J. Austral. Math. Soc. (*Series A*) **26** (1978), 163–168

PSEUDOCOMPLEMENTED ALGEBRAS WITH BOOLEAN CONGRUENCE LATTICES

R. BEAZER

(Received 4 April 1977; revised 10 October 1977)
Communicated by H. Lausch

Abstract

Complemented congruences in the classes of pseudocomplemented semilattices, p-algebras and double p-algebras are described. The descriptions are applied to give intrinsic characterizations of those algebras in the aforementioned classes whose congruence lattice is a Boolean algebra.

Subject classification (*Amer. Math. Soc.* (*MOS*) *1970*): primary 06 A 20, 06 A 25; secondary 06 A 40.

1. Introduction

In this paper we use a technique of Janowitz (1977) to describe complemented congruences on pseudocomplemented semilattices, p-algebras and double p-algebras. The main theorem shows that a congruence relation θ on a pseudocomplemented semilattice L is complemented if and only if it can be described by $a \equiv b(\theta)$ if and only if $a \wedge c = b \wedge c$ for some semicentral element c; that is an element $c \in L$ such that the join $(x \wedge c) \vee (x \wedge c^*)$ exists and is x, for all $x \in L$. Consequently, we show that if L is a pseudocomplemented semilattice then the congruence lattice of L is a Boolean algebra if and only if L is a finite Boolean algebra. The proof of the main theorem can be adapted to show that complemented congruences in p-algebras and double p-algebras can also be described in the aforementioned manner provided that "semicentral" element is replaced by the usual lattice theoretic notion of central element. As an application, we give a new proof of the characterization of those double p-algebras whose congruence lattice is Boolean; a result first obtained by Beazer (1976).

2. Preliminaries

Let L be a lattice. An element $a \in L$ is called *distributive* if and only if $a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)$, for all $x, y \in L$; *dually distributive* if and only if a is a distributive element in the dual of L. An element $a \in L$ is called *standard* if and only if $x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y)$, for all $x, y \in L$. An element $a \in L$ is called *neutral* if and only if the sublattice of L generated by x, y and a is distributive for all

$x, y \in L$. The *centre*, $\operatorname{cen}(L)$, of a bounded lattice L is the set of all complemented, neutral elements of L and is, of course, a Boolean sublattice of L. For the various relationships and properties of these special elements we refer the reader to Grätzer (1976).

An algebra $\langle L; \wedge, {}^*, 0, 1\rangle$ is called a *pseudocomplemented semilattice* if and only if $\langle L; \wedge, 0, 1\rangle$ is a bounded semilattice such that for every $a \in L$ the element $a^* \in L$ is the pseudocomplement of a; that is $x \leqslant a^*$ if and only if $a \wedge x = 0$. An element c in a pseudocomplemented semilattice L is called *semicentral* if and only if the join $(x \wedge c) \vee (x \wedge c^*)$ exists and is x, for all $x \in L$. The set of all semicentral elements of L will be denoted by $C(L)$. If, in any pseudocomplemented semilattice L, we write $B(L) = \{x \in L;\ x = x^{**}\}$ then $\langle B(L); \psi, \wedge, {}^*, 0, 1\rangle$ is a Boolean algebra when $a \cup b$ is defined by $a \cup b = (a^* \wedge b^*)^*$ for any $a, b \in B(L)$. The set $D(L) = \{x \in L;\ x^* = 0\}$ is a filter in L called the *dense filter*. By a congruence relation on a pseudocomplemented semilattice L we mean a semilattice congruence on L preserving the operation *. The relation φ on L defined by $a \equiv b(\varphi)$ if and only if $a^* = b^*$ is a congruence on L and called the *Glivenko congruence*. If θ is a congruence relation on L we write $\operatorname{cok}\theta$ for $\{x \in L;\ x \equiv 1(\theta)\}$.

An algebra $\langle L; \wedge, \vee, {}^*, 0, 1\rangle$ is called a *p-algebra* if and only if $\langle L; \wedge, \vee, 0, 1\rangle$ is a bounded lattice and * is the pseudocomplementation operation on L. A congruence relation on a p-algebra is a lattice congruence preserving *. The Glivenko congruence on any p-algebra is a p-algebra congruence.

