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ON BRAUER'S HEIGHT 0 CONJECTURE

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R. Brauer not only laid the foundations of modular representation theory of finite groups, he also raised a number of questions and made conjectures (see [1], [2] for instance) which since then have attracted the interest of many people working in the field and continue to guide the research efforts to a good extent. One of these is known as the "Height zero conjecture". It may be stated as follows:

CONJECTURE. *Let B be a p-block of the finite group G. All irreducible ordinary characters of G belonging to B are of height* 0 *if and only if a defect group of B is abelian.*

The conjecture is known to be true in special cases: Reynolds [14] treated the case of a normal defect group. Fong [6] proved the "if"-part for p-solvable groups and the "only if"-part for the principal block of a p-solvable group. Very recently, the proof for p-solvable groups has been completed by papers of Wolf and of Gluck and Wolf jointly [15], [8], [9].

The present paper deals with the "if"-part of the conjecture. We show that this part holds true provided it holds for all quasi-simple groups, i.e. for the covering groups of non-abelian simple groups and their factor groups.

Since the finite simple groups are classified, there is good reason to assume that in due time, the conjecture will be verified for, or a counter-example will show up among, these groups. In fact, this should be just a byproduct of a general effort to study the representation theory of simple groups. Relevant results are contained in papers of Fong-Srinivasan [7] and Michler-Olsson [13].

We should mention that, ironically, this way of proving the "if"-part of Brauer's conjecture runs contrary to Brauer's intentions. He seems to have hoped to somehow structure the theory of simple groups using

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the conjecture (see [3]); now a proof will be given using knowledge of simple groups.

Throughout this paper, groups are always finite, p is a fixed prime, "block" means "p-block". By R we denote a complete discrete rank 1 valuation ring with field of quotients of characteristic 0 which is a splitting field for all occuring group and their subgroups and with residue class field of characteristic p. The p-adic valuation is denoted by $\nu = \nu_p$.

DEFINITION. A block B of G is called a *large vertex block* (l. v. block) if for any RG-lattice M affording an irreducible character $\chi \in B$, a vertex of M is a defect group of B.

LEMMA 1. *B is an l. v. block if*

(1) *a defect group of B is abelian, or*

(2) *any irreducible character $\chi \in B$ has height* 0.

Proof. (1) is a restatement of [12], 3.7.

(2) is trivial.

LEMMA 2 (Cline [4]). *Let N be a normal subgroup of G and Let M be an RG-lattice such that $M|_{RN}$ is irreducible. If V is an RG-lattice such that $V|_{RN}$ is M-homogeneous, i.e. $V|_{RN} \cong e \cdot M|_{RN}$ for some e, then $V \cong M \otimes A$ for some RG-lattice A having N in its kernel. If A_1 and A_2 are two such lattices, then $M \otimes A_1 \cong M \otimes A_2$ iff $A_1 \cong A_2$ and $M \otimes A_1 | M \otimes A_2$ iff $A_1 | A_2$; furthermore, $M \otimes A$ is irreducible iff A is irreducible.*

Proof. Put $A = \mathrm{Hom}_{RN}(M, V)$ so A is R-free of rank e. Let G act on A by conjugation, so A is an RG-lattice having N in its kernel; the map $M \otimes A \to V$ given by $m \otimes \alpha \to m\alpha$ is an RG-isomorphism.

If U is any $R(G/N)$-lattice and $u \in U$, let $\alpha_u : M \to M \otimes U$ be defined by $m\alpha_u = m \otimes u$. The map $u \to \alpha_u$ is an RG-isomorphism $U \to \mathrm{Hom}_{RN}(M, M \otimes U)$. The remaining assertions follow easily.

LEMMA 3. *With notation as in the previous lemma, suppose that $M \otimes A$ is D-projective for some subgroup $D \leqslant G$. Then A is DN/N-projective as an RG/N-lattice.*

Proof. Since $M \otimes A$ is D-projective, it is also DN-projective, so there is a direct summand S of $(M \otimes A)|_{DN}$ such that $M \otimes A \,|\, S^G$. Now $S|_N$ is M-homogeneous since $(M \otimes A)|_N$ is, so by Lemma 2, $S \cong M|_{DN} \otimes T$ for some $R(DN)$-lattice T with N in its kernel. Therefore

$$S^G \cong (M|_{DN} \otimes T)^G \cong M \otimes (T^G).$$

Again by Lemma 2, $A \mid T^G$, so A is DN-projective, hence DN/N-projective when considered as G/N-module.

