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ON THE STRUCTURE OF PROJECTIONS AND IDEALS OF CORONA ALGEBRAS

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0. Introduction. If K is the set of all compact bounded operators and $L(H)$ is the set of all bounded operators on a separable Hilbert space H, then $L(H)$ is the multiplier algebra of K. In general we denote the multiplier algebra of a C^*-algebra A by $M(A)$. For more information about $M(A)$, readers are referred to the articles **[1]**, [3], [7], **[9]**, **[14]**, **[18]**, **[20]**, **[23]**, **[26]**, **[27]**, among others. It is well known that in the Calkin algebra $L(H)/K$ every nonzero projection is infinite. If we assume that A is σ-unital (nonunital) and regard the corona algebra $M(A)/A$ as a generalized case of the Calkin algebra, is every nonzero projection in $M(A)/A$ still infinite? Another basic question can be raised: How does the (closed) ideal structure of A relate to the (closed) ideal structure of $M(A)/A$?

In the first part of this note (Sections 1 and 2) we shall give an affirmative answer for the first question if A is a simple σ-unital (nonunital) C^*-algebra with FS. As a consequence, the K-groups of $M(A)/A$ for certain simple C^*-algebras with FS are described. We shall prove that every hereditary C^*-subalgebra of $M(A)$ is the closed linear span of its projections if A is σ-unital with FS. Also, the Murray-von Neumann equivalence classes of projections in $M(A)/A$ are described for separable matroid algebras. In the second part of this note we shall relate the (closed) ideal structure of A to the ideal structure of the corona algebra $M(A)/A$. One way that ideals of $M(A)/A$ arise is via a lifting of ideals from the ideal lattice of A to the ideal lattice of $M(A)$ and then to the ideal lattice of $M(A)/A$; i.e.,

$$I \mapsto A + M(A, I) \mapsto M(A, I)/I.$$

We give necessary and sufficient conditions from different perspectives for the liftability of a nontrivial ideal of A.

We fix some notation first. For a C^*-algebra A we denote the self-adjoint part of A by $A_{s.a.}$, the positive part of A by A_+ and the Banach space double dual of A by A^{**}. '$\sim$' denotes the Murray-von Neumann equivalence of two projections and '$\lesssim$' denotes 'is equivalent to a subprojection of'. We denote by her$(\cdot)$ the hereditary C^*-subalgebra of A generated by $(\cdot)$.

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1. Projections in $M(A)/A$. A C^*-algebra B is called purely infinite if the closure of bBb contains an infinite projection for each $0 \neq b \in B_+$. This definition is an extension of the definition in [**13**, 1.5] for simple C^*-algebras. We are very grateful to L. G. Brown and G. Pedersen for generous help in proving Theorem (1.1), which strengthens our original version.

1.1. THEOREM. *If A is a σ-unital C^*-algebra with* FS, *then for every hereditary C^*-subalgebra B of $M(A)$, positive linear combinations of projections in B are norm dense in B_+. Consequently every hereditary C^*-subalgebra of $M(A)/A$ is the closed linear span of images of projections in $M(A)$.*

Proof. First, if B is a hereditary C^*-subalgebra of $M(A)$ containing A, the same arguments as in the proof of Theorem (2.2) of [**27**], with some minor modifications, prove that the conclusion is true. We leave it to the reader to check.

Second, let B be any hereditary C^*-subalgebra of $M(A)$ and H any nonzero positive element of B. For any $\epsilon > 0$ we define $p_\epsilon = E_{(\epsilon,\infty)}(H)$, where $E_{(\epsilon,\infty)}(H)$ is the spectral projection of H in A^{**} corresponding to (ϵ, ∞). Then p_ϵ is an open projection of A. Let

$$A_\epsilon = \mathrm{her}(p_\epsilon) \quad \text{and} \quad B_\epsilon = M(A) \cap A_\epsilon^{**}.$$

Then $A_\epsilon \subset B_\epsilon \subset B$, A_ϵ satisfies the same hypotheses as A does, and B_ϵ is a hereditary C^*-subalgebra of $M(A_\epsilon)$.

Define a continuous function on R as follows:

$$f_\epsilon(t) = \begin{cases} 0, & \text{if } \quad t \leqq \epsilon, \\ \text{linear}, & \text{if } \epsilon \leqq t \leqq 2\epsilon \\ t, & \text{if } \quad 2\epsilon \leqq t. \end{cases}$$

Let $H_\epsilon = f_\epsilon(H)$. Then H_ϵ is in B_ϵ and $\|H_\epsilon - H\| \leqq \epsilon$. Applying the conclusion in the first paragraph, we obtain a positive linear combination of projections in $B_\epsilon \subset B$ approximating H_ϵ within ϵ. Hence this linear combination of projections approximates H within 2ϵ. This completes the proof of the first sentence.

