A Generalised Hypergeometric Function

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1. *Introduction.*

The hypergeometric function[1] $F(a, b; c; z)$ is analytic in the domain $|\arg(-z)| < \pi$, and, when $|z| < 1$, may be represented by the series

$$\frac{\Gamma(c)}{\Gamma(a)\,\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\,\Gamma(b+n)}{\Gamma(c+n)\,.\,n!} z^n.$$

When $|z| = 1$ in the domain $|\arg(-z)| < \pi$, this series converges[2] to $F(a, b; c; z)$ if $R(a+b-c) < 0$ (integral values of a, b and c are excluded in the present paper).

This function belongs to a more general class of functions which may be represented, under certain conditions, by the series

$$\frac{\Gamma(c)\,\Gamma(\lambda+1)}{\Gamma(a)\,\Gamma(b)} \sum_{n=-\infty}^{\infty} \frac{\Gamma(a+n)\,\Gamma(b+n)}{\Gamma(c+n)\,\Gamma(\lambda+n+1)} z^{\lambda+n}.$$

For a discussion of this class of functions, fractional integrals will be employed.

2. *Fractional integrals.*

A λ-th integral of $F(a, b; c; z)$ along a simple curve l from 0 to z is defined[3] by

$$D^{-\lambda}(l_0)\,F(a, b; c; z) = \frac{1}{\Gamma(\lambda+\gamma)} \left(\frac{d}{dz}\right)^{\gamma} \int_0^z (z-t)^{\lambda+\gamma-1} F(a, b; c; t)\,dt,$$

where γ is the least non-negative integer such that $R(\lambda)+\gamma > 0$; the integration and differentiation being along l.

Theorem 1. *If l lies in $|z| < 1$, then*

$$D^{-\lambda}(l_0)\,F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\,\Gamma(b+n)}{\Gamma(c+n)\,\Gamma(\lambda+n+1)} z^{\lambda+n}.$$

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[1] Whittaker and Watson, *Modern Analysis* (1927), Ch. XIV.

[2] *Ibid.*, pp. 25 and 57.

[3] Fabian, *Quart. J. of Math.*, 7 (1936), 252. *Cf.* the Riemann-Liouville integral.

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This equality continues to hold when $|z| = 1$ *in the domain* $|\arg(-z)| < \pi$, *provided that* $R(a+b-c-\lambda) < 0$.

Proof. The first part of the theorem follows immediately by applying the operator $D^{-\lambda}$ to each term of the series

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\,\Gamma(b+n)}{\Gamma(c+n)\,.\,n!} z^n.$$

To prove the second part of the theorem, we note first that each branch of $D^{-\lambda}(l_0)\,F(a, b; c; z)$ is analytic in and on the circle $|z| = 1$ in the domain $|\arg(-z)| < \pi$, $z \neq 0$, since $F(a, b; c; z)$ is analytic in this region[1]. The required conclusion will then follow if we prove that the series stated in the theorem converges when $|z| = 1$, $|\arg(-z)| < \pi$, and $R(a+b-c-\lambda) < 0$.

To prove this, denote the n-th term of this series by u_n. Then we have, when $|z| = 1$,

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(a+n-1)(b+n-1)}{(c+n-1)(\lambda+n)}\right|$$

$$= \left|1 + \frac{a+b-c-\lambda-1}{n} + O\left(\frac{1}{n^2}\right)\right|$$

$$= \left|1 + \frac{R(a+b-c-\lambda)-1}{n} + O\left(\frac{1}{n^2}\right)\right|.$$

Hence, by a known theorem[2], this series converges absolutely when $|z| = 1$, if $R(a+b-c-\lambda) < 0$.

This completes the proof.

3. *The more general class of functions.*

THEOREM 2. *For non-integral values of* a, b, c *and* λ, *there exists a function* $S(a, b; c, \lambda; z)$ *which consists of branches analytic in the finite part of the domain* $|\arg(-z)| < \pi$, $z \neq 0$; *and which, when* $|z| = 1$ *in this domain and* $R(a+b-c-\lambda) < 0$, *may be represented by the series*

$$\frac{\Gamma(c)\,\Gamma(\lambda+1)}{\Gamma(a)\,\Gamma(b)} \sum_{n=-\infty}^{\infty} \frac{\Gamma(a+n)\,\Gamma(b+n)}{\Gamma(c+n)\,\Gamma(\lambda+n+1)} z^{\lambda+n}.$$

[1] Fabian, *Math. Gazette*, 20 (1936), 249.

[2] Whittaker and Watson, *op. cit.*, p. 23.

Proof. This series may be written

$$\frac{\Gamma(c)\,\Gamma(\lambda+1)}{\Gamma(a)\,\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\,\Gamma(b+n)}{\Gamma(c+n)\,\Gamma(\lambda+n+1)}z^{\lambda+n}$$

$$+\sum_{n=1}^{\infty}\frac{(c-1)(c-2)\ldots(c-n)\,\lambda(\lambda-1)\ldots(\lambda-n+1)}{(a-1)(a-2)\ldots(a-n)(b-1)(b-2)\ \ .\,(b-n)}z^{\lambda-n},$$

that is

$$\frac{\Gamma(c)\,\Gamma(\lambda+1)}{\Gamma(a)\,\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\,\Gamma(b+n)}{\Gamma(c+n)\,\Gamma(\lambda+n+1)}z^{\lambda+n}$$

$$+z^{\lambda-a}\sum_{n=1}^{\infty}\frac{(-c+1)(-c+2)\ldots(-c+n)(-\lambda)(-\lambda+1)\ldots(-\lambda+n-1)}{(-a+1)(-a+2)\ldots(-a+n)(-b+1)(-b+2)\ldots(-b+n)}\left(\frac{1}{z}\right)^{n-a}.$$

By Theorem 1, this represents the function

$$\Gamma(\lambda+1)\,D^{-\lambda}(l_0)\,F(a,\,b\,;\;c\,;\;z)$$

$$+\Gamma(1-a)\,.\,z^{\lambda-a}\,D^{a}(L_0)\,F(1-c,\;-\lambda\,;\;1-b\,;\;w)-z^{\lambda}$$

when $|z|=1$, $|\arg(-z)|<\pi$, and $R(a+b-c-\lambda)<0$; w being $1/z$, and L the path of integration in the w-plane.

If we denote this function by $S(a,\,b\,;\;c,\,\lambda\,;\;z)$, each branch of $S(a,\,b\,;\;c,\;\lambda\,;\;z)$ is analytic in the finite part of the domain $|\arg(-z)|<\pi$, $z\neq 0$, by a previous theorem[1].

Hence the conclusion.

[1] Fabian, *Math. Gazette*, 20 (1936), 249.

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