LEMMA 4. *With notation as before, let A_1 and A_2 be $R(G/N)$-lattices belonging to the same block B_0 of G/N. Then $M \otimes A_1$ and $M \otimes A_2$ belong to the same block B, say, of G. If D is a defect group of B, then DN/N contains a defect group of B_0. If B is a l. v. block, then DN/N is a defect group of B_0 and B_0 is l. v. block.*

Proof. It is enough to consider the case where the A_i's are indecomposable and there is a non-zero homomorphism $A_1 \to A_2$. Then (using Lemma 2) the same is true for the $M \otimes A_i$'s, so they belong to the same block.

If D is a defect group of B, then any A in B_0 is DN/N-projective by Lemma 3, so DN/N contains a defect group of B_0.

Finally, let A be an irreducible lattice in B_0: if EN/N is a vertex of A for some $E \leqslant D$, then A is EN-projective as a G-module and so is $M \otimes A$. But $M \otimes A$ is an irreducible lattice in B by Lemma 2, so, since B is a l.v. block, D is a vertex of $M \otimes A$. Therefore $D \leqslant_G EN$, hence $EN/N = DN/N$. Thus the defect group of B_0 cannot be smaller than DN/N, and a vertex of A is a defect group of B_0, i.e. B_0 is a l. v. block.

PROPOSITION. *Let $G = DN$ where D is a p-group and $N \lhd G$. Then*

(1) *For any block b of N, there is precisely one block B of G covering b.*

(2) *If B has defect group D then b is the only block of N covered by B.*

(3) *If B has defect group D and $\chi \in B$ is an irreducible character such that $vx(M) =_G D$ for any RG-lattice M affording χ, then $\chi|_N$ is irreducible.*

Proof. (1) is V, 3.5 (p. 199) of [5]. Assume that B has defect group D and let χ be an irreducible character in B such that $vx(M) =_G D$ for any M affording χ. Notice that such a χ always exists; any χ of height 0 will do.

If ψ is a constituent of $\chi|_N$ and $T = T_G(\psi)$ its inertia group, then $\chi = \sigma^G$ for some irreducible character σ of T. If S affords σ, then S^G affords χ. Since S^G is T-projective, this implies $D \leq_G T$, hence $T = G$. A fortiori, b is stable under conjugation by G, so (2) follows.

(3) is proved by induction on $|G\colon N|$, the case $G = N$ being trivial.

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If $G > N$, choose a normal subgroup $K \geq N$ of G such that $|K: N| = p$. Then $\chi|_K$ is irreducible by induction. Since $\chi|_N$ is homogeneous as we have already seen, $\chi|_N$ is irreducible by [10], 6.19 (p. 86).

COROLLARY. *Let B be a block of G with abelian defect group D and let N be a normal subgroup of G. Then there exists a block b of N covered by B such that any irreducible character of b extends to DN.*

Proof. Choose an indecomposable lattice M in B such that $vx(M) =_G D$. Since $H = DN \geqslant D$, there is an indecomposable RH-lattice U with $vx(U) =_H D$ and $M | U^G$. Let B_0 be the block of H containing U and let E be a defect group of B_0. Then $D \leq_H E$. If b is a block of N covered by B_0, then b is also covered by B. So both $E \cap N$ and $D \cap N$ are (up to conjugation in G) defect groups of b by [11], 4.2. Therefore $|E \cap N| = |D \cap N|$. Since $E \leqslant H = DN$, it follows that $D =_H E$, so D is a defect group of B_0. We can now apply (3) of the Proposition with (H, B_0) instead of (G, B) to get that any irreducible character χ of H belonging to B_0 is still irreducible when restricted to N. Notice that the extra condition on the lattices affording χ is automatically satisfied by Lemma 1, since B_0 has abelian defect group. Now every irreducible character ψ of N belonging to b is a component of $\chi|_N$ for some irreducible character χ of H with necessarily belongs to B_0 since that is the only block of H covering b by (1) of the Proposition. Therefore $\chi|_N = \psi$ and χ is an extension of ψ.

DEFINITION. A group $G \neq 1$ is called *quasi-simple* if $G = G'$ and every proper normal subgroup is contained in the center of G.

THEOREM. *Suppose that for all quasi-simple groups the characters in blocks with abelian defect groups are all of height* 0. *Then the same is true for all groups.*

Proof. Suppose not. Then there is an irreducible character χ of a group G satisfying

(i) χ belongs to a block B with abelian defect group D,

(ii) χ has positive height, i.e. $\nu\chi(1) > \nu[G: D]$, and

(iii) $(\chi(1), |G|)$ is minimal in the lexicographic order with these properties.

