Canad. Math. Bull. Vol. **50** (2), 2007 pp. 234–242

A Remark on a Modular Analogue of the Sato–Tate Conjecture

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Abstract. The original Sato–Tate Conjecture concerns the angle distribution of the eigenvalues arising from non-CM elliptic curves. In this paper, we formulate a modular analogue of the Sato–Tate Conjecture and prove that the angles arising from non-CM holomorphic Hecke eigenforms with non-trivial central characters are not distributed with respect to the Sate–Tate measure for non-CM elliptic curves. Furthermore, under a reasonable conjecture, we prove that the expected distribution is uniform.

1 Introduction

Let E be an elliptic curve over $\mathbb{Q}$ and Δ_E the discriminant of E. For a rational prime p, coprime to Δ_E, define

$$N_p = p + 1 - a_p = |E(\mathbb{F}_p)|,$$

where $E(\mathbb{F}_p)$ is the set of rational points of E defined over the finite field $\mathbb{F}_p$ and $|E(\mathbb{F}_p)|$ is the cardinality of $E(\mathbb{F}_p)$. For a rational prime $p \nmid \Delta_E$, a result of Hasse [13, Theorem 1.1] states that

$$|a_p| \le 2p^{1/2}.$$

Thus, we can write

$$a_p = 2p^{1/2} \cos\theta_p,$$

for a uniquely defined angle θ_p satisfying $0 \le \theta_p < \pi$. A natural question to ask is how θ_p distributes in the interval $[0, \pi]$. For elliptic curves with complex multiplication, the answer to this question is well known [9]. On the other hand, for elliptic curves without complex multiplication, the problem remains open until today. Sato and Tate independently conjectured that

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \cdot \#\{p : p \le x, \theta_p \in (\alpha, \beta)\} = \Big(\frac{1}{\pi}\int_\alpha^\beta 2\sin^2\theta \, d\theta\Big),$$

where $\pi(x)$ is the number of primes less than or equal to x. It is called the Sato–Tate conjecture and it has many classical origins. For instance, it is related to how often a quadratic form is a prime in a certain region [4] and the distribution of primes in quadratic progressions [8].

We can also extend this conjecture to modular forms. Let

$$\mathcal{H} = \{z \in \mathbb{C}, \operatorname{Im}(z) > 0\}, \quad \mathcal{H}^* = \mathcal{H} \cup \text{cusps},$$

Received by the editors January 15, 2005.
The author's research was partially supported by an NSERC grant.
AMS subject classification: 11F03, 11F25.
Keywords: L-functions, Elliptic curves, Sato–Tate.

https://doi.org/10.4153/CMB-2007-025-7 Published online by Cambridge University Press

be the upper half plane and the upper half plane with cusps, respectively. Let Γ be a modular group.

Definition 1.1 Let ω be a non-trivial primitive Dirichlet character. A (*holomorphic*) *Hecke eigenform f of Γ with the Nebentypus ω* is a complex valued function on $\mathcal{H}^*$ satisfying

(i) There is an integer $k \geq 0$ such that for each $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma$ we have the modular transformation law $f(\gamma z) = \omega(d)(cz + d)^k f(z)$.

(ii) The function f is holomorphic on $\mathcal{H}$ and extends holomorphically to every cusp of Γ. It also vanishes on cusps.

(iii) By (i), we have the Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}.$$

Define the L-function $L(s, f)$ of f as

$$L(s, f) = \sum_{n=1}^{\infty} a_n \cdot n^{-s}.$$

Let $\mathcal{P}$ be the set of rational primes. Then there is a finite subset $\mathcal{P}(f)$ of $\mathcal{P}$ such that

$$L(s, f) = \prod_{p \in \mathcal{P} \setminus \mathcal{P}(f)} (1 - a_p p^{-s} + \omega(p) p^{k-1} p^{-2s})^{-1} \prod_{p \in \mathcal{P}(f)} l_p(s)^{-1},$$

where $l_p(s)$ are polynomials in p^{-s} with $l_p(0) \neq 0$. In other words, f admits an Euler product.

We denote by $H(\Gamma, \omega)$ the set of all Hecke eigenforms of Γ with the Nebentypus ω.