An algebra $\langle L; \wedge, \vee, {}^*, {}^+, 0, 1\rangle$ is called a *double p-algebra* if and only if $\langle L; \wedge, \vee, {}^*, 0, 1\rangle$ is a p-algebra and $\langle L; \wedge, \vee, {}^+, 0, 1\rangle$ is a dual p-algebra; that is $x \geqslant a^+$ if and only if $a \vee x = 1$. If L is a double p-algebra, $a \in L$ and $n < \omega$ then we define an element $a^{n(+*)} \in L$ inductively as follows:

$$a^{0(+*)} = a, \quad a^{(k+1)(+*)} = a^{k(+*)+*} \quad \text{for} \quad k \geqslant 0.$$

In the event that L is distributive, $a^{+*} \leqslant a$ and $\operatorname{cen}(L) = \{a \in L;\ a = a^{+*}\}$. A lattice filter of L is said to be *normal* if it is closed under the operation $^{+*}$. A congruence on a double p-algebra is a p-algebra congruence preserving $^+$. The relation Φ on L defined by $a \equiv b(\Phi)$ if and only if $a^* = b^*$ and $a^+ = b^+$ is a congruence on L called the *determination congruence*.

The standard results and rules of computation in pseudocomplemented semilattices and p-algebras may be found in Grätzer (1976), while those for (distributive) double p-algebras may be found in Beazer (1976) and Katriňák (1973).

Let L be a pseudocomplemented semilattice, or a p-algebra or a double p-algebra. We write $K(L)$ for the (algebra) congruences on L and, as usual, denote the least and greatest elements of $K(L)$ by ω and ι, respectively. If S is any non-empty subset of L then we write $\Theta(S)$ for the smallest congruence on L collapsing S. In the event that $S = \{a, b\}$ we write $\theta(a, b)$ for $\Theta(S)$. Throughout, we denote by θ_a the relation on L defined by $x \equiv y(\theta_a)$ if and only if $x \wedge a = y \wedge a$.

https://doi.org/10.1017/S1446788700011654 Published online by Cambridge University Press

3. Complemented congruences

THEOREM. *If L is a pseudocomplemented semilattice then $\theta \in K(L)$ is complemented if and only if $\theta = \theta_c$ for some $c \in C(L)$.*

PROOF. First, observe that if $a \in L$ then θ_a is a semilattice congruence. That θ_a preserves * is easily seen. Indeed, if $x \wedge a = y \wedge a$ then $y^* \wedge x \wedge a = 0$ and $x^* \wedge y \wedge a = 0$. From the first, $y^* \wedge a \leqslant x^*$ so that $y^* \wedge a \leqslant x^* \wedge a$. From the second, we get $x^* \wedge a \leqslant y^* \wedge a$ and it follows that θ_a is a congruence on L. If $c \in C(L)$ then θ_c is complemented with complement θ_{c^*} in $K(L)$. Indeed, since $c^* \equiv 1(\theta_{c^*})$, $c \equiv 0(\theta_{c^*})$ so that the sequence $0\theta_{c^*}c\theta_c 1$ ensures that $0 \equiv 1(\theta_c \vee \theta_{c^*})$ and therefore $\theta_c \vee \theta_{c^*} = \iota$. Moreover, if $x \equiv y(\theta_c \wedge \theta_{c^*})$ then $x \wedge c = y \wedge c$ and $x \wedge c^* = y \wedge c^*$ so that

$$x = (x \wedge c) \vee (x \wedge c^*) = (y \wedge c) \vee (y \wedge c^*) = y,$$

since $c \in C(L)$. Therefore, $\theta_c \wedge \theta_{c^*} = \omega$.

