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# BLOW UP SEQUENCES AND THE MODULE OF *n*th ORDER DIFFERENTIALS

WILLIAM C. BROWN

**Introduction.** Let $C$ denote an irreducible, algebraic curve defined over an algebraically closed field $k$. Let $P$ be a singular point of $C$. We shall employ the following notation throughout the rest of this paper: $R$ will denote the local ring at $P$, $K$ the quotient field of $R$, $\bar{R}$ the integral closure of $R$ in $K$, $A$ the completion of $R$ with respect to its radical topology, and $\bar{A}$ the integral closure of $A$ in its total quotient ring.

We wish to study the relationships between the $\bar{A}$-module $D^n(\bar{A}/A)$ of $n$th order differentials over $A$ and the multiplicities of the blow up sequence $B$: $R = R_0 < R_1 < \ldots < \bar{R}$ of $R$.

The *module* $D^n(\bar{A}/A)$ *of* $n$th *order differentials* is defined as follows: Let $\sigma : \bar{A} \otimes_A \bar{A} \to \bar{A}$ be the multiplication mapping given by $\sigma(\sum a_i \otimes b_i) = \sum a_i b_i$. Let $I(\bar{A}/A)$ denote the kernel of $\sigma$. Then

$$D^n(\bar{A}/A) = I(\bar{A}/A)/I^{n+1}(\bar{A}/A).$$

The module $D^n(\bar{A}/A)$ is the universal object for $n$th order $A$-derivations and satisfies many functorial properties. We refer the reader to [**5**] or [**7**] for all pertinent properties of $D^n(\bar{A}/A)$ used in this paper.

The *blow up sequence* $B$: $R = R_0 < R_1 < \ldots < \bar{R}$ is defined as in [**4**, p. 669]. Each $R_{i+1}$ is obtained from $R_i$ by blowing up the Jacobson radical of $R_i$. By the multiplicity $\mu(R_i)$ of $R_i$, we shall mean the multiplicity of the Jacobson radical of $R_i$. By the multiplicities of $B$, we shall mean the sequence $\{\mu(R_i)\}$. We similarly define the blow up sequence $B$: $A = A_0 < A_1 < \ldots < \bar{A}$ and multiplicities $\mu(A_i)$. It is easy to show (see Proposition 1) that for each $i$, $A_i = R_i \otimes_R A$, and $\mu(R_i) = \mu(A_i)$. Thus, the multiplicities of $B$ are given by $\hat{B}$.

We note that since $\bar{R}$ is a finitely generated $R$-module, the sequence of $R$-modules in $B$ stabilizes at some point, i.e. $R_n = R_{n+1} = \ldots$ for some $n \gg 1$. Thus, there are only a finite number of different $\mu(R_i)$ for $B$. The problem is to characterize the $\mu(R_i)$ in terms of some suitably defined invariants of $D^n(\bar{A}/A)$ for $n \gg 1$.

Since $\bar{A} = \bar{R} \otimes_R A$, we see that $\bar{A}$ is a finitely generated $A$-module. It then follows from [**3**, Lemma 1.1] that $I(\bar{A}/A)$ is a finitely generated left $\bar{A}$-module. Since $\bar{R}$ is a finitely generated $R$-module, $\bar{R}$ is a semilocal ring. Let $\{m_1, \ldots, m_t\}$ be the maximal ideals of $\bar{R}$. Set $V_i = \bar{R}_{m_i}$ ($\bar{R}$ localized at $m_i$) for $i = 1, \ldots, t$.

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Then each $V_i$ is a discrete rank one valuation ring dominating $R$, and $\bar{A} = \hat{V}_1 \oplus \ldots \oplus \hat{V}_t$. Here $\hat{V}_i$ of course denotes the completion of $V_i$. Thus, $\bar{A}$ is always a finite direct sum of principal ideal domains.

Now suppose $C$ is unibranched at $P$, i.e. $\bar{R}$ is a local ring. Then $t = 1$ in the above discussion, and $\bar{A} = \hat{V}_1$ is a principal ideal domain. In this case, $I(\bar{A}/A)$ (being a finitely generated module over $\bar{A}$) has a set of invariant factors $\delta_1, \delta_2, \ldots, \delta_r$ associated with it. These $\delta_i$ are elements of $\bar{A}$ given by $\delta_1 = \Delta_1$, $\delta_2 = \Delta_2\Delta_1^{-1}$, etc. Here $\Delta_i$ is the greatest common divisor of all $i \times i$ subdeterminates of the relations matrix of $I(\bar{A}/A)$. These $\delta_i$ are unique up to units in $\bar{A}$. We next note that since $k$ is algebraically closed, $\bar{A} = k[[\beta]]$ for some element $\beta$ analytically independent over $k$. Thus, each $\delta_i$ can be written in the following form $\delta_i = \beta^{e_i}$, for some integer $e_i$.

Now consider the blow up sequence $\hat{B}$. We can write $\hat{B}$ as $\hat{B}$: $A = A_0 < A_1 < \cdots < A_N < A_{N+1} = A_{N+2} = \ldots = \bar{A}$. Since $\bar{A}$ is a local ring, each $A_i$ is local. It was shown in [**3**, Lemmas 3.4 and 4.2]† that the decomposition of the module $I(\bar{A}/A)$ over the P.I.D. $\bar{A}$ uniquely determines the multiplicities $\mu(A_i)$ of $\hat{B}$. For, if the decomposition of $I(\bar{A}/A)$ is known, then we can compute the nontrivial invariant factors $\beta^{e_1}, \ldots, \beta^{e_r}$ of $I(\bar{A}/A)$. Then $r = \mu(A) - 1$, and it follows from [**3**, Lemma 3.4] that $\beta^{e_1-\mu(A)}, \ldots, \beta^{e_r-\mu(A)}$ is a set of invariant factors of $I(\bar{A}/A_1)$. Thus, the decomposition of $I(\bar{A}/A)$ determines the decomposition of $I(\bar{A}/A_1)$. From [**3**, Lemma 4.2], $\mu(A_1) = \dim_k\{I(\bar{A}/A_1)/\beta I(\bar{A}/A_1)\}$. So, $I(\bar{A}/A)$ determines $\mu(A_1)$. If we now eliminate the 1's appearing in $\beta^{e_1-\mu(A)}, \ldots, \beta^{e_r-\mu(A)}$, we obtain the nontrivial invariant factors of $I(\bar{A}/A_1)$. There are exactly $\mu(A_1) - 1$ of them, and we may repeat the above process to compute $\mu(A_2)$. Continuing in this fashion, we see that the decomposition of $I(\bar{A}/A)$ determines the multiplicities $\mu(A_i)$ of $\hat{B}$. Conversely, if the multiplicities $\mu(A_i)$ of $\hat{B}$ are known, then it follows from [**3**, Theorem 3.5] that

