BULL. AUSTRAL. MATH. SOC. 26A39,26A42,26A45
VOL. 37 (1988) [1–15]

INTEGRATION BY PARTS FOR SOME GENERAL INTEGRALS

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The present work is concerned with an integration by parts formula for the P^k-integral of De Sarkar and Das, and of the equivalent P^k-integral of Bullen. The process involves a simpler and updated version of that for the Z_{k-1}-integral of Bergin. If f is $P^k-(Z_{k-1})$-integrable and G is of bounded kth variation, then fG is $P^k-(Z_{k-1})$-integrable.

1. INTRODUCTION

As soon as a new integral is defined, it is interesting to investigate the integration by parts formula for that integral. For any integral, I-integral (say), the rôle of integration by parts lies in the following question: if f is I-integrable on $[a,b]$ and $F(x) = (I)\int_a^x f$, then for which G is it true that fG is I-integrable?

For the classical Perron integral we refer to a survey by Bullen [**6**] and also to a simple proof by Bullen [**5**].

If f is P-integrable on $[a,b]$ then $F(x) = (P)\int_a^x f$, and if G is of bounded variation, then fG is P-integrable and

$$(P)\int_a^b fG = F(b)G(b) - F(a)G(a) - (R)\int_a^b FG'$$

or equivalently,

$$(P)\int_a^b fG = F(b)G(b) - F(a)G(a) - (RS)\int_a^b FdG,$$

where in the second formula, the right-hand side is to be interpreted as follows:

$$G(a) = G(a+), \qquad G(b) = G(b-), \qquad (RS)\int_a^b fdG = \lim_{\substack{\alpha\to a+\\ \beta\to b-}} (RS)\int_\alpha^\beta FdG.$$

Bullen [**3**] and also De Sarkar and Das [**14**] obtained a kth order generalisation of the Perron integral which they called the P^k-integral. The former used Peano

Received 22 January 1987
The authors wish to express their heartiest thanks to Professor P.S. Bullen of the University of British Columbia, Canada. In fact the problem originated when Professor Bullen visited the University of Kalyani in 1983 and highlighted the integration by parts formula for the classical Perron integral.

derivatives and the latter used equivalent Riemann* derivatives. Peano, Riemann* and ordinary derivatives of a function f at x, of order r, will be denoted by $f_{(r)}(x)$, $D^r f(x)$, and $f^{(r)}(x)$, respectively.

According to De Sarkar and Das [14], a function M continuous on $[a,b]$ is called a P^k-major function of f on $[a,b]$ if:

(i) $D^r M$ exists and is finite on $[a,b]$ for $1 \leqslant r \leqslant k-1$;

(ii) $\underline{D}^k M(x) \geqslant f(x)$ a.e. in $[a,b]$;

(iii) $\underline{D}^k M(x) > -\infty$ n.e. in $[a,b]$;

(iv) $D^r M(a) = 0$, $0 \leqslant r \leqslant k-1$.

If $-m$ is a P^k-major function of $-f$, then m is called a P^k-minor function of f on $[a,b]$. If $-\infty < \inf\{M(b)\} = \sup\{m(b)\} < +\infty$, then f is P^k-integrable on $[a,b]$ and the common value is called the P^k-integral of f on $[a,b]$, and is denoted by $(P^k)\int_a^b f$.

Following Bergin [1] and Remark 6 of De Sarkar and Das [14], we can say that $D^{k-1}M$ is a $(k-1)$-majorant and $D^{k-1}m$ is a $(k-1)$-minorant of f on $[a,b]$ and the finite common value $\inf\{D^{k-1}M(b)\} = \sup\{D^{k-1}m(b)\}$ is the Z_{k-1}-integral of f, $(Z_{k-1})\int_a^b f$. Bergin, however, does not assume condition (iv). If $M^\star$ is a pre-majorant of Bergin, it is sufficient to consider $M(x) = M^\star(x) - \sum_{r=0}^{k-1}((x-a)^r/r!)D^r M^\star(a)$. Bergin's Z_k-integral is equivalent to Burkill's C_kP-integral (Proposition 6.1 of Bergin [1]).

It is now evident that f is P^k-integrable if and only if it is Z_{k-1}-integrable. Further, if $F(x) = (P^k)\int_a^x f$, then

$$D^{k-1}F(x) = (Z_{k-1})\int_a^x f; \tag{1}$$

$$F(x) = (Z_0)\int_a^x (Z_1)\int_a^{x_1} \cdots (Z_{k-2})\int_a^{x_{k-2}} (Z_{k-1})\int_a^{x_{k-1}} f. \tag{2}$$

(see Bullen [3, Theorem 16]).

Russell [15] introduced the kth order generalisation of the classical concept of functions of bounded variation which he calls functions of bounded kth variation, BV_k functions.