Let $\bar{B}$ be any nonzero hereditary C^*-subalgebra of $M(A)/A$. Apply the above to $B = \pi^{-1}(\bar{B})$ to obtain the last sentence of the theorem.

The following lemma is a consequence of the Riesz decomposition property of C^*-algebras with FS ([**27**]).

1.2. LEMMA. *If A is a simple C^*-algebra with* FS *and p, q are two nonzero projections in A, then there exist an integer n and mutually orthogonal projections $r_i (1 \leqq i \leqq n)$ in A such that $p = \sum_{i=1}^n r_i$ and $r_i \lesssim q$ for all $1 \leqq i \leqq n$.*

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Proof. Since q is full, there are x_i's and y_i's in A such that

$$\left\| \sum_{i=1}^{n} x_i q y_i - p \right\| \leqq \epsilon < 1.$$

By the same argument as in the proof of Theorem (2.3) of [**27**] we can show that

$$\sum_{i=1}^{n} r_i = p$$

for some projections r_i in A such that $r_i \lesssim q$ $(1 \leqq i \leqq n)$.

1.3. THEOREM. *If A is a σ-unital simple C^*-algebra with* FS, *then*

(a) *Every nonzero projection in $M(A)/A$ is infinite.*

(b) *$M(A)/A$ is purely infinite.*

Consequently every nonzero hereditary C^*-subalgebra of $M(A)/A$ contains a nonzero stable subalgebra.

Proof. If A is elementary, then the conclusions are well known. We assume that A is non-elementary. By Theorem (1.1) every nonzero projection of $M(A)/A$ has a nonzero subprojection which is the image of a projection in $M(A)$. Hence to prove (a), this suffices to show that $\pi(P)$ is infinite for any projection P in $M(A)\backslash A$. Similarly this suffices to prove (b).

Let P be a projection of $M(A)\backslash A$ and set $\bar{p} = \pi(P)$. Since A is σ-unital with FS, we can write $P = \sum_{i=1}^{\infty} e_i$ for some nonzero mutually orthogonal projections e_i in A. Since A is non-elementary with FS, A does not have minimal nonzero projections. By Lemma (1.2) we can find a nonzero projection p_1 such that $p_1 < e_2$ and $p_1 \sim f_1 < e_1$. For the same reason we can find a nonzero projection p_2 such that $p_2 < e_3$ and $p_2 \sim f_2 < p_1$. Recursively we can find nonzero projections p_i such that $p_i < e_{i+1}$ and $p_i \sim f_i < p_{i-1}$ for $i \geqq 1$ $(p_0 = e_1)$. Let v_i be a partial isometry in A such that $v_i v_i^* = p_i$ and $v_i^* v_i = f_i$ for $i \geqq 1$. Define

$$V = \sum_{i=1}^{\infty} (v_i + e_{i+1} - p_i).$$

Then V is a partial isometry in $M(A)$ such that

$$VV^* = P - e_1 \quad \text{and}$$

$$V^*V = f_1 + \sum_{i=2}^{\infty} (f_i + e_i - p_{i-1}) = P - \sum_{i=1}^{\infty} (p_{i-1} - fi).$$

Since $p_{i-1} - f_i \neq 0$ for $i \geqq 1$, $\sum_{i=1}^{\infty} (p_{i-1} - f_i)$ is in $M(A)\backslash A$. Thus

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$$\pi\left(\sum_{i=1}^{\infty}(p_{i-1} - f_i)\right) \neq \bar{0}.$$

Thus $\pi(V)^*\pi(V) < \bar{p}$ and $\pi(V)\pi(V)^* = \bar{p}$.

In [**13**], the K-groups of simple purely infinite C^*-algebras were described. For B such a C^*-algebra, $D(B)\backslash\{[0]\}$ becomes a group under the group operation defined in [**13**]. $K_0(B)$ turns out to be isomorphic to $D(B)\backslash\{[0]\}$. $K_1(B)$ turns out to be isomorphic to the group $U(B)/U_0(B)$ without stabilizing, where $U(B)$ denotes the group of unitaries of $\tilde{B}$ and $U_0(B)$ denotes the path component containing the identity of $\tilde{B}$. The following corollary is an easy consequence of Theorem (1.3) and the results in [**13**].

