banach contraction theorem (see [5, Proposition 1.1]) and it is easy to verify that the metric d defined by

$$d(x, y) = \sum_{1}^{\infty} \frac{1}{2^n} \frac{d_n(x, y)}{1 + d_n(x, y)}$$

has the required property.

https://doi.org/10.4153/CMB-1975-118-8 Published online by Cambridge University Press

REMARK. If we postulate the convergence of the sequence $\{f^n(x)\}$ for some $x \in X$ we may drop the requirement of the completeness of X relative to the family $\{d_n \mid n \geq 1\}$ from the hypotheses of the above theorem.

This result can be quoted saying that the Banach contraction via pseudometrics implies Edelstein contractivity relative to a suitable metric. The conjecture is whether the converse is also true.

The relevance of the Edelstein concept of contractivity and of the ideas developed in this note to the theory of differential and operator equations will be the topic of the forthcoming paper by W. Derrick and the author.

REFERENCES

1. J. Bryant and L. F. Guseman, Jr., *Extensions of contractive mappings and Edelstein's iterative test*, Canad. Math. Bull. (2) **16** (1973) 185–192.

2. J. Dugundji, *Topology*, Allyn and Bacon, Boston 1966.

3. M. Edelstein, *On fixed and periodic points under contractive mappings*, J. London Math. Soc. 381–386.

4. L. Janos, *A converse of Banach's contraction theorem*, Proc. Amer. Math. Soc. **18** (1967) 287–289.

5. ——, *A converse of the generalized Banach contraction theorem*, Ardiv der Math. **21** (1970) 69–71.

6. ——, *Contraction property of the operator of integration*, Canad. Math. Bull. (to appear).

7. Sam N. Nadler, Jr., *A note on an iterative test of Edelstein*, Canad. Math. Bull. (3) **15** (1972) 381–386.