Let f be a real-valued function defined in the closed interval $[a,b]$ and let k be a positive integer greater than one. If $x_0, x_1, \ldots, x_k$ are any $k+1$ distinct points, not necessarily in linear order, in $[a,b]$, then the *kth divided difference* of f is defined by

$$Q_k(f; x_0, x_1, \ldots, x_k) = \sum_{i=0}^{k}\left\{f(x_i)\Big/\prod_{\substack{j=0\\ j\neq i}}^{k}(x_i - x_j)\right\}.$$

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If, for all choices of distinct points $x_0, x_1, \dots, x_k$ in the interval $[a,b]$ we have $Q(f; x_0, x_1, \dots, x_k) \geqslant 0$, then f is called k-convex on $[a,b]$. The number

$$V_k(f; a, b) = \sup_{\pi} \sum_{i=0}^{n-k} (x_{i+k} - x_i) \big| Q_k(f; x_i, x_{i+1}, \dots, x_{i+k}) \big|,$$

where the supremum is taken for all π-subdivisions in $[a,b]$ of the form $a \leqslant x_0 < x_1 < \dots < x_n \leqslant b$, is called the *total kth variation* of f on $[a,b]$. If $V_k(f; a,b) < +\infty$, then f is said to be of *bounded kth variation*, BV_k on $[a,b]$ and we write $f \in BV_k[a,b]$.

In view of Therorm 1 of Russell [**17**], the class $BV_k[a,b]$ is given by

$$BV_k[a,b] = \{f : f = f_1 - f_2\} \tag{3}$$

where f_1 and f_2 are $0-, 1-, \dots, k$-convex functions having right and left $(k-1)$th ordinary derivatives at a and b respectively.

So, by Theorem 7 of Bullen [**2**], $f^{(k-1)}$ exists n.e. in $[a,b]$. Consequently, by Theorems 9 and 12 of Russell [**15**], f^{k-1} is BV on E, where $[a,b] \backslash E$ is countable.

Again by Russell [**18**], if $f \in BV_k[a,b]$ and $k \geqslant 1$, then $F(x) = \int_a^x f(t)dt \in BV_{k+1}[a,b]$.

Das and Lahiri [**10**] introduced the definition of absolutely kth continuous functions, AC_k functions, and showed that every AC_k function is BV_k. De Sarkar and Das [**11**] showed that $f \in BV_{k+1}[a,b]$ implies $f \in AC_k[a,b]$, for $k \geqslant 1$. The present authors [**9**] showed that the first integral of an AC_k function is AC_{k+1}, for $k \geqslant 1$ and also, that every k-fold Lebesgue integral is AC_k. An equivalent descriptive definition of the k-fold integral given by them is as follows:

A function f on $[a,b]$ is L^k-integrable on $[a,b]$ if there is a function F on $[a,b]$ such that:

(i) $F^{(k)}(x) = f(x)$ a.e. in $[a,b]$ and
(ii) F is AC_k on $[a,b]$.

The function F (thus uniquely determined except for a polynomial of degree $k-1$, Das and Lahiri [**10**, Theorem 2]) is called the L^k*-integral* of f on $[a,b]$.

It is desirable to reproduce the definitions of AC_k functions and Riemann* derivative for easy reference.

The function f is said to be *absolutely kth continuous*, AC_k on $[a,b]$ if, for arbitary $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that, for any system $\{x_{i,j} \in [a,b] : i = 1, 2, \dots, n;\ j = 0, 1, \dots, k\}$ with $\sum_{i=1}^{n} (x_{i,k} - x_{i,0}) < \delta(\varepsilon)$ and with $x_{i,j} < x_{i,j+1}$ and $x_{i,k} \leqslant x_{i+1,0}$, the inequality

$$\sum_{i=1}^{n} (x_{i,k} - x_{i,0}) \left| Q_k(f; x_{i,0}, x_{i,1}, \dots, x_{i,k}) \right| < \varepsilon$$

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holds.

Let $x, x_1, \ldots, x_k$ be points of $[a,b]$ and let $h_i = x_i - x$, for $i = 1, 2, \ldots, k$, with $0 < |h_1| < |h_2| < \cdots < |h_k|$. Then define the *k th Riemann* derivative* of f at x by

$$D^k f(x) = k! \lim_{h_k \to 0} \quad \lim_{h_{k-1} \to 0} \quad \cdots \quad \lim_{h_1 \to 0} Q_k(f; x, x_1, \ldots, x_k)$$