$$(1)\quad \underbrace{\beta^{\mu(A_0)}, \ldots, \beta^{\mu(A_0)}}_{\mu(A_0)-\mu(A_1)}, \underbrace{\beta^{\mu(A_0)+\mu(A_1)}, \ldots, \beta^{\mu(A_0)+\mu(A_1)}}_{\mu(A_1)-\mu(A_2)}, \ldots, \underbrace{\beta^{\mu(A_0)+\cdots+\mu(A_N)}, \ldots, \beta^{\mu(A_0)+\cdots+\mu(A_N)}}_{\mu(A_N)-1}$$

is a set of invariant factors of $I(\bar{A}/A)$. Thus, the decomposition of the module $I(\bar{A}/A)$ uniquely determines the multiplicities of the blow up sequence $\hat{B}$, and, therefore, the multiplicities of $B$ as well. It was also shown in [**3**, Theorem 1.1] that for $n$ sufficiently large, $D^n(\bar{A}/A) = I(\bar{A}/A)$. Thus, if $C$ is unibranched at $P$, the decomposition of the module $D^n(\bar{A}/A)$ $(n \gg 1)$ uniquely determines the multiplicities $\mu(R_i)$ of the blow up sequence $B$ at $P$.

†The proofs of the main results in [**3**] are not quite complete if $k$ has characteristic $p \neq 0$. However, slight modifications of the techniques in [**3**] will give complete proofs in the characteristic $p$ case.

The purpose of this paper is to study how much of this theory remains intact if we remove the assumption that $C$ is unibranched at $P$. Surprisingly, most of the theory survives. We shall show that for $n$ sufficiently large, $D^n(\bar{A}/A) = \bigoplus_{i=1}^t I(\hat{V}_i/A)$, and each $I(\hat{V}_i/A)$ is nilpotent. We shall examine two cases at this point. Either $\bar{R}$ is unramified over $R$ or $\bar{R}$ is ramified over $R$.

We shall show that $\bar{R}$ is unramified over $R$ if and only if $D^n(\bar{A}/A) = 0$ for $n$ sufficiently large. In this case, the multiplicity sequence for $B$ is particularly simple. We have $\mu(R_i) = \mu(\bar{R}) = t$ for all $i$. In other words, the number of branches of $C$ centered at $P$ gives the multiplicities of the blow up sequence $B$ when $C$ is unramified at $P$.

If $\bar{R}$ is ramified over $R$, then $D^n(\bar{A}/A) \neq 0$ for any $n$. In this case, $B$ is considerably more complicated. For example, the unibranched case considered in [**3**] is a subcase of this case.

In general, we shall be able to attach a set of invariant factors to $D^n(\bar{A}/A)$ which in either case (ramified or unramified) uniquely determine the multiplicities in the blow up sequence $B$. The general theory developed in this paper will include and actually come from the unibranch theory discussed in [**3**].

**1. Some preliminary results.** We use the same notation as in the introduction. Thus, $R$ denotes the local ring at a singular point $P$ of some irreducible algebraic curve $C$ defined over an algebraically closed field $k$. For the time being, we make no assumptions about the nature of the singularity at $P$. We shall let $m$ denote the maximal ideal of $R$. All topological statements about $R$ and related rings will be made relative to the $m$-adic topology on $R$.

Now let $\bar{R}$ denote the integral closure of $R$ in its quotient field $K$, and let $B$: $R < R_1 < R_2 < \ldots < \bar{R}$ be the blow up sequence of $R$. Each $R_{i+1}$ is obtained from $R_i$ by blowing up the Jacobson radical $J_i$ of $R_i$. Since $k$ is infinite, any open ideal in $R_i$ has a transversal element. In particular, $J_i$ has a transversal element say $x(i)$. Then $R_{i+1} = R_i[x_1/x(i), \ldots, x_r/x(i)]$ where $\{x_1, \ldots, x_r\}$ are elements in $R_i$ which generate $J_i$. Thus, each $R_i$ in $B$ is a semilocal ring which is finitely generated as an $R$-module. We note that since $\bar{R}$ is a Noetherian $R$-module, there exists an integer $n$ such that $R_n = R_{n+1} = \ldots$. Now $R_n = \bar{R}$. For, $R_{n+1}$ is the blow up of $R_n$ along its Jacobson radical $J_n$. Thus, $R_{n+1} = R_n$ implies that $J_n$ is principal. But this immediately implies that every localization of $R_n$ (at maximal ideals) is a regular local ring. Thus, $R_n$ is normal and hence $R_n = \bar{R}$. Therefore, $B$ always has the form

(2) $B$: $R = R_0 < R_1 < \ldots < R_n = \bar{R} = \bar{R} = \bar{R} \ldots$

for some $n \gg 1$.

If $A$ is the completion of $R$ and $\bar{A}$ the integral closure of $A$ in its total quotient ring, then similar remarks can be made about the blow up sequence $\hat{B}$: $A < A_1 < \ldots < \bar{A}$. For a detailed discussion of blow up sequences, we refer the reader to [**4**].

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In the introduction, we mentioned that the multiplicities of $B$ and $\hat{B}$ are the same. This is part of the following proposition:

PROPOSITION 1. *Let $B$ and $\hat{B}$ denote the blow up sequences of $R$ and $A$ respectively. Let $J_i$ denote the Jacobson radical of $R_i$. Then*

(a) *$J_iA_i$ is the Jacobson radical of $A_i$.*

(b) $A_i = R_i \otimes_R A \quad i = 0, 1, \ldots$

(c) $\mu(A_i) = \mu(R_i) \quad i = 0, 1, \ldots$

*Proof*. This proposition follows from the proof of Proposition 2.8 in [**4**]. We proceed via induction on $i$. If $i = 0$, then clearly (a), (b) and (c) hold for $R_0 = R$ and $A_0 = A$, the completion of $R$. Thus, assume the proposition is proven for $i$ and consider $A_{i+1}$. Since $A_i = R_i \otimes_R A$, and $A$ is flat over $R$, we have $A_i$ is flat over $R_i$. Denoting blow-ups with superscripts and using [**4**, Corollary 1.2], we have