The Ramanujan conjecture on $H(\Gamma, \omega)$ can be stated as follows:

Conjecture 1.2 (Ramanujan) *For each $f \in H(\Gamma, \omega)$, we can rewrite the Euler product as*

$$L(s, f) = \prod_{p \in \mathcal{P} \setminus \mathcal{P}(f)} (1 - \alpha_p \cdot p^{(k-1)/2} p^{-s})^{-1} (1 - \beta_p \cdot p^{(k-1)/2} p^{-s})^{-1} \prod_{p \in \mathcal{P}(f)} l_p(s)^{-1},$$

where α_p and β_p are the roots of the quadratic polynomial $x^2 - (a_p / p^{(k-1)/2})x + \omega(p)$. Then $|\alpha_p| = |\beta_p| = 1$.

The above conjecture is proved by Deligne [1]. Therefore, for each $f \in H(\Gamma, \omega)$, we have $|\alpha_p| = |\beta_p| = 1$ for all rational primes $p \in \mathcal{P} \setminus \mathcal{P}(f)$. Since $|\alpha_p| = |\beta_p| = 1$, we can write α_p and β_p as polar forms

$$\alpha_p = e^{i\theta_p}, \quad \beta_p = e^{i\psi_p}, \quad 0 \leq \theta_p, \psi_p < 2\pi,$$

The question now is how θ_p, ψ_p distribute on $[0, 2\pi]$.

2 Distributions and L-Functions

Definition 2.1 ([10, Appendix to Ch. 1]) Let X be a compact topological space and $C(X)$ the set of all continuous functions on X. Let S be a sequence $\{x_i\}_{i\in I} \subseteq X$ with the index set I equipped with a norm map $N\colon I \longrightarrow \mathbb{N}$ satisfying the property that for all $n \in \mathbb{N}$, $N^{-1}(n)$ is a finite set. For all positive real numbers x, define

$$\mathcal{N}^S(x) := \{i \in I \mid N(i) \leq x\}.$$

Let μ be a distribution on X and for all $g \in C(X)$, define

$$\mu_x(g) := \frac{1}{|\mathcal{N}^S(x)|} \sum_{i\in\mathcal{N}^S(x)} g(x_i).$$

We say that S is *distributed with respect to a distribution μ on X*, if for all $g \in C(X)$

$$\lim_{x\to\infty} \mu_x(g) = \mu(g).$$

In our case, $X = S^1 \cong \mathbb{R}/2\pi$. We have the following handy criterion [10, Corollary 2, Appendix to Chapter 1]:

Theorem 2.2 (Generalized Weyl Criterion) *Let f be a piece-wise continuous function on $\mathbb{R}$ of period 2π whose the Fourier expansion is*

$$f(\theta) = \frac{1}{2\pi} \sum_{m=-\infty}^{m=\infty} c_m e^{-im\theta}, \quad \text{and} \sum_{m=-\infty}^{m=\infty} |c_m|^2 < \infty.$$

Let S be a sequence $\{x_i\}_{i\in I}$ of real numbers between 0 and 2π with a norm map $N\colon I \longrightarrow \mathbb{N}$. Then S is distributed with respect to a distribution $\int f(\theta)\,d\theta$ if and only if for all $m \in \mathbb{Z}, x \in \mathbb{R}^+$,

$$\sum_{i\in\mathcal{N}^S(x)} e^{imx_i} = c_m|\mathcal{N}^S(x)| + \mathbf{o}\big(|\mathcal{N}^S(x)|\big),$$

as x tends to infinity. In particular, if all $c_m = 0$ except for $m = 0$, then S is distributed with respect to the standard Lebesgue measure. In this case, we say that S is uniformly distributed.