if the iterated limit exists. The *right* and the *left Riemann* derivatives* $D^k_+ f(x)$ and $D^k_- f(x)$, are defined in the obvious way. Taking lim sup (respectively lim inf) at each stage, we get the upper derivative $\overline{D}^k f(x)$ (respectively the lower derivative $\underline{D}^k f(x)$). The one sided derivatives $\overline{D}^k_+ f(x)$, $\underline{D}^k_+ f(x)$ and so on are obtained in the usual way. It is worth noting that simply $D^k_+ f(x) = D^k_- f(x)$ does not ensure the existence of $D^k f(x)$. However, if in addition, $D^{k-1} f(x)$ exists, the existence of $D^k f(x)$ is ensured. Also, if $D^{k-1} f(x)$ exists, then $\underline{D}^k f(x) = \inf\{\underline{D}^k_+ f(x), \underline{D}^k_- f(x)\}$ and $\overline{D}^k f(x) = \sup\{\overline{D}^k_+ f(x), \overline{D}^k_- f(x)\}$. Note that, if $f^{(k)}(x)$ exists, then $D^k f(x)$ exists and equals $f^{(k)}(x)$. The converse is true only when $k = 1$.

The purpose of the present paper is to formulate an integration by parts formula for the P^k-integral, namely, if f is P^k-integrable and G is BV_k on $[a,b]$, then fG is P^k-integrable. The process involves a simple and up-dated version of the integration by parts formula for the Z_{k-1} integral of Bergin [**1**]. Furthermore, it is observed that G can be allowed to be of bounded essential kth variation as defined by De Sarkar and Das [**13**].

2. Integration by parts

We shall prove the following integration by parts formula for the P^k-integral.

Theorem 1. *Let $k > 1$. Let f be P^k-integrable on $[a,b]$ and let $F(x) = (P^k)\int_a^x f$. If $G \in BV_k[a,b]$, then fG is P^k-integrable, and*

$$(P^k)\int_a^x fG + (P)\int_a^b (P^{k-1})\int_a^x D^{k-1}FG' = F(b)G(b) - F(a)G(a) + \sum_{r=1}^{k-1} (-1)^r \binom{k-1}{r} (L^r)\int_a^b FG^{(r)}.$$

In the process of the proof we shall also obtain the following theorem.

Theorem 2. (see [**1**, Proposition 5.1]) *Let $k > 1$. Let f be Z_{k-1}-integrable on $[a,b]$, and let $F(x) = (Z_{k-1})\int_a^x f$. If $G \in BV_k[a,b]$, then fG is Z_{k-1}-integrable, and*

$$(Z_{k-1})\int_a^b fG + (Z_{k-2})\int_a^b FG' = F(b)G(b) - F(a)G(a).$$

We remark that if Theorems 1 and 2 hold for G_1 and G_2, then they hold for all $\lambda_1 G_1 + \lambda_2 G_2$ where λ_1 and λ_2 are real constants. In view of (3) we can therefore assume that G and G' are non-negative; however in the case $k = 2$, G' exists n.e. in $[a, b]$.

We first prove two lemmas.

LEMMA 1. *Let $k > 1$, let M be a function on $[a, b]$ such that $M^{(k-1)}$ is continuous on $[a, b]$, and let $G \in BV_k[a, b]$. Define*

$$S(x) = M(x)G(x) + \sum_{r=1}^{k-1} (-1)^r \binom{k-1}{r} (L^r) \int_a^x MG^{(r)}, \quad a \leqslant x \leqslant b.$$

Then

$$S^{(k-1)}(x) = M^{(k-1)}(x)G(x) \text{ for all } x \text{ in } [a, b].$$

PROOF: The integrals on the right exist. In particular, $(M, G) \in RS_r^*[a, b]$, $1 \leqslant r \leqslant k$ (Russell [**16**] and/or Das and Das [**7**]) and

$$(4) \qquad (r-1)!(RS_r^*) \int_a^b M \frac{d^r G(x)}{dx^{r-1}} = (R) \int_a^b MG^{(r)} = (L) \int_a^b MG^{(r)}.$$

Using induction, it is not difficult, (see the proof of Lemma 5.4 of Bergin [**1**]), to show that

$$S^{(p)} = \sum_{r=0}^{p} (-1)^r \binom{k-1+r-p-1}{r} M^{(p-r)} G^{(r)} + \sum_{r=p+1}^{k-1} (-1)^r \binom{k-1}{r} (L^{r-p}) \int_a^x MG^{(r)},$$

for $p = 0, 1, \ldots, k-3$.