By a sequence of reductions, we show that this forces G to be quasi-simple, contradicting the hypothesis.

Step 1. χ is primitive.

Proof. If $\chi = \psi^G$ for an irreducible character ψ of a proper subgroup H of G, let b denote the block of H containing ψ. Then b^G is defined and in fact $b^G = B$ by [5], V, 1.2 (p. 193). Therefore a defect group Δ of b is, up to conjugation in G, contained in D by [5], III, 9.6 (p. 137). If U is an RH-lattice affording ψ then U is Δ-projective. Therefore U^G is Δ-projective. Since U^G affords χ, this implies $D \leqslant_G \Delta$ (remember that B is a l.v. block).

Therefore $D =_G \Delta$, in particular Δ is abelian. By induction, ψ has height 0, so $\nu\psi(1) = \nu[H: \Delta]$, hence $\nu\chi(1) = \nu\psi(1) + \nu[G: H] = \nu[G: \Delta]$, i.e. χ has height 0, a contradiction.

Step 2. χ is faithful.

Proof. Let $K = \ker \chi$ and let M be an RG-lattice affording χ. We may consider M as $R\bar{G}$-lattice, where $\bar{G} = G/K$. Let V/K be a vertex of M in $\bar{G}$. Then clearly M is V-projective as a G-module; again by using the fact that B is a l.v. block, it follows that $D \leqslant_G V$, so $DK \leqslant_G V$. On the other hand, DK/K contains a defect group $\bar{D}$ of the block $\bar{B}$ of $\bar{G}$ containing M by [5], V, 4.2 (p. 203). Therefore

$$DK/K \leqslant_{\bar{G}} V/K \leqslant_{\bar{G}} \bar{D} \leqslant DK/K\,,$$

so equality holds.

In particular, $\bar{D}$ is abelian. If $K \neq 1$, induction shows that χ is of height 0 as a character of $\bar{G}$, i.e.

$$\nu\chi(1) = \nu[\bar{G}: \bar{D}] = \nu[G: DK] \leqslant \nu[G: D]\,,$$

a contradiction.

Step 3. Let $N \lhd G$ such that $\chi|_N$ is irreducible. Then $N = G$.

Proof. If $\chi|_N$ is irreducible for some proper normal subgroup N, let b denote the block of N containing $\chi|_N$. By [11], 4.2, the abelian group $D \cap N$ is, up to conjugation in G, a defect group of b. By induction then

$$\nu\chi(1) = \nu[N: D \cap N] = \nu[DN: D] \leqslant \nu[G: D]\,,$$

a contradiction.

Step 4. $G = G'$.

Proof. If $G' < G$, there is a normal subgroup N of prime index in G. It follows from Step 1 and the elements of Clifford theory (see [10], 6.19,

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p. 86, for instance) that $\chi|_N$ is irreducible, contradicting Step 3.

Step 5. Let N be a proper non-central normal subgroup of G. There exists an irreducible character ψ of N and an $1 < e < \chi(1)$ such that

$$\chi|_N = e\psi .$$

Moreover, there is a finite central extension

$$1 \longrightarrow Z \longrightarrow G^* \longrightarrow G \longrightarrow 1$$

and irreducible characters ψ^* and ϕ^* of G^* such that

(1) $\psi^*\phi^*$ is irreducible, has Z in its kernel and equals χ when considered as a character of G;

(2) DN splits in the above sequence, i.e. there is a homomorphism $\alpha\colon DN \to G^*$ such that $Z \times \operatorname{im}\alpha$ is the inverse image of DN in G^*. Moreover,

(3) $N\alpha \lhd G^*$, ϕ^* has $N\alpha$ in its kernel and degree e, and $(\psi^*|_{N\alpha})\alpha = \psi$.

Proof. By Step 1, $\chi|_N$ is homogeneous, i.e. $\chi|_N = e\psi$. By Step 3, $e > 1$. If $e = \chi(1)$, then ψ is linear and so, by Step 2, $N \leqslant Z(G)$ against the assumption.

The remaining assertions are just the results of Schur's method (see [10], 11.2, 11.17 for instance) of lifting projective representations, the only extra information being that not only N but DN splits. This follows from the Corollary since B covers only one block of N, so ψ extends to DN.

Step 6. Keeping the notation of Step 5, let B_0 be the block of $G^*/N\alpha$ containing ϕ^*. Then a defect group of B_0 is isomorphic to $Z_p \times DN/N$, where Z_p is the Sylow p-subgroup of Z.