Let f be a Hecke eigenform with a non-trivial Nebentypus ω and

$$S_f = \{\theta_p, \psi_p\}_{p\in\mathcal{P}\setminus\mathcal{P}(f)},$$

the set of angles arising from f, with the index set

$$\bigcup_{p\in\mathcal{P}\setminus\mathcal{P}(f)} \{p, p\},$$

https://doi.org/10.4153/CMB-2007-025-7 Published online by Cambridge University Press

and the natural norm map N defined by

$$N(p) = p.$$

Thus, studying the distribution of S_f is equivalent to studying the asymptotic behavior of

$$A^m(x) = \sum_{\substack{p \le x \\ p \in \mathcal{P} \setminus \mathcal{P}(f)}} (e^{im\theta_p} + e^{im\psi_p}) = \sum_{\substack{p \le x \\ p \in \mathcal{P} \setminus \mathcal{P}(f)}} (\alpha_p^m + \beta_p^m),$$

where m is an integer. Note that

$$A^{-m}(x) = \sum_{\substack{p \le x \\ p \in \mathcal{P} \setminus \mathcal{P}(f)}} (e^{-im\theta_p} + e^{-im\psi_p}) = \overline{\sum_{\substack{p \le x \\ p \in \mathcal{P} \setminus \mathcal{P}(f)}} (e^{im\theta_p} + e^{im\psi_p})} = \overline{A^m(x)}.$$

Therefore, we only need to consider the case when $m \ge 0$. We need the following lemma.

Lemma 2.3 *Let $F(s)$ be a Dirichlet series of the Euler product*

$$F(s) = \prod_{p \in \mathcal{P} \setminus \mathcal{G}} \Big(\prod_{i=1}^{m} \big(1 - \alpha_p^{(i)} \cdot p^{-s}\big)^{-1} \Big), \quad |\alpha_p^{(i)}| = 1,$$

where $\mathcal{G}$ is a finite subset of rational primes. If $F(s)$ has an analytic continuation to $\mathrm{Re}(s) \ge 1$ *and is non-vanishing at* $\mathrm{Re}(s) = 1$*, then*

$$\sum_{\substack{p \le x \\ p \in \mathcal{P} \setminus \mathcal{G}}} \Big(\sum_{i=1}^{m} \alpha_p^{(i)} \Big) = \mathbf{o}(\pi(x)).$$

Proof Consider $F'(s)/F(s) = (\log F(s)t)'$.

$$\begin{aligned} -F'(s)/F(s) &= -(\log F(s))' \\ &= -\Big(\sum_{p \in \mathcal{P} \setminus \mathcal{G}} \Big(\sum_{i=1}^{m} \log(1 - \alpha_p^{(i)} \cdot p^{-s}) \Big) \Big)' \\ &= \sum_{p \in \mathcal{P} \setminus \mathcal{G}} \Big(\sum_{i=1}^{m} \Big(\sum_{k=1}^{\infty} k(\alpha_p^{(i)})^k (\log p) p^{-ks} \Big) \Big). \end{aligned}$$

Using the condition $|\alpha_p^{(i)}| = 1$ to estimate the term for $k \ge 2$ and applying a Tauberian theorem, we obtain our result. ■

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Let $\mathbb{A}_\mathbb{Q}$ be the ring of adeles of $\mathbb{Q}$. Let $\pi = \bigotimes_p \pi_p$ be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with central character ω_π. Fix a positive integer m, and let

$$\mathrm{Sym}^m\colon \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_{m+1}(\mathbb{C})$$

be the m-th symmetric power representation of $\mathrm{GL}_2(\mathbb{C})$ on symmetric tensors of rank m (*cf.* [11, 12]). By the local Langlands correspondence, $\mathrm{Sym}^m(\pi_p)$ is well defined for every p. The Langlands functoriality in this case is equivalent to the fact that $\mathrm{Sym}^m(\pi) = \bigotimes_p \mathrm{Sym}^m(\pi_p)$ is an automorphic representation of $\mathrm{GL}_{m+1}(\mathbb{A}_\mathbb{Q})$. Let $\mathcal{P}(\pi)$ be the set of places where π is ramified. One can define the L-function $L(s,\pi)$ associated to π as follows:

$$L(s,\pi) = \prod_{p\in\mathcal{P}\setminus\mathcal{P}(\pi)} (1-\alpha_p p^{-s})^{-1}(1-\beta_p p^{-s})^{-1} \prod_{p\in\mathcal{P}(\pi)} h_p(s)^{-1},$$

where $l_p(s)$ are polynomials in p^{-s} with $l_p(0) \neq 0$. Then it follows that

$$L(s,\mathrm{Sym}^m(\pi)) = \prod_{p\in\mathcal{P}\setminus\mathcal{P}(\pi)} \prod_{i=0}^{m} (1-\alpha_p^{m-i}\beta_p^i p^{-s})^{-1} \prod_{p\in\mathcal{P}(\pi)} g_p(s)^{-1}.$$

where $g_p(s)$ are polynomials in p^{-s} with $g_p(0) \neq 0$.