For $p = k-3$, we have

$$S^{(k-3)} = \sum_{r=0}^{k-3} (-1)^r (r+1) M^{(k-3-r)} G^{(r)} + (-1)^{k-2}(k-1)(L)\int_a^x MG^{(k-2)} + (-1)^{k-1}(L^2)\int_a^x MG^{(k-1)}$$

$$= \sum_{r=0}^{k-3} (-1)^r (r+1) M^{(k-3-r)} G^{(r)} + (-1)^{k-2}(k-2)(L)\int_a^x MG^{(k-2)} + (-1)^{k-2}(L^2)\int_a^x M'G^{(k-2)}$$

$$= \sum_{r=0}^{k-4} (-1)^r (r+1) M^{(k-3-r)} G^{(r)} + (-1)^{k-3}(k-3)(L)\int_a^x M'G^{(k-3)} + (-1)^{k-3}(L)\int_a^x M'G^{(k-3)} + (-1)^{k-2}(L^2)\int_a^x M'G^{(k-2)}.$$

Simplifying, we obtain

$$S^{(k-3)} = (L)\int_a^x \{\sum_{s=0}^{k-3} (-1)^s M^{(k-2-s)} G^{(s)} + (-1)^{k-2}(L)\int_a^x M'G^{(k-2)}\}.$$

Hence

$$S^{(k-2)} = \sum_{s=0}^{k-3} (-1)^s M^{(k-2-s)} G^{(s)} + (-1)^{k-2}(L)\int_a^x M'G^{(k-2)} = (L)\int_a^x M^{(k-1)}G,$$

using integration by parts. Since $M^{(k-1)}$ and G are continuous, it follows that

$$S^{(k-1)} = M^{(k-1)}G$$

and thus the lemma is proved. ∎

LEMMA 2. *Let $k > 1$, let $D^{k-1}M$ exist on $[a,b]$ and let $G \in BV_k[a,b]$. Then there is a function S on $[a,b]$ such that, for all x in $[a,b]$*

$$D^{k-1}S(x) = D^{k-1}M(x)G(x).$$

PROOF: Let $x \in [a,b]$ be arbitary. Define $\overline{M}(t) = M(t) - P(t)$, where $P(t) = \sum_{r=0}^{k-1} ((t-x)^r/r!) D^r M(x)$. Clearly then, $D^r \overline{M}(x) = 0$ for $r = 0, 1, \ldots, k-1$ so that $\overline{M}(t) = o\Big((t-x)^{k-1}\Big)$ as $t \to x$. Set

$$S(t) = M(t)G(t) + \sum_{r=1}^{k-1} (-1)^r \binom{k-1}{r} (L^r) \int_a^t MG^{(r)};$$

$$\overline{S}(t) = \overline{M}(t)G(t) + \sum_{r=1}^{k-1} (-1)^r \binom{k-1}{r} (L^r) \int_a^t \overline{M}G^{(r)}.$$

Then

$$(S - \overline{S})(t) = P(t)G(t) + \sum_{r=1}^{k-1} (-1)^r \binom{k-1}{r} (L^r) \int_a^t PG^{(r)}.$$

Since $P^{(k-1)}(t) = D^{k-1}M(x)$ for all t in $[a,b]$, we can apply Lemma 1 so as to obtain

$$(5) \qquad (S - \overline{S})^{(k-1)}(t) = D^{k-1}M(x)G(t).$$

In particular,

$$(S - \overline{S})^{(k-1)}(x) = D^{k-1}M(x)G(x),$$

which yields

$$(6) \qquad D^{k-1}(S - \overline{S})^{(x)} = D^{k-1}M(x)G(x).$$

Now, since $\overline{M}(t) = o\Big((t-x)^{k-1}\Big)$ as $t \to x$ it follows that $\overline{S}(t) = o\Big((t-x)^{k-1}\Big)$ as $t \to x$ so that $D^{k-1}\overline{S}(x) = \overline{S}_{(k-1)}(x) = 0$. Consequently, from (6), we obtain

$$D^{k-1}S(x) = D^{k-1}M(x)G(x).$$

This proves the lemma. ∎

COROLLARY 1. *Let $M, G, S, \overline{S}$ be as above. Then for $x, x_1, \ldots, x_k$ in $[a,b]$,*

$$Q_k(S; x, x_1, \ldots, x_k) = Q_k(\overline{S}; x, x_1, \ldots, x_k) + D^{k-1}M(x) \int_0^1 \int_0^{y_1} \cdots \int_0^{y_{k-1}} G'(u_k)\, dy_k$$

where

$$u_k = (1 - y_1)x + (y_1 - y_2)x_1 + \cdots + (y_{k-1} - y_k)x_{k-1} + y_k x_k.$$

PROOF: From (5), $(S - \overline{S})^{(k-1)}$ is AC on $[a,b]$ and so the proof is a simple adaption of Theorem 16 of Russell [**15**]. ∎

We prove the case $k = 2$ of Theorems 1 and 2 separately in a lemma.

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LEMMA 3. *(a) Let f be P^2-integrable on $[a,b]$ and let $F(x) = (P^2)\int_a^x f$. If $G \in BV_2[a,b]$, then fG is P^2-integrable on $[a,b]$ and*

$$(P^2)\int_a^b fG + (P)\int_a^b (P)\int_a^b F'G' = F(b)G(b) - F(a)G(a) - (L)\int_a^b FG'.$$

(b) Let f be Z_1-integrable on $[a,b]$ and let $F(x) = (Z_1)\int_a^x f$. If $G \in BV_2[a,b]$, then fG is Z_1-integrable on $[a,b]$ and

$$(Z_1)\int_a^b fG + (Z_0)\int_a^b FG' = F(b)G(b) - F(a)G(a).$$

(We recall that the Z_0-integral is the classical P-integral.)