(3) $\quad A_{i+1} = A_i^{J_iA_i} = R_{i+1} \otimes_{R_i} A_i.$

But, $R_{i+1} \otimes_{R_i} A_i = R_{i+1} \otimes_{R_i} (R_i \otimes_R A) = R_{i+1} \otimes_R A$. Thus, we have established (b) in the $i + 1$ case. As for (a), we first note that $A_{i+1}$ is integral over $A$ since $A_{i+1} \subset \bar{A}$. Thus, every maximal ideal of $A_{i+1}$ contracts to $mA$ in $A$ and consequently to $m$ in $R$. Since $R_{i+1}$ is integral over $R$, we see every maximal ideal in $A_{i+1}$ contracts to a maximal idea in $R_{i+1}$. Therefore, if $J(A_{i+1})$ denotes the Jacobson radical of $A_{i+1}$, we have $J_{i+1}A_{i+1} \subset J(A_{i+1})$. But

(4) $\quad A_{i+1}/J_{i+1}A_{i+1} \cong (A/mA) \otimes_{R/m} (R_{i+1}/J_{i+1}) \cong k \otimes \ldots \otimes k.$

Thus, $J_{i+1}A_{i+1} = J(A_{i+1})$ and the proof of (a) is complete. Since each $A_i$ is just the completion of $R_i$ with respect to its radical topology, (c) follows directly from [**9**, Lemma 1, p. 285].

Thus, to compute the multiplicities in the blow-up sequence $B$, we may use the sequence $\hat{B}$.

We now set up the notation for the main theorem of this section. As in the introduction, let $\{m_1, \ldots, m_t\}$ be the maximal ideals of $\bar{R}$. Set $V_i = \bar{R}_{m_i}$, $i = 1, \ldots, t$. Then each $V_i$ is a discrete rank one valuation ring which dominates $R$. We shall let $\hat{V}_i$ denote the completion of $V_i$ with respect to its maximal ideal $m_iV_i$. Since $k$ is algebraically closed, we have the integral closure $\bar{A}$ of $A$ in its total quotient ring is just the completion of $\bar{R}$ [**9**, Theorem 33, p. 320]. Thus, $\bar{A} = \hat{V}_1 \oplus \cdots \oplus \hat{V}_t$. Let $\pi_i$ denote the natural projection of $\bar{A}$ onto $\hat{V}_i$. Set $p_i = \ker \pi_i \cap A$ for $i = 1, \ldots, t$. Then $p_1, \ldots, p_t$ are exactly the minimal primes of $A$, and we have $(0) = p_1 \cap \cdots \cap p_t$. Thus, the image of $A$ in $\hat{V}_i$ is just $A/p_i$. When we write $\hat{V}_i \otimes_A \hat{V}_i$, $I(\hat{V}_i/A)$ etc., we shall mean $\hat{V}_i \otimes_{A/p_i} \hat{V}_i$, $I(\hat{V}_i/A/p_i)$, etc.

As in the introduction, $I(\hat{V}_i/A)$ will denote the kernel of the multiplication mapping $\sigma_i$: $\hat{V}_i \otimes_A \hat{V}_i \rightarrow \hat{V}_i$. Since $\bar{A}$ is a finitely generated $A$-module, each $\hat{V}_i$ is a finitely generated $A/p_i$-module. Consequently, $I(\hat{V}_i/A)$ is a finitely generated left $\hat{V}_i$-module as well as a finitely generated left $\hat{V}_i$-algebra.

We can now prove the main result of this section.

THEOREM 1. *Let $A$ be the complete local ring at a singular point $P$ of an irreducible algebraic curve $C$ defined over an algebraically closed field $k$. Let $\bar{A} = \hat{V}_1 \oplus \ldots \oplus \hat{V}_t$ be the integral closure of $A$ where $\{V_1, \ldots, V_t\}$ are the discrete rank one valuation rings in $K$ which dominate the local ring $R$ at $P$. Then for all $n$ sufficiently large, $D^n(\bar{A}/A) = I(\hat{V}_1/A) \oplus \ldots \oplus I(\hat{V}_t/A)$ and each $I(\hat{V}_i/A)$ is nilpotent.*

*Proof.* We first note that for any natural number $n$, $D^n(\bar{A}/A) = D^n(\hat{V}_1/A) \oplus \ldots \oplus D^n(\hat{V}_t/A)$. For, let $\sigma$: $\bar{A} \otimes_A \bar{A} \rightarrow \bar{A}$ be the multiplication map. Since the $\hat{V}_i$ are pairwise orthogonal in $\bar{A}$, we have $\bar{A} \otimes_A \bar{A} = \bigoplus_{i,j=1}^{t}(\hat{V}_i \otimes_A \hat{V}_j)$. Thus, $I(\bar{A}/A)$, which is the kernel of $\sigma$, is given by

$$(5) \quad I(\bar{A}/A) = I(\hat{V}_1/A) \oplus \ldots \oplus I(\hat{V}_t/A) \oplus \{\oplus_{i \neq j}(\hat{V}_i \otimes_A \hat{V}_j)\}$$

Thus,

$$D^n(\bar{A}/A) = I(\bar{A}/A)/I^{n+1}(\bar{A}/A) = I(\hat{V}_1/A)/I^{n+1}(\hat{V}_1/A) \oplus \ldots \oplus I(\hat{V}_t/A)/I^{n+1}(\hat{V}_t/A) = D^n(\hat{V}_1/A) \oplus \ldots \oplus D^n(\hat{V}_t/A).$$

Thus, to prove the theorem, it suffices to show that each $I(\hat{V}_i/A)$ is nilpotent.

Let $\bar{R}$ denote the integral closure of $R$ in $K$. Since $\bar{R}$ is a Dedekind domain with finitely many maximal ideals, $\bar{R}$ is a principal ideal domain. Thus, the Jacobson radical $J = m_1 \ldots m_t$ of $\bar{R}$ is principal. Let $\beta \in \bar{R}$ such that $\beta\bar{R} = J$. Then $\beta$ generates the maximal ideal ideal $m_i V_i$ in each valuation ring $V_i$. Hence, $\beta$ is a common uniformizing parameter for the $V_i$ $i = 1, \ldots, t$. Since $k$ is algebraically closed, we conclude that $\hat{V}_i \cong k[[\beta]]$ for each $i = 1, \ldots, t$.

Now let $c$ denote the conductor of $R$ in $\bar{R}$. Since $P$ is a singular point of $C$, $R \neq \bar{R}$. Thus, $c$ is a proper ideal in $R$. Since $R$ is Noetherian with $m$ as its only proper prime, we see that $\sqrt{c} = m$. Thus, some power, say $n_0$, of $m$ falls in $c$, i.e., $m^{n_0} \subset c$. Now consider $m\bar{R}$. Since every $m_i$ is an associated prime of $m\bar{R}$, we have $J = \sqrt{m\bar{R}}$. Thus, some power of $\beta$ falls inside of $m\bar{R}$, and, consequently, some possibly larger power falls in $c$. Suppose $\beta^n \in c$.