Remarks 1

(1) The generalized Ramanujan conjecture predicts that if π is a cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$, then for all $p \in \mathcal{P}\setminus\mathcal{P}(\pi)$, $|\alpha_p| = |\beta_p| = 1$.

(2) Let ω be a non-trivial primitive Dirichlet character. By Deligne [1], for any $f \in H(\Gamma,\omega)$, f is attached to a cuspidal representation π of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ such that $L(s,f) = L(s,\pi)$.

3 Main Theorems

Definition 3.1 Let f be a Hecke eigenform with the Nebentypus ω, where ω is a non-trivial primitive character. We say that f is non-CM if there is no Grössen-character χ such that $L(s,\chi)$ is equal to $L(s,f)$.

As in the case of elliptic curves, we consider only those f's which are non-CM. Now we can state and prove our theorems.

Theorem 3.2 *Let f be a non-CM Hecke eigenform with the Nebentypus ω, for a non-trivial primitive character ω. Then the sequence $S_f = \{\theta_p,\psi_p\}_{p\in\mathcal{P}\setminus\mathcal{P}(f)}$ is not distributed with respect to the Sato–Tate measure $(1/2\pi)\int 2\sin^2\theta\,d\theta$.*

Remark 2 Note that in the original Sato–Tate conjecture, the values of the sequences are between 0 and π. However, in our setting, those are between 0 and 2π. Therefore, the corresponding Sato–Tate measure is $(1/2\pi)\int 2\sin^2\theta\,d\theta$.

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Proof Suppose that S_f is distributed with respect to the Sato–Tate measure

$$(1/2\pi)\int 2\sin^2\theta\,d\theta.$$

Since $2\sin^2\theta = 1-\cos 2\theta$, and $|\mathcal{N}^{S_f}(x)| = 2\pi(x)+\mathbf{O}(1)$, we have

$$A^0(x) = 2\pi(x)+\mathbf{o}(\pi(x)),\quad A^1(x)=\mathbf{o}\big(\pi(x)\big),\quad \text{and}\quad A^2(x) = -\pi(x)+\mathbf{o}(\pi(x)),$$

For $m=0$,

$$A^0(x) = \sum_{\substack{p\le x\\ p\in\mathcal{P}\setminus\mathcal{P}(f)}} (\alpha_p^0+\beta_p^0) = |\mathcal{N}^{S_f}(x)| = 2\pi(x)+\mathbf{O}(1).$$

For $m=1$, let π be the cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ attached to f. By [3, 5], $L(s,\pi)$ is entire and non-vanishing at $\mathrm{Re}(s)=1$. We can ignore the ramified places since there are only finitely many of them. Therefore, by Lemma 2.3,

$$A^1(x) = \sum_{\substack{p\le x\\ p\in\mathcal{P}\setminus\mathcal{P}(f)}} t(\alpha_p+\beta_p) = \mathbf{o}(\pi(x)).$$

For $m=2$, we claim first that π is not monomial, *i.e.,* there is no non-trivial Grössencharacter η such that $\pi\otimes\eta\cong\pi$. If there is such a Grössencharacter η, then $\eta^2=1$ and η determines a quadratic extension E. According to [7], there is a Grössencharacter χ of E such that $L(s,\pi)=L(s,\chi)$, which is impossible since f is non-CM.

Since π is not monomial, by [2], $\mathrm{Sym}^2(\pi)$ is cuspidal. Therefore, by [3, 5], $L(s,\mathrm{Sym}^2(\pi))$ is entire and non-vanishing at $\mathrm{Re}(s)=1$. Thus, by Lemma 2.3

$$\sum_{\substack{p\le x\\ p\in\mathcal{P}\setminus\mathcal{P}(f)}} (\alpha_p^2+\alpha_p\beta_p+\beta_p^2) = \mathbf{o}(\pi(x)).$$