PROOF: (a) Let M be any P^2-major function of f on $[a,b]$. By Lemma 2, there is $S = MG - (L)\int_a^t MG'$ such that

$$S'(x) = M'(x)G(x) \quad \text{for all } x \text{ in } [a,b].$$

It is also clear that $S(a) = S'(a) = 0$. Again, by Corollary 1, for $x \in [a,b]$ where $G'(x)$ exists, we have, for $x_1, x_2 \in [a,b]$, that

$$Q_2(S;x,x_1,x_2) = Q_2(\overline{S};x,x_1x_2) + M'(x)\int_0^1\int_0^{y_1} G'(u_2)\,dy_2,$$

where $u_2 = (1-y_1)x + (y_1-y_2)x_1 + y_2x_2$, and $\overline{S} = \overline{M}G - (L)\int_a^t \overline{M}G'$. Since $\overline{M}(t) = M(t) - \{M(x) + (t-x)M'(x)\}$, we have $\overline{S} = \overline{M}G + o\left((t-x)^2\right)$ as $t \to x$.

Since the functions S and $\overline{S}$ are continuous, we may assume $x_1 = x + h$ and $x_2 = x + 2h$. Then applying Lemma 4 of Russell [19] and noting that $\Delta_h^2\overline{S}(x) = h^2 2!\, Q_2(\overline{S};x,x_1,x_2)$, we obtain

$$\begin{aligned}
2!\,Q_2(S;x,x_1,x_2) &= \frac{1}{h^2}\Delta_h^2\overline{S}(x) + 2!\,M'(x)\int_0^1\int_0^{y_1} G'(u_2)\,dy_2 \\
&= \overline{M}(x_2)\frac{\Delta_h^2 G(x)}{h^2} + \frac{1}{h}\Delta_h^1\overline{M}(x_1)\frac{1}{h}\Delta_h^1 G(x) \\
&\quad + \frac{1}{h^2}\Delta_h^2\overline{M}(x)G(x) + 2M'(x)\int_0^1\int_0^{y_1} G'(u_2)\,dy_2 + o(1).
\end{aligned}$$

In view of (3), we may assume $G(x)$ and $G'(x)$ both non-negative. We note that $G'(x)$ exists n.e. in $[a,b]$. Since $\overline{M}(x) = \overline{M}'(x) = 0$ and $\overline{S}(x)$ exists, we have

$$\underline{D}^2 S(x) \geqslant \underline{D}^2 M(x)G(x) + M'(x)G'(x) \qquad \text{n.e. in } [a,b].$$

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Consequently, since M is a P^2-major function of f on $[a,b]$, we have

$$\underline{D}^2 S(x) \geqslant f(x)G(x) + M'(x)G'(x) \qquad \text{a.e. in } [a,b]$$

and $\underline{D}^2 S(x) > -\infty$ n.e. in $[a,b]$. Since F' is the Z_1-integral (see(1)) and M' is the Z_1-major function of f on $[a,b]$, it follows that

$$(7) \qquad \begin{aligned} &\underline{D}^2 S(x) \geqslant f(x)G(x) + F'(x)G'(x) && \text{a.e. in } [a,b];\\ &\underline{D}^2 S(x) > -\infty && \text{n.e. in } [a,b]. \end{aligned}$$

Obviously then $S(x)$ is a P^2-major fucntion of $fG + F'G'$. We recall that G' exists n.e. in $[a,b]$ and for the P^2-integrability of a function it need only be finite or indeed defined a.e.

Similarly, for any P^2-minor function m, the function

$$s = mG - (L)\int_a^t mG'$$

is a P^2-monor function of $fG + F'G'$. Given $\varepsilon > 0$ we can choose M and m such that $0 \leqslant S(b) - s(b) < \varepsilon$. It therefore follows that $fG + F'G'$ is P^2-integrable, and

$$(P^2)\int_a^b (fG + F'G') = [FG]_a^b - (L)\int_a^b FG'.$$

It is obvious that F' is Z_0-integrable, that is, P-integrable and G' is of bounded essential variation on $[a,b]$. Hence $F'G'$ is P-integrable (see Bullen [**6**, Section 12, p357]), and $(P)\int_a^b (P)\int_a^x F'G' = (P^2)\int_a^b F'G'$.

Consequently, fG is P^2-integrable and

$$(P^2)\int_a^b fG + (P)\int_a^b (P)\int_a^x F'G' = F(b)G(b) - F(a)G(a) - (L)\int_a^b FG'.$$

This proves (a).