We note that $\beta^{n+l} \in c \subset R \subset A$ for $l = 0, 1, \ldots$. Let $p_1, \ldots, p_t$ be the minimal primes of (0) in $A$. Since $\beta^n$ is not a zero-divisor in $R$, $\beta^n$ is not a zero-divisor in $A$. Thus, $\beta^n \notin \bigcup_{i=1}^{t} p_i$. Therefore, $\pi_i(\beta^n) = (\pi_i(\beta))^n$ is a nonzero element of $A/p_i$. For simplicity of notation, we shall identify $\beta$ with $\pi_i(\beta)$. Then since $\hat{V}_i = k[[\beta]]$, we see $\hat{V}_i$ is a finitely generated module over $A/p_i$ with generators $1, \beta, \cdots, \beta^{n-1}$.

Let $\delta_i$: $\hat{V}_i \rightarrow I(\hat{V}_i/A)$ be the canonical Taylor series given by $\delta_i(x) = 1 \otimes_A x - x \otimes_A 1$. It now follows from [**5**, Lemma 1.1] that $I(\hat{V}_i/A)$ is a left $\hat{V}_i$-algebra generated by $\{\delta_i(\beta), \delta_i(\beta^2), \ldots, \delta_i(\beta^{n-1})\}$. Since $\beta^n \in c \subset A$, we have $\delta_i(\beta^n) = 0$. But, then

$$(6)\quad 0 = \delta_i(\beta^n) = \binom{n}{1}\beta^{n-1}\delta_i(\beta) + \ldots + [\delta_i(\beta)]^n.$$

Solving (6) for $[\delta_i(\beta)]^n$, we get

$$(7)\quad [\delta_i(\beta)]^n = -\beta\left\{\binom{n}{1}\beta^{n-2}\delta_i(\beta) + \ldots + \binom{n}{n-1}[\delta_i(\beta)]^{n-1}\right\}.$$

Now any element of $c$ annihilates $I(\hat{V}_i/A)$. Consequently, raising Equation (7) to the $n$th power gives $[\delta_i(\beta)]^{n^2} = 0$. Thus, $\delta_i(\beta)$ is nilpotent. If we apply the same argument to $\beta^2, \beta^3 \cdots \beta^{n-1}$, we see that each generator $\delta_i(\beta^j)$ $j = 1, \ldots, n-1$ of $I(\hat{V}_i/A)$ is nilpotent. Thus, $I(V_i/A)$ is nilpotent and the proof of Theorem 1 is complete.

We conclude this section with a proposition which will be useful in both the ramified and unramified case.

For each $j = 1, \ldots, t$, we can consider the blow up sequence $\hat{B}_j$ of $A/p_j$ in $\hat{V}_j$. Thus,

$$(8)\quad \hat{B}_j\colon A/p_j = (A/p_j)_0 < (A/p_j)_1 < \ldots < \hat{V}_j.$$

One can easily check that $\hat{V}_j$ is the integral closure of $A/p_j$ in its quotient field. Since $\hat{V}_j$ is a local ring, each term in the chain $\hat{B}_j$ is a local ring. We note that if $A/p_j = \hat{V}_j$, then $\hat{B}_j$ is just the trivial sequence $\hat{B}_j$: $V_j = \hat{V}_j = \ldots.$

Now let $B$: $R < R_1 < R_2 < \ldots < \bar{R}$ denote the blow up sequence of $R$. We wish to relate the multiplicities occurring in $B$ with the multiplicities of the $\hat{B}_j$. Since the multiplicities of $B$ are the same as the multiplicities of $\hat{B}$: $A < A_1 < A_2 < \ldots < \bar{A}$, the following proposition gives us the relationship.

PROPOSITION 2. *Let $A$ be the completion of the local ring of a singular point $P$ of an irreducible algebraic curve $C$ defined over an algebraically closed field $k$. Let $\bar{A} = \hat{V}_1 \oplus \ldots \oplus \hat{V}_t$ be the integral closure of $A$ in its total quotient ring, and let $\{p_1, \ldots, p_t\}$ be the minimal primes of $A$. Let $\hat{B}$: $A < A_1 < \ldots < \bar{A}$ and $B_j$: $A/p_j < (A/p_j)_1 < \ldots < \hat{V}_j$, $j = 1, \ldots, t$ be the blow up sequences for $A$ and $A/p_j$ respectively. Then $\mu(A_i) = \sum_{j=1}^{t}\mu((A/p_j)_i)$ for each $i = 0, 1, \ldots.$*

*Proof.* Consider a fixed ring $A_i$ in the blow up sequence $\hat{B}$. Then $A_i \subset \bar{A}$, and we can consider the kernel of the projection map $\pi_j$ of $\bar{A}$ onto $\hat{V}_j$ when restricted to $A_i$. Set $p_j^{(i)} = \ker \pi_j \cap A_i$. Then a simple argument shows that $p_1^{(i)}, \ldots, p_t^{(i)}$ are exactly the minimal primes of $A_i$. Since $A_i$ is reduced, $(A_i)_{p_j^{(i)}}$ (the localization of $A_i$ at $p_j^{(i)}$) is a reduced, Noetherian local ring of dimension zero. Thus, $(A_i)_{p_j^{(i)}}$ is a field. Consequently, the length of the Artinian local ring $(A_i)_{p_j^{(i)}}$ is one. We also note that if $\hat{J}_i$ is the Jacobson radical of $A_i$, then for each $j = 1, \ldots, t$, $\hat{J}_i(A_i/p_j^{(i)})$ is the Jacobson radical of $A_i/p_j^{(i)}$. It now follows from the projection formula [**6**, (23.5)] that $\mu(A_i) = \sum_{j=1}^{t}\mu(A_i/p_j^{(i)})$. Thus, the proposition will be proven if we can show that

$$(9)\quad A_i/p_j^{(i)} \cong (A/p_j)_i \quad j = 1, \ldots, t.$$

If $i = 0$, then (9) certainly holds. Now consider $A_1$ and $(A/p_j)_1$. If $x$ is a regular element of $A$, then $x \notin \bigcup_{l=1}^{t} p_l$. In particular $x \notin p_j$. Therefore, $\pi_j(x)$ is a regular element of $A/p_j$. Thus, $\pi_j$ has a natural extension to a map $\theta_j$: $A^m \rightarrow (A/p_j)^{\pi_j(m)}$. Now $A^m = A_1$, $(A/p_j)^{\pi_j(m)} = (A/p_j)_1$ and $\theta_j$ is just $\pi_j$ restricted to $A_1$. Since $\pi_j$: $A \rightarrow A/p_j$ is surjective, we have $\theta_j$: $A_1 \rightarrow (A/p_j)_1$ is also surjective. Finally, since $\theta_j$ is just $\pi_j$ restricted to $A_1$, the kernel of $\theta_j$ is exactly $p_j^{(1)}$. Thus, $A_1/p_j^{(1)} \cong (A/p_j)_1$.