Now suppose that θ is complemented with complement θ' in $K(L)$. Then there exists a chain $0 = c_0 < c_1 < \ldots < c_{n-1} < c_n = 1$ such that $c_{i-1} \equiv c_i(\theta \vee \theta')$, $1 \leqslant i \leqslant n$. Obviously we can assume that n is the length of a shortest chain guaranteeing that $\theta \vee \theta' = \iota$. In addition, we can assume that each $c_i \in B(L)$, since θ and θ' both preserve **. We claim that $n \leqslant 2$. Assuming that $n \geqslant 3$, we have $0 = c_0 \Theta c_1 \Theta' c_2 \Theta c_3$ where $\Theta \in \{\theta, \theta'\}$. Let $[0, c_2]_{B(L)}$ denote the interval $\{x \in B(L); 0 \leqslant x \leqslant c_2\}$ in the Boolean algebra $\langle B(L); \cup, \wedge, {}^*, 0, 1\rangle$. Then $[0, c_2]_{B(L)}$ is a Boolean lattice under $\cup$ and $\wedge$. Let $\bar{c}_1 \in B(L)$ denote the complement of c_1 in $[0, c_2]_{B(L)}$ so that $c_1 \cup \bar{c}_1 = c_2$ and $c_1 \wedge \bar{c}_1 = 0$. Then $c_2 = (c_1^* \wedge \bar{c}_1^*)^*$ and so, since $c_1 \equiv 0(\Theta)$, it follows that $\bar{c}_1^{**} \equiv c_2(\Theta)$; that is $\bar{c}_1 \equiv c_2(\Theta)$. We also have $\bar{c}_1 \equiv 0(\Theta')$, since $c_1 \equiv c_2(\Theta')$ and $\bar{c}_1 < c_2$. Thus, $\bar{c}_1 \in B(L)$, $0 < \bar{c}_1 < c_2$ and $0\Theta'\bar{c}_1\Theta c_2$. But now the chain

$$0 = c_0 < \bar{c}_1 < c_3 \ldots \leqslant c_n = 1$$

with $0\Theta'\bar{c}_1\Theta_3$ also guarantees that $\theta \vee \theta' = \iota$ but has length $n-1$ contrary to the minimality of n. Thus, $n \leqslant 2$. If $n = 1$ then either $\theta = \theta_0$ or $\theta = \theta_1$ and we are done. If $n = 2$ then we have $0 = c_0 < c_1 < c_2 = 1$ with $0\Theta c_1 \Theta' c_2 = 1$. Without loss of generality we can, by the above, take $\Theta = \theta'$. Thus, there exists $c \in B(L)$ such that $0 < c < 1$ and $0\theta' c \theta 1$. It follows that $\theta = \theta_c$. Indeed, $\theta_c \leqslant \theta$, since $c \equiv 1(\theta)$, and if $x \equiv y(\theta)$ then $x \wedge c \equiv y \wedge c(\theta)$ and $x \wedge c \equiv y \wedge c(\theta')$, since $c \equiv 0(\theta')$. Therefore, $x \wedge c \equiv y \wedge c(\theta \wedge \theta')$ and so $x \wedge c = y \wedge c$; that is $x \equiv y(\theta_c)$. Similarly, since $0 < c^* < 1$ and $0\theta c^* \theta' 1$, we have $\theta' = \theta_{c^*}$. Finally, we show that $c \in C(L)$. Let u be any upper bound for $\{x \wedge c, x \wedge c^*\}$. Then $(u \wedge x) \wedge c = x \wedge c$ and $(u \wedge x) \wedge c^* = x \wedge c^*$ so that $u \wedge x \equiv x(\theta \wedge \theta')$ and therefore $u \wedge x = x$; that is $x \leqslant u$. Hence the least upper bound for $\{x \wedge c, x \wedge c^*\}$ exists and is x, for all $x \in L$.

COROLLARY 1. *If L is a pseudocomplemented semilattice then $K(L)$ is a Boolean algebra if and only if L is a finite Boolean algebra.*

Proof. If $K(L)$ is Boolean then the Glivenko congruence $\varphi = \theta_c$ for some $c \in C(L)$. Thus, $D(L) = \operatorname{Cok}\varphi = \operatorname{Cok}\theta_c = [c)$ so that $c^* = 0$ and, since $c \in C(L)$ implies $c \vee c^* = 1$, it follows that $c = 1$. Consequently, $\varphi = \theta_1 = \omega$ and therefore, since $x^* = x^{***}$ for any $x \in L$, $x = x^{**}$; that is $L = B(L)$ and $K(B(L))$ is Boolean. It follows, by a well-known result, that $L = B(L)$ is a finite Boolean algebra. The converse is obvious.