(b) Now let f be Z_1-integrable and let $F(x) = (Z_1)\int_a^x f$. If M is a pre-majorant and m is a pre-minorant, then define $S = MG - (L)\int_a^x MG'$ and $s = mG - (L)\int_a^x mG'$.

It is sufficient to note that S' is a Z_1-major function and s' is a Z_1-minor function of $fG + FG'$, and

$$(Z_1)\int_a^b (fG + FG') = [FG]_a^b.$$

Since F is Z_0-integrable and G' is of bounded essential variation, it follows that FG' is Z_0-integrable, and since $(Z_0)\int_a^b FG' = (Z_1)\int_a^b FG'$, we have

$$(Z_1)\int_a^b fG + (Z_0)\int_a^b FG' = F(b)G(b) - F(a)G(a).$$

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This proves (b). ∎

Proof of Theorem 1: $(k > 2)$ Let M be any P^k-major function of f on $[a,b]$ so that $D^{k-1}M$ exists everywhere in $[a,b]$ and $M(a) = D^rM(a) = 0$, for $r = 1,2,\dots,k-1$. By Lemma 2, there is

$$S(t) = M(t)G(t) + \sum_{r=1}^{k-1}(-1)^r\binom{k-1}{r}(L^r)\int_a^t MG^{(r)}$$

such that, for all t in $[a,b]$,

$$D^{k-1}S(t) = D^{k-1}M(t)G(t).$$

Since $M(t) = o\Big((t-a)^{k-1}\Big)$ as $t \to a$, it follows that $S(a) = D^rS(a) = 0$ for $r = 1,2,\dots,k-1$. For arbitary but fixed $x \in [a,b]$ define (as in the proof of Lemma 2)

$$\overline{M}(t) = M(t) - P(t), \qquad P(t) = \sum_{r=0}^{k-1}\frac{(t-x)^r}{r!}D^rM(x),$$

and

$$\overline{S}(t) = \overline{M}(t)G(t) + \sum_{r=1}^{k-1}(-1)^r\binom{k-1}{r}(L^r)\int_a^t \overline{M}G^{(r)}.$$

Since $\overline{M}(t) = o\Big((t-x)^{k-1}\Big)$ as $t \to x$, it follows that $S(t) = o\Big((t-x)^{k-1}\Big)$ as $t \to x$ (see the proof of Lemma 2).

By Corollary 1, since S and $\overline{S}$ are continuous in $[a,b]$, using the relation $\Delta_h^k\overline{S}(x) = h^k k!\,Q_k\big(\overline{S};x,x_1,\dots,x_k\big)$, Russell [**19**, p.458], we obtain

$$\begin{aligned} k!\,Q_k(S;x,x_1,\dots,x_k) &= \frac{1}{h^k}\Delta_h^k\big(\overline{M}G\big)(x) \\ &\qquad + k!\,D^{k-1}M(x)\int_0^1\int_0^{y_1}\cdots\int_0^{y_{k-1}} G'(u_k)\,dy_k + o(1) \\ &= \sum_{s=0}^{k}\binom{k}{s}\Delta_h^s\overline{M}(x+(k-s)h)\Delta_h^{k-s}G(x)/h^k \\ &\qquad + k!\,D^{k-1}M(x)\int_0^1\int_0^{y_1}\cdots\int_0^{y_{k-1}} G'(u_k)\,dy_k + o(1), \end{aligned}$$

using Lemma 4 of Russell [**19**]. By (3), $G(x)$ can be taken as non-negative. Since $\overline{M}(x) = D^r\overline{M}(x) = 0$ for $r = 1,\dots,k-1$, and since $D^{k-1}S(x)$ exists, it follows that

$$\begin{aligned} \underline{D}^kS(x) &\geqslant \underline{D}^k\overline{M}(x)G(x) + D^{k-1}M(x)G'(x) \\ &= \underline{D}^kM(x)G(x) + D^{k-1}M(x)G'(x) \qquad \text{for all } x \text{ in } [a,b]. \end{aligned}$$

Since M is a P^k-major function of f, we obtain

$$\underline{D}^k S(x) \geqslant f(x)G(x) + D^{k-1}M(x)G'(x) \quad \text{a.e. in } [a,b];$$
$$\underline{D}^k S(x) > -\infty \quad \text{n.e. in } [a,b].$$

Also, $D^{k-1}M$ is a Z_{k-1}-major function of f on $[a,b]$ and $D^{k-1}F$ is the Z_{k-1}-integral, relation (1), and so we have

$$(8) \qquad \begin{aligned} &\underline{D}^k S(x) \geqslant f(x)G(x) + D^{k-1}F(x)G'(x) \quad &&\text{a.e. in } [a,b];\\ &\underline{D}^k S(x) > -\infty &&\text{n.e. in } [a,b]. \end{aligned}$$