We now proceed by induction on $i$. Thus, we may assume that $\pi_j$ when restricted to $A_{i-1}$ maps $A_{i-1}$ onto $(A/p_j)_{i-1}$ and has kernel $p_j^{(i-1)}$. If $x$ is a regular element in $A_{i-1}$, then, $x \notin \bigcup_{l=1}^{t} p_l^{(i-1)}$. In particular, $\pi_j(x)$ is a regular element in $(A/p_j)_{i-1}$. Thus, as in the case $i = 1$, $\pi_j$ has a unique extension

$$\theta_j: A_{i-1}^{\hat{J}_{i-1}} \rightarrow (A/p_j)_{i-1}^{\pi_j(\hat{J}_{i-1})}.$$

Again we have

$$A_{i-1}^{\hat{J}_{i-1}} = A_i, \quad (A/p_j)_{i-1}^{\pi_j(\hat{J}_{i-1})} = (A/p_j)_i$$

and $\theta_j$ is just $\pi_j$ restricted to $A_i$. Thus, $\theta_j$ is surjective and has kernel $p_j^{(i)}$. Hence, (9) is proven and the proof of Proposition 2 is complete.

**2. The unramified case.** In this section, we shall assume that $C$ has no ramification at $P$. In other words, we shall assume that $\bar{R}$ is unramified over $R$. Recall this means that $m$ generates the maximal ideal in each $V_i$, $i = 1, \ldots, t$, and that $V_i/mV_i$ is a seprable field extension of $R/m$ for every $i = 1, \ldots, t$. Since $k = R/m = V_i/mV_i$, the last part of the definition is always satisfied. The following theorem completely characterizes when $\bar{R}$ is unramified over $R$.

THEOREM 2. *Let $R$ be the local ring at a singular point $P$ of an irreducible algebraic curve $C$ defined over an algebraically closed field $k$. Let $\bar{R}$ be the integral closure of $R$, $A$ the completion of $R$ and $\bar{A}$ the integral closure of $A$. Then the following statements are equivalent:*

(a) *$\bar{R}$ is unramified over $R$.*

(b) *$\bar{R}$ is a separable $R$-algebra, i.e. $\bar{R}$ is projective as a left $\bar{R} \otimes_R \bar{R}$-module.*

(c) *The Jacobson radical of $\bar{R}$ is generated by an element of $R$.*

(d) *For all $n$ sufficiently large, $D^n(\bar{A}/A) = 0$.*

*Proof*: The fact that (a) and (b) are equivalent is well known. A proof can be found in [**1**, Theorem 2.5]. We show (c) and (b) are equivalent. First, assume $\bar{R}$ is separable over $R$. Then by [**2**, Theorem 7.1], $\bar{R}/m\bar{R}$ must be separable over $R/m = k$. Thus, $m\bar{R}$ must be the Jacobson radical of $\bar{R}$. Since $k$ is infinite, $m$ has a transversal element, say $x$. Then letting $R^m$ denote the blow up of $R$ by $m$, we have $m\bar{R} = mR^m\bar{R} = xR^m\bar{R} = x\bar{R}$. Thus, the Jacobson radical of $\bar{R}$ is generated by an element $x \in m$. Conversely, assume $m_1 \ldots m_t$ (the Jacobson radical of $\bar{R}$) is generated by some element $x \in R$. Then necessarily $x \in m$, and we have $xV_i = x\bar{R}_{m_i} = (m_1 \ldots m_t)\bar{R}_{m_i} = m_i\bar{R}_{m_i} = m_iV_i$. Thus, $mV_i = m_iV_i$. So, $\bar{R}$ is unramified over $R$ and therefore separable over $R$.

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Finally, we argue that (d) is equivalent to the rest. Suppose first that $\bar{R}$ is unramified over $R$. Then by (c), the Jacobson radical of $\bar{R}$ is generated by some element of $R$. Thus, in the proof of Theorem 1, we can take $\beta$ to lie in $R$. But then $\pi_i(\beta)$ is a nonzero element in $A/p_i$. This implies that $A/p_i = \hat{V}_i$, $i = 1, \ldots, t$. Therefore, $I(\hat{V}_i/A) = 0$ for all $i = 1, \ldots, t$. So, by Theorem 1, $D^n(\bar{A}/A) = 0$ for all $n$ sufficiently large.

Conversely, assume (d) holds. Then by Theorem 1 $I(\hat{V}_i/A) = 0$, $i = 1, \ldots, t$. Thus, $\sigma_i : \hat{V}_i \otimes_{A/p_i} \hat{V}_i \rightarrow \hat{V}_i$ is an isomorphism. It now follows from [**8**; Theorem 1.1] that the inclusion map $A/p_i \rightarrow \hat{V}_i$ is an epimorphism in the category or rings. Since $\hat{V}_i$ is a finitely generated $A/p_i$-module, [**8**, Proposition 1.6] implies that $A/p_i = \hat{V}_i$. Thus, Proposition 2 implies that $\mu(R) = \mu(A) = t$.

Now let $x$ be a transversal for $m$. By the remarks in [**4**, p. 657], we have $t = \mu(R) = \lambda_R(\bar{R}/x\bar{R})$. Here $\lambda_R(M)$ denotes the length of the $R$-module $M$. But, $\lambda_R(\bar{R}/m_1 \ldots m_t) = \lambda_R(k^t) = t$. Since $x\bar{R} \subset m_1 \ldots m_t$, we have $\lambda_R(m_1 \ldots m_t/x\bar{R}) = 0$. So $x\bar{R} = m_1 \ldots m_t$. Thus, the Jacobson radical of $\bar{R}$ is generated by $x$. Therefore, (d) implies (c), and the proof of Theorem 2 is complete.

In the introduction of this paper, we claimed that if $\bar{R}$ is unramified over $R$ then the multiplicities of the blow-up sequence $B$ are particularly simple. It is clear from Theorem 2 and Proposition 2 that if $\bar{R}$ is unramified over $R$, then the multiplicities of $B$ are given by the constant sequence $\{t\}$.