Proof. Let B^* the block of G^* containing χ (as a character of G^*), so $B \subset B^*$. If E^* is a defect group of B^*, it follows easily from [5], V, 4.3, 4.5 (pp. 203, 204) that E^*Z/Z is a defect group of B, so we may assume $D = E^*Z/Z$. Therefore E^*Z is the inverse image of D in G^*, i.e. $E^*Z = Z \times D\alpha$ (see Step 5 (2)). Since E^* contains the normal p-subgroup Z_p of G^*, this implies $E^* = Z_p \times D\alpha$. In particular, E^* is abelian, so B^* is a l. v. block by Lemma 1. Since $\psi^*\phi^* \in B^*$ by Step 5 (1) and $\psi^*|_{N\alpha}$ is irreducible by Step 5 (3), we may apply Lemma 4 to conclude that $E^*N\alpha/N\alpha$ is a defect group of B_0. The assertion follows.

Step 7. G is quasi-simple.

Proof. Suppose N is a proper non-central normal subgroup. Then Steps 5 and 6 apply. Since $Z_p \times DN/N$ is abelian and

$$e = \phi^*(1) < \chi(1) ,$$

we have by induction

$$\begin{aligned}\nu(e) &= \nu[G^*/N\alpha] - \nu[Z_p \times DN/N] \\ &= \nu[G: DN] .\end{aligned}$$

Again by induction, we have

$$\nu\psi(1) = \nu[N: D \cap N] = \nu[DN: D] ,$$

since ψ belongs to a block of N with abelian defect group isomorphic to $D \cap N$. Therefore $\nu\chi(1) = \nu(e) + \nu\psi(1) = \nu[G: D]$, a contradiction.

Hence there is no such N in G; together with Step 4, this is the assertion.

Remark. If $\mathscr{K}$ is a class of finite groups which is closed under taking subgroups (i.e. $H \leq G \in \mathscr{K} \Rightarrow H \in \mathscr{K}$), factor groups (i.e. $N \trianglelefteq G \in \mathscr{K} \Rightarrow G/N \in \mathscr{K}$) and central extensions (i.e. $Z \leq Z(G)$, $G/Z \in \mathscr{K} \Rightarrow G \in \mathscr{K}$) and every quasi-simple group in $\mathscr{K}$ satisfies the "if"-part of Brauer's height 0 conjecture, then so does every group in $\mathscr{K}$, as is clear from the proof above.

This applies for instance for the class of solvable groups (there are no quasi-simple solvable groups) and, more generally, for p-solvable groups, since if G is a quasi-simple p-solvable group, then $G/Z(G)$ is a p'-group. By Ito's Theorem [10], 6.15, every irreducible character of G has p'-degree, hence height 0. One obtains therefore an independent proof of Fong's result mentioned in the introduction.

References

[1] Brauer, R., Number theoretical investigations on groups of finite order, Proc. Internat. Symp. on Algebraical Number Theory, Tokyo and Nikko 1955, 55–62.

[2] ——, On some conjectures concerning finite simple groups, Studies in Math. Analysis and Related Topics, Stanford Univ. Press 1962, 56–61.

[3] ——, Representations of finite groups, Lectures on Modern Mathematics, Vol. 1, Wiley, New York 1963, 133–175.

[4] Cline, E., Stable Clifford Theory, J. Algebra, **22** (1972), 350–364.

[5] Feit, W., The representation theory of finite groups, North-Holland, Amsterdam 1982.

[6] Fong, P., Some properties of characters of finite solvable groups, Bull. Amer. Math. Soc., **66** (1960), 116–117.

[7] Fong, P. and Srinivasan, B., The blocks of finite general linear and unitary groups, Invent. math., **69** (1982), 109–153.
[8] Gluck, D. and Wolf, T. R., Defect groups and character heights in blocks of solvable groups, II. Preprint 1983.
[9] Gluck, D. and Wolf, T. R., Brauer's height conjecture for p-solvable groups, preprint 1983.
[10] Isaacs, I. M., Character theory of finite groups, Academic Press, New York 1976.
[11] Knörr, R., Blocks, vertices and normal subgroups, Math. Z., **148** (1976), 53–60.
[12] ——, On the vertices of irreducible modules, Ann. of Math., (2) **110** (1979), 487–499.
[13] Michler, G. and Olsson, J. B., Character correspondences in finite general linear, unitary and symmetric groups, preprint 1983.
[14] Reynolds, W. F., Blocks and normal subgroups of finite groups, Nagoya Math. J., **22** (1963), 15–32.
[15] Wolf, T. R., Defect groups and character heights in blocks of solvable groups, J. Algebra, **72** (1981), 183–209.

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