Then

$$\begin{aligned} A^2(x) &= \sum_{\substack{p\le x\\ p\in\mathcal{P}\setminus\mathcal{P}(f)}} (\alpha_p^2+\beta_p^2)\\ &= \sum_{\substack{p\le x\\ p\in\mathcal{P}\setminus\mathcal{P}(f)}} (\alpha_p^2+\alpha_p\beta_p+\beta_p^2-\alpha_p\beta_p)\\ &= \mathbf{o}\big(\pi(x)\big) - \sum_{\substack{p\le x\\ p\in\mathcal{P}\setminus\mathcal{P}(f)}} \omega(p). \end{aligned}$$

By classical theorems, $L(s,\omega)$ is entire and non-vanishing at $\mathrm{Re}(s) = 1$. Thus, applying Lemma 2.3, we get

$$A^2(x) = \mathbf{o}(\pi(x)) - \sum_{\substack{p \le x \\ p \in \mathcal{P} \setminus \mathcal{P}(f)}} \omega(p) = \mathbf{o}(\pi(x)) \ne -\pi(x) + \mathbf{o}(\pi(x));$$

it contradicts the fact that $A^2(x) = -\pi(x) + \mathbf{o}(\pi(x))$ if the set $\{\theta_p, \psi_p\}$ distributes with respect to the Sato–Tate measure. This completes the proof of the theorem. ■

Now it is natural to ask what is the expected distribution of the sequence $S_f = \{\theta_p, \psi_p\}_{p \in \mathcal{P} \setminus \mathcal{P}(f)}$ arising from a Hecke eigenform f with non-trivial Nebentypus. We need the following lemma.

Lemma 3.3 *Let $\pi = \bigotimes_p \pi_p$ be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with central character ω_π. For two positive integers m and n, define*

$$S^m(x) = \sum_{\substack{p \le x \\ p \in \mathcal{P} \setminus \mathcal{P}(\pi)}} \sum_{i=0}^{m} \alpha_p^{m-i} \beta_p^i, \quad \textit{and} \quad \tilde{S}^n(x) = \sum_{\substack{p \le x \\ p \in \mathcal{P} \setminus \mathcal{P}(\pi)}} \omega(p) \Big(\sum_{i=0}^{n} \alpha_p^{n-i} \beta_p^i \Big).$$

Assume that the L-functions $L(s, \mathrm{Sym}^m(\pi))$ and $L(s, \mathrm{Sym}^n(\pi) \otimes \omega_\pi)$ have analytic continuation for $\mathrm{Re}(s) \ge 1$, and are non-vanishing for $\mathrm{Re}(s) \ge 1$. Then

$$S^m(x) = \mathbf{o}(\pi(x)), \quad \tilde{S}^n(x) = \mathbf{o}(\pi(x)).$$

Proof It is an application of Lemma 2.3. Note that we can ignore the contribution from ramified places. ■

Then we have the following theorem.

Theorem 3.4 *Let π be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ which satisfies the Ramanujan conjecture. Assume that for all positive integers m, the L-functions*

$$L(s, \mathrm{Sym}^m(\pi)) \quad \textit{and} \quad L(s, \mathrm{Sym}^m(\pi) \otimes \omega)$$

have analytic continuation for $\mathrm{Re}(s) \ge 1$, and are non-vanishing for $\mathrm{Re}(s) \ge 1$. Then the sequence $S_f = \{\theta_p, \psi_p\}_{p \in \mathcal{P} \setminus \mathcal{P}(\pi)}$ is uniformly distributed.

Proof According to Theorem 2.2, we need to prove $c_0 = 1$ and $c_m = 0$ for all positive integers m's. For c_0, we have

$$A^0(x) = \sum_{\substack{p \le x \\ p \in \mathcal{P} \setminus \mathcal{P}(\pi)}} (\alpha_p^0 + \beta_p^0) = 1 \cdot |\mathcal{N}^{S_f}(x)|.$$

Thus, $c_0 = 1$.

For $m = 1$, by [3, 5], $L(s, \pi)$ is entire and non-vanishing at $\mathrm{Re}(s) = 1$. By Lemma 2.3,

$$A^1(x) = \sum_{\substack{p \le x \\ p \in \mathcal{P} \setminus \mathcal{P}(\pi)}} (\alpha_p + \beta_p) = \mathbf{o}(\pi(x)) = \mathbf{o}\big(|\mathcal{N}^{S_f}(x)|\big) = 0 \cdot |\mathcal{N}^{S_f}(x)| + \mathbf{o}\big(|\mathcal{N}^{S_f}(x)|\big).$$

It implies that $c_1 = 0$.