1.4. COROLLARY. *If A is a σ-unital simple C^*-algebra with* FS *and a simple quotient $M(A)/A$, then*

$$K_0(M(A)/A) \cong D(M(A)/A)\backslash\{[0]\} \quad \textit{and}$$

$$K_1(M(A)/A) \cong U(M(A)/A)/U_0(M(A)/A).$$

1.5. *Remarks*. (1) All simple separable nonunital AF algebras, Bunce-Deddens algebras and all stabilized factors satisfy the conditions in Theorem (1.3).

(2) Examples satisfying the conditions in Corollary (1.4) and the following corollary can be derived from [**23**]. In [**23**], it was proved that if A is a simple separable nonunital AF algebra, then $M(A)/A$ is simple if and only if either A is elementary or A has a continuous scale. Note that in either of these cases it is obvious that the set $T(A)$ of tracial states of A is compact.

Combining recent results of [**22**] and [**23**], we obtain the following corollary:

1.6. COROLLARY. *If A is a σ-unital (nonunital) simple C^*-algebra with* FS *and a simple quotient $M(A)/A$, then $M(A)/A$ contains two isometries with orthogonal ranges. In particular, if A is* AF, *any two extensions of A by a C^*-algebra B can be added; and moreover $T(A)$ is compact.*

Proof. The first conclusion follows from Theorem (1.3) and [**4**, 3.12.1], or from Corollary (1.4). The consequences follow from the first sentence and results of [**22**].

2. Equivalence of projections in $M(A)/A$. In [**18**, 2.9] it was proved that if A is a separable nonunital matroid algebra, then the unitary group of $M(A)/A$ is connected if and only if A is finite. In [**18**, 3.1] it was proved that $M(A)/A$ is simple if A is a separable finite matroid algebra. If A is an infinite separable matroid algebra, it was proved in [**18**, 3.2] that $M(A)/A$

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has a unique nonzero proper closed ideal $\pi(J)$, where J is the shell ideal of

$$L(H_A) \cong M(A \otimes K);$$

i.e., the largest closed proper ideal of $L(H_A)$ ([**27**]).

If A is a nonunital separable matroid algebra, let $\bar{\tau}$ be the extension of the essentially unique trace τ on A to $M(A)$ ([**16**]). We shall use this notation without further comment. Note that it applies also to $A \otimes K$ and $M(A \otimes \mathrm{K})$. One question is asked: Does $\bar{\tau}(P) = \bar{\tau}(Q)$ imply $P \sim Q$ if P and Q are two projections in $M(A)$? We shall give a negative answer to the question as follows:

2.1. PROPOSITION. *If A is a separable nonunital matroid algebra, then there always exist projections $P \in M(A)\backslash A$ and $p \in A$ such that $\bar{\tau}(P) = \bar{\tau}(p)$. P and p can not be equivalent.*

Proof. First we claim that if the conclusion were not true, then any projection $\bar{q}$ in $M(A)/A$ with $\bar{q} \sim \bar{1}$ would be $\bar{1}$ itself. In fact, since $\bar{q} \sim \bar{1}$, there exists a projection Q in $\pi^{-1}(\bar{q})$ and a projection $p_0 \in A$ such that $Q \sim 1 - p_0$ by Lemma (2.8) of [**28**]. Then $\bar{\tau}(Q) = \bar{\tau}(1 - p_0)$ and so $\bar{\tau}(1 - Q) = \bar{\tau}(p_0)$. Hence $1 - Q \in A$ and $\bar{q} = \pi(Q) = \bar{1}$.

Secondly we claim that any nonzero projection $\bar{q} \in M(A)/A$ would be $\bar{1}$ if A is finite, and any projection

$$\bar{q} \in M(A)/A\backslash\pi(J)$$

would be $\bar{1}$ if A is infinite. Thus we reach a contradiction.

If A is finite, $M(A)/A$ is simple by [**18**, 3.1]. Let $\bar{q}$ be any nonzero projection in $M(A)/A$. Then $\bar{q}$ is infinite by Theorem (1.3). [**13**, 1.5] implies that $\bar{1} \lesssim \bar{q}$. Then $\bar{1} \leqq \bar{q}$ by the first claim and so $\bar{1} = \bar{q}$.