**3. The general case.** As usual, we shall assume $R$ is the local ring at a singular point $P$ of $C$. We shall let $A$ denote the completion of $R$, and $\bar{A}$ the integral closure of $A$ in its total quotient ring. Throughout this section, we shall make no assumptions about the nature of the singularity at $P$. Thus, $\bar{R}$ could be ramified or unramified over $R$.

By Theorem 1, $D^n(\bar{A}/A) = I(\hat{V}_1/A) \oplus \ldots \oplus I(\hat{V}_t/A)$ for $n \gg 1$. Recall that $I(\hat{V}_i/A)$ means $I(\hat{V}_i/A/p_i)$ where $\{p_1, \ldots, p_t\}$ are the minimal primes of $A$.

Now for any $i = 1, \ldots, t$, $I(\hat{V}_i/A)$ is a finitely generated module over the principal ideal domain $\hat{V}_i$. Thus, the decomposition of the $\hat{V}_i$-module $I(\hat{V}_i/A)$ is uniquely determined by a set of invariant factors $\{\delta_1{}^i, \ldots, \delta_{r(i)}{}^i\}$ which are unique up to units in $\hat{V}_i$. By the invariant factors of $D^n(\bar{A}/A)$, we shall mean the set $\bigcup_{i=1}^t \{\delta_1{}^i, \ldots, \delta_{r(i)}{}^i\}$. Note, that if $I(\hat{V}_i/A) = 0$ for some $i$, then we can and do take for $\{\delta_1{}^i \ldots \delta_{r(i)}{}^i\}$, the set $\{1_{\hat{V}_i}\}$. Here $1_{\hat{V}_i}$ denotes the identity of $\hat{V}_i$.

We can now state the general result.

THEOREM 3. *Let $A$ be the completion of the local ring $R$ at a singular point $P$ of an irreducible algebraic curve $C$ defined over an algebraically closed field $k$. Let $\bar{A}$ be the integral closure of $A$ in its total quotient ring. Then the decomposition of* *the module $D^n(\bar{A}/A)$ for $n \gg 1$ uniquely determines the multiplicities of the blow up sequence $B$ of $R$.*

*Proof.* Theorem 3 follows easily from Proposition 2 and [**3**, Theorem 3.5]. Let $B : R < R_1 < \ldots < \bar{R}$ be the blow up sequence of $R$. By Proposition 1, $\mu(R_i) = \mu(A_i)$ where $\hat{B}$: $A < A_1 < \ldots < \bar{A}$ is the blow up sequence of $A$. Thus, by Proposition 2 the multiplicities of $B$ are uniquely determined by the multiplicities of $\hat{B}_j$, $j = 1, \ldots, t$.

Now for $n$ sufficiently large, Theorem 1 implies that $D^n(\bar{A}/A) = I(\hat{V}_1/A) \oplus \ldots \oplus I(\hat{V}_t/A)$. If $D^n(\bar{A}/A) = 0$, then the invariant factors of $D^n(\bar{A}/A)$ are just $F = \{1_{V_1}, \ldots, 1_{V_t}\}$. Then as shown in Theorem 2, for each $i = 1, \ldots, t$, $A/p_i = \hat{V}_i$. Consequently, $\hat{B}_i$ has the form $\hat{B}_i$: $V_i = \hat{V}_i = \ldots$. So, the multiplicities of $\hat{B}_i$ are identically one, and Proposition 2 implies that the multiplicities of $B$ are identically $t$. Thus, if the invariant factors $F$ of $D^n(\bar{A}/A)$ for $n \gg 1$ are trivial, i.e., $F = \{1_{V_1}, \ldots, 1_{V_t}\}$, then the multiplicities of $B$ are identically $t$.

Let us suppose $D^n(\bar{A}/A) \neq 0$. Then after suitably relabeling, we may suppose $I(\hat{V}_i/A) \neq 0$ for $i = 1, \ldots, h$, and $I(\hat{V}_i/A) = 0$ for $i > h$. Here, of course, $1 \leqq h \leqq t$. Thus, the invariant factors of $D^n(\bar{A}/A)$ can be written a

$$F = \{\delta_1^1, \ldots, \delta_{r(1)}^1, \ldots, \delta_1^h, \ldots \delta_{r(h)}^h, 1_{\hat{V}_{h+1}}, \ldots, 1_{\hat{V}_t}\}.$$

Now the multiplicities of the local rings in $\hat{B}_i$, $i = 1, \ldots, h$, are by [**3**, Lemmas 3.4 and 4.2] uniquely determined by the invariants $\{\delta_1{}^i, \ldots, \delta_{r(i)}{}^i\}$. The exact relationship was discussed in the introduction of this paper. The multiplicities of the local rings in $\hat{B}_i$, $i > h$, are identically one. Thus, the multiplicities of the $\hat{B}_i$, $i = 1, \ldots, t$, are uniquely determined by the decomposition of $D^n(\bar{A}/A)$. Consequently, by Proposition 2, the module $D^n(\bar{A}/A)$ uniquely determines the multiplicities of the blow up sequence $B$.

We note that Theorem 3 gives the correct result if $C$ is unibranched at $P$. In this case, $t = 1$, $D^n(\bar{A}/A) = I(\hat{V}_1/A)$ for $n \gg 1$, and we return to the setting in [**3**].

The reader may be wondering why we don't consider $I(\bar{R}/R)$ and its invariants when studying the multiplicities of the blow up sequence $B$ of $R$. Note that $\bar{R}$ is a principal ideal domain, and thus, $I(\bar{R}/R)$ has a natural set of invariant factors associated with it.

One reason we don't study $I(\bar{R}/R)$ is that when we pass to the completion, the branches of $C$ at $P$ get separated, and the computations for $I(\hat{V}_i/A)$, $i = 1, \ldots, t$ are a bit easier to make. For example, if $\bar{R}$ is unramified over $R$, then $I(\hat{V}_i/A) = 0$ for every $i = 1, \ldots, t$. On the other hand, since $R \neq \bar{R}$, [**8**, Theorem 1.1 and Proposition 1.6] imples that $I(\bar{R}/R)$ is never zero for any singular point $P$. Thus, $I(\bar{R}/R)$ always has associated with it a set of non-trivial invariant factors. A second reason we avoid $I(\bar{R}/R)$ is that its invariants don't seem to give us the multiplicities of the blow up sequence $B$ in any natural way as in Theorem 3. We conclude this section with an example which illustrates this last point.