Corollary 2. *If L is a p-algebra then $\theta \in K(L)$ is complemented if and only if $\theta = \theta_c$ for some $c \in \operatorname{Cen}(L)$.*

Proof. If $\theta \in K(L)$ is complemented then, as in the proof of the theorem, we can assert the existence of an element $c \in C(L)$ such that $0 < c < 1$, $0\theta' c\theta 1$ and $\theta = \theta_c$. It follows, since $c \in C(L)$, that c has complement c^* in L. Moreover, since $\theta = \theta_c$, it follows that θ_c is a lattice congruence and therefore (see Grätzer, 1976) c is dually distributive. To show that c is central, it remains only to show that c is a standard element (see Grätzer, 1976). To effect this, observe that if $x, y \in L$ then $x \wedge (c \vee y) \equiv (x \wedge c) \vee (x \wedge y)\,(\theta)$, since $c \equiv 1(\theta)$, and $x \wedge (c \vee y) \equiv (x \wedge c) \vee (x \wedge y)\,(\theta')$, since $c \equiv 0(\theta')$. Therefore, $x \wedge (c \vee y) \equiv (x \wedge c) \vee (x \wedge y)\,(\theta \wedge \theta')$; that is,

$$x \wedge (c \vee y) = (x \wedge c) \vee (x \wedge y)$$

and so c is standard.

For the sufficiency, we show that if $c \in \operatorname{Cen}(L)$ has complement c' in $\operatorname{Cen}(L)$ and $\theta \in K(L)$ is of the form θ_c then θ has complement $\theta_{c'}$ in $K(L)$. Indeed, $\theta_{c'}$ is a p-algebra congruence of L because it is a pseudocomplemented semilattice congruence which preserves joins, since c' is dually distributive. Moreover, $\theta_c \vee \theta_{c'} = \iota$ is guaranteed by the sequence $0\theta_{c'} c\theta_c 1$ and $\theta_c \wedge \theta_{c'} = \omega$ is guaranteed by the neutrality of c.

Corollary 3. *If L is a p-algebra then $K(L)$ is a Boolean algebra if and only if L is a finite Boolean algebra.*

Proof. As in the proof of Corollary 1, the Glivenko congruence $\varphi = \theta_c$ for some $c \in \operatorname{Cen}(L)$ satisfying $c^* = 0$. However, if c' is the complement of c in $\operatorname{Cen}(L)$ then $c' \leqslant c^*$, since $x = c^*$ is the largest solution of the equation $c \wedge x = 0$. It follows that $c' = 0$ and therefore $\varphi = \theta_1 = \omega$. Hence L is a finite Boolean algebra.

Remark. Corollary 3 is a special case of a theorem of Janowitz (1975) concerning annihilator preserving congruences on bounded 0-distributive lattices.

Corollary 4. *If L is a double p-algebra then $\theta \in K(L)$ is complemented if and only if $\theta = \theta_c$ for some $c \in \operatorname{Cen}(L)$.*

PROOF. The necessity follows exactly as in the proof of Corollary 2. For the sufficiency, we show that if $c \in \mathrm{Cen}(L)$ has complement c' in $\mathrm{Cen}(L)$ and $\theta \in K(L)$ is of the form θ_c then θ has complement $\theta_{c'}$ in $K(L)$. Clearly we need only show that $\theta_{c'}$ is a double p-algebra congruence on L. Indeed, since $x \wedge c = y \wedge c$ if and only if $x \vee c' = y \vee c'$, it follows by Corollary 2 and its dual that $\theta_{c'}$ is a double p-algebra congruence on L.

Beazer (1976) gave an intrinsic characterization of those distributive double p-algebras whose congruence lattice is Boolean. Close scrutiny of the proof of that theorem together with the fact that $\Phi = \omega$ implies distributivity (see Katriňák, 1973) shows the assumption of distributivity may be dropped. We give an alternative proof of this result using Corollary 4.

COROLLARY 5. *If L is a double p-algebra then $K(L)$ is a Boolean algebra if and only if the following conditions hold*:

(1) $\Phi = \omega$.