Consequently, S is a P^k-major function of $fG + D^{k-1}FG'$ on $[a,b]$. Similarly, the function

$$s(t) = m(t)G(t) + \sum_{r=1}^{k-1} (-1)^r \binom{k-1}{r} (L^r) \int_a^t mG^{(r)}$$

is a P^k-major function of $fG + D^{k-1}FG'$ on $[a,b]$. Hence, $fG + D^{k-1}FG'$ is P^k-integrable on $[a,b]$ and

$$(P^k) \int_a^b (fG + D^{k-1}FG') = F(b)G(b) - F(a)G(a) + \sum_{r=1}^{k-1} (-1)^r \binom{k-1}{r} (L^r) \int_a^b FG^{(r)}.$$

If $k = 3$, we have that $fG + D^2FG'$ is P^3-integrable on $[a,b]$. Also, since D^2F is P^2-integrable and $G' \in BV_2[a,b]$, using Lemma 3(a), D^2FG' is P^2-integrable. By Theorem 15 of Bullen [**3**], D^2FG' is P^3-integrable on $[a,b]$ and

$$(P^3) \int_a^b D^2FG' = (P) \int_a^b (P^2) \int_a^x D^2FG'.$$

Hence fG is P^3-integrable on $[a,b]$ and

$$(P^3) \int_a^b fG + (P) \int_a^b (P^2) \int_a^x D^2FG' = F(b)G(b) - F(a)G(a) + \sum_{r=1}^{2} (-1)^r \binom{2}{r} (L^r) \int_a^b FG^{(r)}.$$

So, using induction, since $D^{k-1}F$ is P^{k-1}-integrable and $G' \in BV_{k-1}[a,b]$ we have that $D^{k-1}FG'$ is P^{k-1}-integrable. By Theorem 15 of Bullen [**3**], $D^{k-1}FG'$ is P^k-integrable and $(P^k) \int_a^b D^{k-1}FG' = (P) \int_a^b (P^{k-1}) \int_a^x D^{k-1}FG'$. Hence fG is P^k-integrable, and

$$(P^k) \int_a^b fG + (P) \int_a^b (P^{k-1}) \int_a^x D^{k-1}FG' = [FG]_a^b + \sum_{r=1}^{k-1} (-1)^r \binom{k-1}{r} (L^r) \int_a^b FG^{(r)},$$

so the theorem is proved. ∎

We note that $(P)\int_a^b (P^{k-1})\int_a^x D^{k-1}FG'$ is stronger than $(P^k)\int_a^b D^{k-1}FG'$, $k \geqslant 2$, since there are functions which are Z_{k-1}-integrable on $[a,b]$ but not Z_{k-2}-integrable on $[a,b]$. Furthermore, since G' can be taken to be non-negative whenever it exists, the second integral on the left can be replaced by $(L^k)\int_a^b D^{k-1}FG'$ whenever $D^{k-1}F$ is non-negative (see Proposition 4.9 of Bergin [**1**]).

Proof of Theorem 2: $(k > 2)$ Let f be Z_{k-1}-integrable and let $F(x) = (Z_{k-1})\int_a^x f$. If M is a pre-majorant and m is a pre-minorant for the Z_{k-1}-integral of f on $[a,b]$, then $D^{k-1}M$ and $D^{k-1}m$ are respectively Z_{k-1}-major and Z_{k-1}-minor functions of f on $[a,b]$. Define S and s as in the proof of Theorem 1 $(k > 2)$. Then $D^{k-1}S$ and $D^{k-1}s$ are Z_{k-1}-major and minor functions of $fG+FG'$ and so $fG+FG'$ is Z_{k-1}-integrable on $[a,b]$. Obviously then,

$$(Z_{k-1})\int_a^b (fG + FG') = [FG]_a^b.$$

In view of Lemma 3(b), we can assume that if $f^\star$ is Z_{k-2}-integrable on $[a,b]$ and $G^\star \in BV_{k-1}[a,b]$, then $f^\star G^\star$ is Z_{k-2}-integrable. Here, since F is Z_{k-2}-integrable and G' is BV_{k-1}, it follows that FG' is Z_{k-2}-integrable on $[a,b]$. Consequently, by Propositions 4.8 and 4.10 of Bergin [**1**], fG is Z_{k-1}-integrable on $[a,b]$ and

$$(Z_{k-1})\int_a^b fG + (Z_{k-2})\int_a^b FG' = [FG]_a^b.$$

This proves the Theorem. ∎

We remark that the proofs of Lemma 3(b) and Theorem 2 of this paper seem to be simpler than those of Propositions 5.1(a) and 5.1(b) of Bergin [**1**]. However, we cannot obtain Propositions 5.6 and 5.8 of Bergin [**1**] with $G \in BV_{k-1}[a,b]$ and $G \in BV[a,b]$ respectively. But the aim of the integration by parts formula is to express $(I)\int_a^b fG$ in terms of stronger integrals and thus our consideration is consistent.