If A is infinite and $\bar{q} \in [M(A)/A]\backslash\pi(J)$, there exists a projection $Q \in M(A)\backslash J$ such that $\pi(Q) = \bar{q}$ by [**7**] or [**28**, Section 2]. It follows that $\bar{\tau}(Q) = \infty$. We show that $\bar{\tau}(Q) = \infty$ is necessary and sufficient for $Q \sim 1$ in $M(A)$. By Theorem (2.1) of [**27**],

$$Q \sim \sum_{i=1}^{\infty} p_i \otimes e_{ii}$$

for some nonzero projections $p_i \in A$. Fix any nonzero projection $p_0 \in A$. Since

$$\sum_{i=1}^{\infty} \tau(p_i \otimes e_{ii}) = \infty,$$

there are $n_i \nearrow \infty$ such that

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$$\tau(p_0 \otimes e_{ii}) \leqq \sum_{j=n_{i-1}+1}^{n_i} \tau(p_j \otimes e_{jj}) \quad \text{for each } i \geqq 1.$$

Then by [**16**, 2.9], there exists $v_i \in A$ for each $i \geqq 1$ such that

$$v_i v_i^* = p_0 \otimes e_{ii} \quad \text{and} \quad v_i^* v_i \leqq \sum_{j=n_{i-1}+1}^{n_i} p_j \otimes e_{jj}.$$

Let $V = \sum_{i=1}^{\infty} v_i$; then

$$V \in M(A), \quad VV^* = p_0 \otimes 1, \quad \text{and}$$

$$V^*V \leqq \sum_{i=1}^{\infty} p_i \otimes e_{ii} \sim Q.$$

Thus $p_0 \otimes 1 \lesssim Q$. On the other hand, $1 \sim p_0 \otimes 1$ by Theorem (2.5) of [**8**]. It follows from [**27**, 3.5] that $Q \sim 1$ and so $\bar{q} \sim \bar{1}$. By the first claim, $\bar{q} = \bar{1}$.

Although in general for two projections P and Q in $M(A)$, $\bar{\tau}(P) = \bar{\tau}(Q)$ does not imply $P \sim Q$, we have the following weaker conclusion:

2.2. Proposition. *If A is a separable nonunital matroid algebra, and if P and Q are two projections in $M(A)$, then*

(a) $P \lesssim Q \Rightarrow \bar{\tau}(P) \leqq \bar{\tau}(Q)$.

(b) *If either $\{P, Q\} \subset A$ or $\{P, Q\} \subset M(A)\backslash A$, then*

$$\bar{\tau}(P) = \bar{\tau}(Q) \Rightarrow P \sim Q.$$

(c) $\bar{\tau}(P) < \bar{\tau}(Q) \Rightarrow P \lesssim Q$ *unless $P \in M(A)\backslash A$ and $Q \in A$.*

Proof. (a) is trivial.

(b) If $P, Q \in A$, this is a part of [**16**, 2.9]. We may assume that both P and Q are in $M(A)\backslash A$. If $\bar{\tau}(P) = \bar{\tau}(Q) = \infty$, $P \sim 1 \sim Q$ as shown in the proof of Proposition (2.1). We may assume that $\bar{\tau}(P) = \bar{\tau}(Q) < \infty$. Choose increasing sequences $\{p_n\}$ and $\{q_n\}$ of projections in A such that $p_n \nearrow P$ and $q_n \nearrow Q$ in the strict topology. Then $\tau(p_n) \nearrow \bar{\tau}(P)$ and $\tau(q_n) \nearrow \bar{\tau}(Q)$. Let $n_1 > 1$ be large enough so that $\tau(q_1) < \tau(p_{n_1})$. By [**16**, 2.9] there exists $v_1 \in A$ such that $v_1^* v_1 = q_1$, $v_1 v_1^* < p_{n_1}$ and so

$$\bar{\tau}(P - v_1 v_1^*) = \bar{\tau}(Q - q_1).$$

Choose $m_1 > 1$ such that

$$\tau(q_{m_1} - q_1) > \tau(p_{n_1} - v_1 v_1^*).$$

Then again by [**16**, 2.9], there exists $v_2 \in A$ such that

$$v_2 v_2^* = p_{n_1} - v_1 v_1^*, \quad v_2^* v_2 < q_{m_1} - q_1$$

and so

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$$\overline{\tau}(Q - q_1 - v_2^* v_2) = \overline{\tau}(P - p_{n_1}).$$

Proceeding in this way, we can find a sequence of partial isometries $v_i \in A$ such that

$$V = \sum_{i=1}^{\infty} v_i \in M(A), \quad VV^* = P, \quad \text{and} \quad V^*V = Q.$$

Hence $P \sim Q$ in $M(A)$.