(2) *For all $a \in L$, there exists $n < \omega$ such that $a^{(n+1)(+*)} = a^{n(+*)}$.*

(3) $\mathrm{Cen}(L)$ *is finite.*

PROOF. If $K(L)$ is Boolean then, by Corollary 4, the determination congruence $\Phi = \theta_c$ for some $c \in \mathrm{Cen}(L)$. Therefore, $\{1\} = \mathrm{Cok}\,\Phi = \mathrm{Cok}\,\theta_c = [c)$ and so $c = 1$ which implies that $\Phi = \omega$. Consequently, L is distributive. Next, if $a \in L$ then the normal filter F_a generated by a in L is given by $F_a = \{x \in L;\ x \geqslant a^{n(+*)}$ for some $n < \omega\}$. Moreover, $F_a = \mathrm{Cok}\,\Theta(F_a)$ by Beazer (1976). It follows, since $K(L)$ is Boolean, that $\Theta(F_a) = \theta_c$ for some $c \in \mathrm{Cen}(L)$ and, therefore, $F_a = \mathrm{Cok}\,\theta_c = [c)$. Hence, $a \geqslant c \geqslant a^{n(+*)}$, for some $n < \omega$, which implies that $a^{n(+*)} \geqslant c^{n(+*)} = c \geqslant a^{n(+*)}$; that is $c = a^{n(+*)}$ and, therefore, $a^{(n+1)(+*)} = a^{n(+*)}$. For the necessity of condition (3), suppose that $\mathrm{Cen}(L)$ is not finite. Then there exists a non-principal filter F of $\mathrm{Cen}(L)$. Let

$$\theta_F = \bigcup \{\theta_a;\ a \in F\}.$$

It follows, since $\{\theta_a;\ a \in F\}$ is a directed subset of $K(L)$, that $\theta_F \in K(L)$ and so $\theta_F = \theta_c$ for some $c \in \mathrm{Cen}(L)$. Hence, $\mathrm{Cok}\,\theta_F = \mathrm{Cok}\,\theta_c$ and so $x \geqslant c$ if and only if $x \geqslant a$, for some $a \in F$, which implies that F is the principal filter of $\mathrm{Cen}(L)$ generated by c; contrary to hypothesis.

Now suppose that conditions (1), (2) and (3) all hold. It follows from (1) and Beazer (1976) that every congruence of L is of the form $\Theta(F)$ for some normal filter F of L. Clearly $\Theta(F) = \bigvee \{\theta(a, 1);\ a \in F\}$. However, condition (2) implies that for any $a \in L$ there exists a least integer n_a such that $a^{(n_a+1)(+*)} = a^{n_a(+*)}$. It follows, since $a^{n_a(+*)} \leqslant a$, that $\theta(a, 1) = \theta(a^{n_a(+*)}, 1)$, for any $a \in L$. Therefore, $\theta(a, 1) = \theta_{c_a}$ for some $c_a \in \mathrm{Cen}(L)$; namely $c_a = a^{n_a(+*)}$. Now condition (3) implies that $\Theta(F)$ is a finite join of congruences of the form θ_{c_a}, where $c_a \in \mathrm{Cen}(L)$.

Therefore, since the formula $\theta_{c_1} \vee \theta_{c_2} = \theta_{c_1 \wedge c_2}$ holds for any $c_1, c_2 \in \mathrm{Cen}(L)$ and $\mathrm{Cen}(L)$ is closed under finite meets, $\Theta(F) = \theta_c$ for some $c \in \mathrm{Cen}(L)$. It follows, from Corollary 4, that $K(L)$ is a Boolean algebra.

REMARK. Beazer (1976) obtained as a corollary to Theorem 4 of that paper a characterization of the simple algebras in the class of distributive double p-algebras. Specifically, it was shown that a distributive double p-algebra L is simple if and only if $\Phi = \omega$ and for all $a \in L \setminus \{1\}$, there exists an integer n such that $a^{n(+*)} = 0$. We finish with the remark that the very same characterization holds if distributivity is dropped.

References

R. Beazer (1976), "The determination congruence on double p-algebras", *Algebra Universalis* **6**, 121–129.

G. Grätzer (1976), *General Lattice Theory* (Birkhäuser Verlag, Basel).

M. F. Janowitz (1975), "Annihilator preserving congruence relations of lattices", *Algebra Universalis* **5**, 391–394.

M. F. Janowitz (1978), "Complemented congruences on complemented lattices" (to appear).

T. Katriňák (1973), "The structure of distributive double p-algebras. Regularity and congruences", *Algebra Universalis* **3**, 238–246.

The University of Glasgow and The University of Manitoba
Glasgow G12 8QW Canada R3T 2N2