*Example.* Consider the curve $C$: $Y^2 = X^2 + X^3$ defined over the complex numbers $\mathbf{C}$. Let $R$ denote the local ring at the origin $(0, 0)$. If we let $x$ and $y$ denote the images of $X$ and $Y$ in the coordinate ring of $C$, then we can write $R = \mathbf{C}[x, y]_{(x,y)}$ where $y^2 = x^2(x + 1)$. If we set $z = y/x$, then we can easily check that $R[z]$ is the integral closure $\bar{R}$ of $R$ in $\mathbf{C}(x, y)$. $\bar{R} = R[z]$ has exactly two maximal ideals $M_1 = (z - 1)$ and $M_2 = (z + 1)$ which lie over $m = (x, y)$ in $R$. Since $M_1M_2 = (z^2 - 1) = (x) = m\bar{R}$, we see $\bar{R}$ is unramified over $R$. Thus, the blow up sequence $B$ for $R$ is trivial, i.e., $B$: $R < \bar{R} = \bar{R} = \ldots$, and the multiplicities of $B$ are identically 2.

Let us now investigate $I(\bar{R}/R)$. Since $\bar{R}$ is a separable $R$-algebra, $I(\bar{R}/R)$ is generated by an idempotent. By pulling back the separability idempotent from $(\bar{R}/m\bar{R}) \otimes_{\mathbf{C}} (\bar{R}/m\bar{R})$ to $\bar{R} \otimes_R \bar{R}$, the reader can easily verify that the idempotent $e$ which generates $I(\bar{R}/R)$ is exactly $e = (-z/2)(1 \otimes_R z - z \otimes_R 1)$. Since $I(\bar{R}/R)$ is a cyclic $\bar{R}$-module generated by $1 \otimes_R z - z \otimes_R 1$, we see $I(\bar{R}/R) = \bar{R}e$. One can easily check that $x\bar{R}$ is the annihilator of $I(\bar{R}/R)$. Thus, the set of invariant factors for $I(\bar{R}/R)$ is just $\{x\}$.

How we are to decide that the multiplicities of $B$ are $\{2, 2, \ldots\}$ by looking at the set $\{x\}$ is unclear. However, since the invariants of $D^n(\bar{A}/A)$ (for $n \gg 1$) are just $\{1_{\hat{V}_1}, 1_{\hat{V}_2}\}$, we would know immediately from the discussion in Theorem 3 that $B$ is trivial with constant multiplicity 2.

**4.** $D^n(\bar{A}/A)$ **and isomorphism classes of** $A$. Let $C$ as usual denote an irreducible algebraic curve defined over an algebraically closed ground field $k$. Let $A$ denote the completion of the local ring at a singular point $P$ of $C$. Then as we have seen, $\bar{A}$ always has the form $k[[\beta]] \oplus \ldots \oplus k[[\beta]]$. The number of summands present here is equal to the number of branches of $C$ centered at $P$. Now suppose that $\mathscr{D}$ is another irreducible algebraic curve defined over $k$, and let $Q$ be a singular point of $\mathscr{D}$. Let $E$ denote the completion of the local ring at $Q$. Then if the number of branches of $C$ centred at $P$ is the same as the number of branches of $\mathscr{D}$ centered at $Q$, then $\bar{A} \cong \bar{E}$. In this case, it makes sense to inquire when $D^n(\bar{A}/A) \cong D^n(\bar{E}/E)$ for $n \gg 1$.

Let $\Gamma_t$ denote the collection of complete local rings $A$ such that $A$ is the completion of the local ring at a singular point $P$ of some irreducible algebraic curve $C$ (defined over $k$) which has exactly $t$ branches at $P$. Thus, if $A$ and $E$ are members of $\Gamma_t$, then their integral closures $\bar{A}$ and $\bar{E}$ are isomorphic to $k[[\beta]] \oplus \ldots \oplus k[[\beta]]$ ($t$ summands). We wish to briefly discuss when $D^n(\bar{A}/A) \cong D^n(\bar{E}/E)$ for $A, E \in \Gamma_t$.

It would be nice if $D^n(\bar{A}/A) \cong D^n(\bar{E}/E)$ as $k[[\beta]] \oplus \ldots \oplus k[[\beta]]$ - modules implies that $A$ and $E$ are isomorphic. Unfortunately, it is well known that this is false even in the unibranch case $t = 1$. For example, if $A \in \Gamma_1$, and $A'$ denotes the Arf closure of $A$ in $\bar{A}$, then $A' \in \Gamma_1$, and $D^n(\bar{A}/A) = D^n(\bar{A}/A')$

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for all $n$. Since every $A \in \Gamma_1$ is not necessarily an Arf ring, we cannot hope that $D^n(\bar{A}/A)$ determines $A$ up to isomorphism. The reader is urged to consult [**4**] for the pertinent facts about Arf rings used in this section.

If $A$ and $E \in \Gamma_1$ satisfy some order relationship such as $A \subset E$ or $E \subset A$, then we do have a positive result concerning $A'$ and $E'$, the Arf closures of $A$ and $E$. Namely:

PROPOSITION 3. *Suppose* $A, E \in \Gamma_1$ *such that* $A \subset E$. *Then* $D^n(\bar{A}/A) \cong D^n(\bar{E}/E)$ as $k[[\beta]]$ - *modules if and only if the* Arf *closures* $A'$ *and* $E'$ *of* $A$ *and* $E$ *in* $k[[\beta]]$ *are equal.*

*Proof.* This proposition is the main content of [**3**, Theorem 4.7]. In the unibranch case, $D^n(\bar{A}/A) = I(k[[\beta]]/A)$ for $n \gg 1$. Thus, by Theorem 3, if $D^n(\bar{A}/A)$ is isomorphic to $D^n(\bar{E}/E)$, then the multiplicities of the branch sequences for $A$ and $E$ are identical. Since the multiplicities of the branch sequences for $A$ and $A'$ are the same, and $A' \subset E' \subset \bar{A}$, it follows from [**4**, Corollary 3.10] that $A' = E'$.

Conversely, suppose $A' = E'$. Since $A$ contains the field $k$, the Arf closure $A'$ of $A$ is the same as the strict closure of $A$ in $\bar{A}$. Thus, for all $n$, $D^n(\bar{A}/A) = D^n(\bar{A}/A') = D^n(\bar{A}/E') = D^n(\bar{A}/E)$.