(c) Assume $\overline{\tau}(P) < \infty$. If $P, Q \in A$, use [**16**, 2.9]. If $P \in A$ and $Q \in M(A)\backslash A$, choose projections $q_n \nearrow Q$. Then

$$\tau(q_n) \nearrow \overline{\tau}(Q).$$

Since $\overline{\tau}(P) < \overline{\tau}(Q)$, there exists n_0 such that $\tau(q_{n_0}) > \overline{\tau}(P)$ and so $P \lesssim q_{n_0} < Q$ by [**16**, 2.9]. We may assume $P, Q \in M(A)\backslash A$ from now on. If $\overline{\tau}(Q) = \infty$, the conclusion is clear since $Q \sim 1$.

We may assume $\overline{\tau}(Q) < \infty$ and $\overline{\tau}(P) < \infty$ from now on.

Let $P = \sum_{i=1}^{\infty} e_i$ for some mutually orthogonal nonzero projections $e_i \in A$ with the sum converging in the strict topology, and similarly $Q = \sum_{i=1}^{\infty} f_i$. Since

$$\overline{\tau}(P) = \sum_{i=1}^{\infty} \tau(e_i) < \infty \quad \text{and}$$

$$\tau(f_i) > 0 \quad \text{for each } i \geqq 1,$$

there exist $n_i \nearrow \infty$ such that

$$\sum_{j=n_{i-1}+1}^{\infty} \tau(e_j) < \tau(f_i) \quad \text{for } i \geqq 1.$$

Then

$$\sum_{j=n_{i-1}+1}^{n_i} \tau(e_j) < \tau(f_i) \quad \text{for } i \geqq 1.$$

It follows from [**16**, 2.9] that there exist $\{v_i\} \subset A$ such that

$$v_i v_i^* = \sum_{j=n_{i-1}+1}^{n_i} e_j \quad \text{and}$$

$$v_i^* v_i = g_i < f_i \quad \text{for } i \geqq 1.$$

Let $V = \sum_{i=1}^{\infty} v_i$; then

$$V \in M(A) \quad \text{and} \quad VV^* = P - \sum_{j=1}^{n_0} e_j \quad \text{and}$$

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$$V^*V = \sum_{i=1}^{\infty} g_i < \sum_{i=1}^{\infty} f_i = Q.$$

Therefore

$$\tau\left(\sum_{j=1}^{n_0} e_j\right) = \overline{\tau}(P - VV^*) < \overline{\tau}(Q - V^*V) = \overline{\tau}\left(Q - \sum_{i=1}^{\infty} g_i\right).$$

Since $\sum_{j=1}^{n_0} e_j \in A$, the cases we have discussed previously imply that

$$\sum_{j=1}^{n_0} e_j \lesssim Q - \sum_{i=1}^{\infty} g_i.$$

Therefore

$$P = \sum_{i=1}^{\infty} e_i \lesssim \sum_{i=1}^{\infty} f_i = Q.$$

2.3. *Remarks*. (1) In the proof of the second claim of Proposition (2.1) we have proved: If A is a separable infinite matroid algebra and $Q \in M(A)$ is a projection, then $Q \sim 1$ if and only if $\overline{\tau}(Q) = \infty$ if and only if $Q \notin J$, where we assume that $J = A$ if A is elementary. This answers the question asked by G. Elliott in [**18**, 3.3]. As an easy consequence of this, we can prove: Two projections P and Q in J are Murray-von Neumann equivalent if and only if P and Q are unitarily equivalent in $M(A)$. In fact,

$$\overline{\tau}(1 - P) = \overline{\tau}(1 - Q) = \infty.$$

Then $1 - P \sim 1 \sim 1 - Q$. By Lemma (2.8) of [**28**] it follows that two projections $\overline{p}$ and $\overline{q}$ in $\pi(J)$ are equivalent in the sense of Murray-von Neumann if and only if $\overline{p}$ and $\overline{q}$ are unitarily equivalent.

(2) If A is a separable finite matroid algebra, and if P and Q are two projections in $M(A)$, then

(i) Whenever either $\{1 - P, 1 - Q\} \subset A$ or $\{1 - P, 1 - Q\} \subset M(A)\backslash A$, $P \sim Q$ if and only if P and Q are unitarily equivalent.

This follows from Proposition (2.2). We leave it, and also the following, to the reader.