For $m = 2$, by our assumption, $L(s, \mathrm{Sym}^2(\pi))$ has analytic continuation for $\mathrm{Re}(s) \ge 1$, non-vanishing at $\mathrm{Re}(s) = 1$. Therefore, by Lemma 2.3,

$$A^2(x) = \sum_{\substack{p \le x \\ p \in \mathcal{P} \setminus \mathcal{P}(\pi)}} (\alpha_p^2 + \beta_p^2) = \mathbf{o}(\pi(x)) = \mathbf{o}\big(|\mathcal{N}^{S_f}(x)|\big).$$

We obtain $c_2 = 0$. Now we consider $m \ge 3$. We have the following identity

$$a^m + b^m = \sum_{i=0}^{m} a^{m-i} b^i - \sum_{i=1}^{m-1} a^{m-i} b^i = \sum_{i=0}^{m} a^{m-i} b^i - ab \sum_{i=0}^{m-2} a^{m-2-i} b^i.$$

Therefore, for $m \ge 3$

$$A^m(x) = S^m(x) - \tilde{S}^{m-2}(x) = \mathbf{o}(\pi(x)) = \mathbf{o}\big(|\mathcal{N}^{S_f}(x)|\big).$$

This completes the proof of the theorem. ∎

It is widely believed that the Ramanujan conjecture is true for cuspidal representations of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. However, the assumption in Theorem 3.4 is not always true. For instance, by [6], there is a cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ such that the L-functions of its symmetric powers might have poles at $s = 1$. Yet, for the L-functions attached to Hecke eigenforms, it is expected to be true. More precisely,

Conjecture 3.5 *Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ attached to a Hecke eigenform with the Nebentypus ω_π. Then for all positive integer m, the L-functions $L(s, \mathrm{Sym}^m(\pi))$ and $L(s, \mathrm{Sym}^n(\pi) \otimes \omega_\pi)$ have analytic continuation for $\mathrm{Re}(s) \ge 1$, and are non-vanishing for $\mathrm{Re}(s) \ge 1$.*

Combining Theorem 3.4 and the conjecture above, we have

Theorem 3.6 *Let ω be a non-trivial primitive Dirichlet character and f a Hecke eigenform with the Nebentypus ω. Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ attached to f. Assume that for all positive integers m, the L-functions $L(s, \mathrm{Sym}^m(\pi))$ and $L(s, \mathrm{Sym}^m(\pi) \otimes \omega)$ have analytic continuation for $\mathrm{Re}(s) \ge 1$, and are non-vanishing for $\mathrm{Re}(s) \ge 1$. Then the sequence $S_f = \{\theta_p, \psi_p\}_{p \in \mathcal{P} \setminus \mathcal{P}(f)}$ is uniformly distributed.*

We conclude this paper with several remarks.

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(1) Our results can be extended to any number field.

(2) The Ramanujan conjecture in Theorem 3.4 is a part of the Langlands program. It is conjectured to be held for any cuspidal representation of $GL_2(\mathbb{A}_{\mathbb{Q}})$.

(3) For Conjecture 3.5, the existence of the meromorphic continuation of L-functions for symmetric powers is also a part of the Langlands program. Therefore, it is conjectured to be true in general. As we remarked before, the holomorphic condition on $\mathrm{Re}(s) = 1$ is not true in general. However, a deeper conjecture predicts that if π is a cuspidal representation attached to a non-CM Hecke eigenform, then for all positive integers n, $\mathrm{Sym}^n(\pi)$ are cuspidal as well. This explains why it is a general belief that Conjecture 3.5 should be true even if it is not true in general.

(4) In the cases of non-CM elliptic curves, by [9], the assumption of non-vanishing can be removed. It should be possible to remove this assumption from Theorem 3.6. We plan to investigate it in future work.

4 Acknowledgment

I would like to thank Ram Murty for valuable discussions and comments about this work. This work was started during my stay at the Fields institute in 2003. I would like to thank the institute for their hospitality.

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