We cannot hope for such a nice result in the general situation $t \geqq 1$. This is because the module $D^n(\bar{A}/A)$ cannot distinguish between unramified extensions. For suppose, $A \in \Gamma_t$ $(t > 1)$ is unramified. Then by Theorem 2, $\bar{A}$ is a separable algebra over $A$. If $E$ is any ring such that $A \subset E \subset \bar{A}$, then $\bar{A}$ is also separable over $E$. Thus, $D^n(\bar{A}/A) = D^n(\bar{A}/E) = 0$ for $n \gg 1$. Since $A'$ need not be equal to $E'$, we see that Proposition 3 is false if $t > 1$.

However, if $A, E \in \Gamma_t$ are special enough, we can state a generalization of Proposition 3. Let $A_i$ as usual denote the $i$th blow up of $A$. Let us say that the local rings $A \subseteq E$ in $\Gamma_t$ are *compatible* if

(a) $A_i \subseteq E_i$ for all $i = 0, 1, \ldots,$ and

(b) For all maximal ideals $M \subset \bar{A}$ and for all $i$, the number of minimal primes in $A_i$ which are contained in $M \cap A_i$ is exactly the same as the number of minimal primes of $E_i$ contained in $M \cap E_i$.

From the remarks made above, it is clear that in order to state any analog of Proposition 3, we must avoid the unramified situation. Since $D^n(\bar{A}/A) \cong I(\hat{V}_1/A) \oplus \ldots \oplus I(\hat{V}_t/A)$ $(n \gg 1)$, due care must also be made to match up proper components of $D^n(\bar{A}/A)$ and $D^n(\bar{E}/E)$. Thus, a correct analog of Proposition 3 is as follows:

PROPOSITION 4. *Let* $A \in \Gamma_t$ *have minimal primes* $\{p_1, \ldots, p_t\}$ *and assume* $A/p_i \subsetneq \hat{V}_i$ *for all* $i = 1, \ldots, t$. *Let* $E \in \Gamma_t$ *such that* $A \subset E$, *and* $A$ *and* $E$ *are compatible. Assume that we have labeled the minimal primes* $\{q_1, \ldots, q_t\}$ *of* $E$ *so that* $A/p_i \subset E/q_i \subset \hat{V}_i$ $i = 1, \ldots, t$. *If there exists a* $k[[\beta]] \oplus \ldots \oplus k[[\beta]]$-

*isomorphism* $T: D^n(\bar{A}/A) \to D^n(\bar{E}/E)$ (for $n \gg 1$) *such that* $T(I(\hat{V}_i/A)) = I(\hat{V}_i/E)$ *for all* $i = 1, \ldots, t$, *then the* Arf *closures of* $A$ *and* $E$ *in* $\bar{A}$ *coincide.*

*Proof.* Since $A/p_i \neq \hat{V}_i$, $I(\hat{V}_i/A) \neq 0$. Therefore, $I(\hat{V}_i/E) \neq 0$, and $E/q_i \neq \hat{V}_i$. Since $I(\hat{V}_i/A) \cong I(\hat{V}_i/E)$, the multiplicity sequences of $A/p_i$ and $E/q_i$ are identical. Thus, using the notation of Proposition 2, we have $\mu\{(A/p_i)_j\} = \mu\{(E/q_i)_j\}$ for all $i$ and $j$.

Now let $M$ be a maximal ideal of $\bar{A}$. We wish to compute the multiplicity of the local ring $(E_1)_{M \cap E_1}$. We proceed as in the proof of Proposition 2. Let $\{q_1^{(1)}, \ldots, q_t^{(1)}\}$ denote the minimal primes of $E_1$. We can assume that $q_1^{(1)}, \ldots, q_l^{(1)} \subset M \cap E_1$, and $q_{l+1}^{(1)}, \ldots, q_t^{(1)} \not\subset M \cap E_1$. Here $1 \leqq l \leqq t$. Then the minimal primes in $(E_1)_{M \cap E_1}$ are just $\{q_1^{(1)}(E_1)_{M \cap E_1}, \ldots, q_l^{(1)}(E_1)_{M \cap E_1}\}$. A simple calculation shows that each localization $\{(E_1)_{M \cap E_1}\}q_i^{(1)}(E_1)_{M \cap E_1}$, $i = 1, \ldots, l$, is a field, and that

$$(E_1)_{M \cap E_1}/q_i^{(1)}(E_1)_{M \cap E_1} \cong E_1/q_i^{(1)} \cong (E/q_i)_1.$$

Thus by the projection formula, $\mu\{(E_1)_{M \cap E_1}\} = \sum_{i=1}^{l} \mu\{(E/q_i)_1$.

Since $A$ and $E$ are compatible, $A_1 \subset E_1$. Thus, $q_i^{(1)}$ contracts to $p_i^{(1)}$ in $A_1$. Since the number of minimal primes of $E_1$ contained in $M \cap E_1$ is exactly the same as the number of minimal primes of $A_1$ in $M \cap A_1$, we see that $\{p_1^{(1)}, \ldots, p_l^{(1)}\}$ are exactly the minimal primes of $A_1$ contained in $M \cap A_1$. Thus, a similar computation as in the preceding paragraph gives $\mu\{(A_1)_{M \cap A_1}\} = \sum_{i=1}^{l} \{(A/p_i)_1\}$. Therefore, $\mu\{(A_1)_{M \cap A_1}\} = \mu\{(E_1)_{M \cap E_1}\}$. Continuing in this fashion, we can show that for all $i = 0, 1, \ldots, \mu\{(A_i)_{M \cap A_1}\} = \mu\{(E_i)_{M \cap E_i}\}$. Since $M$ was arbitrary, we conclude that $A$ and $E$ have the same multiplicity sequence along each maximal ideal of $\bar{A}$. It now follows from [**4**, Corollary 3.10] that the Arf closures of $A$ and $E$ in $\bar{A}$ coincide.

Finally, we note that Proposition 4 is a true generalization of Proposition 3. For suppose $A, E \in \Gamma_1$ with $A \subset E$, and $D^n(\bar{A}/A) \cong D^n(\bar{E}/E)$. Then $\mu(A_i) = \mu(E_i)$ for every $i = 0, 1, \ldots$. Each ring $A_i$ or $E_i$ is local, and a transversal for either is just an element of minimum positive order (relative to the canonical valuation of $k[[\beta]]$). Since $\mu(A_i) = \mu(E_i)$, a common transversal for both $A_i$ and $E_i$ can be chosen out of $A_i$. But this immediately implies that $A_{i+1} \subset E_{i+1}$. Thus, $A$ and $E$ satisfy condition (a) in the definition of compatibility. Since condition (b) is trivial, we see that $A$ and $E$ are compatible. Thus, Proposition 4 implies that the Arf closures of $A$ and $E$ in $\bar{A}$ are the same.

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*Michigan State University,*
*East Lansing, Michigan*