Since D^k- and $\mathcal{P}^k$-integrals of De Sarkar and Das [**14**] and of Bullen and Mukhopadhyay [**4**] are equivalent to the P^k-integral, Theorem 1 above also provides an integration by parts formula for each of these integrals.

Furthermore, the L^r-integrals, (throughout), $1 \leqslant r \leqslant k-1$, can be replaced by the r-fold Riemann integral, R^r-integral, say (see (4)). Thus we obtain:

Theorem 3. *Let* $k > 1$. *Let* f *be* P^k*-integrable on* $[a,b]$ *and let* $F(x) =$

https://doi.org/10.1017/S0004972700004093 Published online by Cambridge University Press

$(P^k)\int_a^x f$. *If* $G \in BV_k[a,b]$, *then* fG *is* P^k*-integrable on* $[a,b]$ *and*

$$(P^k)\int_a^b fG + (P)\int_a^b (P^{k-1})\int_a^x D^{k-1}FG' = [FG]_a^b + \sum_{r=1}^{k-1}(-1)^r\binom{k-1}{r}(R^r)\int_a^b FG^{(r)}.$$

De Sarkar and Das [13] gave the definition of functions of bounded essential kth variation, BAV_k functions. It has been proved that a function f is BAV_k on $[a,b]$ if and only if it is BV_k on $E \subset [a,b]$ with $mE = b-a$. Also, to each $f \in BAV_k[a,b]$ there exists $F \in BV_k[a,b]$ such that $F = f$ on some $E \subset [a,b]$ with $mE = b-a$. We shall call F an *extension* of f.

Theorems 1, 2 and 3 can easily be extended to $G \in BAV_k[a,b]$. We demonstrate an analogue of Theorem 1 only; the others follow similarly.

THEOREM 4. *Let* $k > 1$. *Let* f *be* P^k*-integrable on* $[a,b]$ *and let* $F(x) = (P^k)\int_a^x f$. *If* $G \in BAV_k[a,b]$, *then* fG *is* P^k*-integrable on* $[a,b]$. *If* $\overline{G}$ *is the extension of* G, *then*

$$(P^k)\int_a^b fG + (P)\int_a^b (P^{k-1})\int_a^x D^{k-1}F\overline{G}' = [F\overline{G}]_a^b + \sum_{r=1}^{k-1}(-1)^r\binom{k-1}{r}(L^r)\int_a^b F\overline{G}^{(r)}.$$

PROOF: We proceed as in the proofs of Lemma 3(a) and Theorem 1 $(k > 2)$ with G replaced by $\overline{G} \in BV_k[a,b]$ and obtain relations analogous to (7) and (8), namely

$$\begin{aligned} &\underline{D}^k S(x) \geqslant f(x)\overline{G}(x) + D^{k-1}F(x)\overline{G}'(x) && \text{a.e. in } [a,b];\\ &\underline{D}^k S(x) > -\infty && \text{n.e. in } [a,b], \end{aligned}$$

for $k > 1$. Since $\overline{G}(x) = G(x)$ a.e. in $[a,b]$, we obtain, for $k > 1$

$$\begin{aligned} &\underline{D}^k S(x) \geqslant f(x)G(x) + D^{k-1}F(x)\overline{G}'(x) && \text{a.e. in } [a,b];\\ &\underline{D}^k S(x) > -\infty && \text{n.e. in } [a,b]. \end{aligned} \tag{9}$$

Obviously then, $S(x)$ is a P^k-major $(k > 1)$ function of $fG + D^{k-1}F\overline{G}'$ on $[a,b]$. The rest is clear and thus the theorem is proved. ∎

We remark that Theorem 6 of Bullen [5] can now be stated as follows:

Let $f \in P^{\star}_{ap}(a,b)$ *and let* $F(x) = \left(P^{\star}_{ap}\right) \int_a^x f$. *If* $F \in P(a,b)$ *and* $G \in BV_2[a,b]$, *then* fG *is* $P^{\star}_{ap}$*-integrable and*

$$\left(P^{\star}_{ap}\right) \int_a^b fG = F(b)G(b) - F(a)G(a) - (P) \int_a^b FG'.$$

(The integral on the right exists, see Section 12 of Bullen [**6**].)

Recently, De Sarkar, Das and Lahiri [**12**] introduced approximate extensions of D^k- and $\mathcal{P}^k$-integrals, the AD^k- and $A\mathcal{P}^k$-integrals respectively. The present authors [**8**] introduced approximate extensions of the P^k- and C_kD-integrals, the AP^k- and A_kD-integrals respectively. Integration by parts formulae for such approximate integrals will be considered in a subsequent paper.

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