(ii) If two projections $\overline{p}$ and $\overline{q}$ in $M(A)/A$ are not equal to $\overline{1}$, then $\overline{p} \sim \overline{q}$ if and only if $\overline{p}$ and $\overline{q}$ are unitarily equivalent.

2.4. COROLLARY. (a) *If A is a separable nonunital finite matroid algebra, then*

(i) $K_1(M(A)/A) = K_1(M(A)) = \{0\}$.

(ii) $0 \to K_0(A) \to K_0(M(A)) \to K_0(M(A)/A) \to 0$

is exact.

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(b) *If A is an infinite separable matroid algebra, then*

(i) $K_0(J/A) \cong D(J/A)\backslash\{[0]\}$.

(ii) $0 \to K_0(A) \to K_0(J) \to K_0(J/A) \to 0$.

(iii) $K_0(M(A)/J) \cong K_1(J) \cong K_1(J/A)$ *and* $K_1(M(A)/J) \cong K_0(J)$.

Proof. (a) follows from [**18**, 2.9], Corollary (1.4), and the K-theory long exact sequence for

$$0 \to A \to M(A) \to M(A)/A \to 0.$$

(b) (i) follows from Theorem (1.3) and [**13**].
In (iii)

$$K_1(M(A)/J) \cong K_0(J) \quad \text{and} \quad K_0(M(A)/J) \cong K_1(J)$$

follows from the K-theory long exact sequence for

$$0 \to J \to M(A) \to M(A)/J \to 0$$

and the fact that

$$K_0(M(A)) \cong K_1(M(A)) \cong \{0\}$$

(see [**3**, 12.2.1]). Consider the exact sequence

$$0 \to A \to J \to J/A \to 0.$$

$K_1(J/A) \cong K_1(J)$ and (ii) will follow from the six term exact sequence of K-theory if we show that $K_0(A) \to K_0(J)$ is injective. This follows from the fact that τ induces an injective map from $K_0(A)$ to $\mathbf{R}$ which factors through $K_0(J)$.

As we have proved, for certain C^*-algebras including all σ-unital AF algebras, that two equivalent projections in $M(A)/A$ lift to equivalent projections in $M(A)$ by Lemma (2.8) of [**28**], a question comes up naturally: If A is separable matroid, can we describe equivalence classes of projections in $M(A)/A$ by the values of the trace on projections in the preimages in $M(A)$? We shall give such a description in the following theorem. We need to recall some related matters first. By [**3**, 5.3.1 and 5.5.5] $K_0(A)$ is isomorphic to the Grothendieck group of $D(A \otimes K)$. Hence every element in $K_0(A)$ is in the form $[e] - [f]$ for some projections e and f in $A \otimes K$. We shall denote by $\tilde{\tau}$ also the natural extension of $\tilde{\tau}$ from $D(A \otimes K)$ to $K_0(A)$ defined by

$$\tilde{\tau}([e] - [f]) = \tilde{\tau}([e]) - \tilde{\tau}([f])$$

and agree that

$$\infty \equiv \infty \bmod[\tilde{\tau}(K_0(A))].$$

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2.5. THEOREM. *If A is a separable matroid algebra without unit and $\bar{p}$ and $\bar{q}$ are two projections in $[M(A)/A]\backslash\{\bar{0}\}$, then $\bar{p} \sim \bar{q}$ if and only if*

$$\bar{\tau}(P) \equiv \bar{\tau}(Q) \bmod[\tilde{\tau}(K_0(A))]$$

for any projections $Q \in \pi^{-1}(\bar{q})$ and $P \in \pi^{-1}(\bar{p})$.

To prove this theorem, we need the following lemma:

2.6. LEMMA. *If A is a nonunital C^*-algebra and two equivalent projections $\bar{p}$ and $\bar{q}$ in $M(A)/A$ lift to projections P and Q in $M(A)$ respectively such that* her(P) *has an approximate identity consisting of projections, then $\bar{p}$ lifts to a projection $P_1 \leqq P$ and $\bar{q}$ lifts to a projection $Q_1 \leqq Q$ such that $Q_1 \sim P_1$.*

Proof. Since $\bar{p} \sim \bar{q}$, there is a partial isometry $\bar{v} \in M(A)/A$ such that $\bar{v}\bar{v}^* = \bar{q}$ and $\bar{v}^*\bar{v} = \bar{p}$. Let $V \in M(A)$ be a preimage of $\bar{v}$. Let $W = QVP$, then

$$\pi(W) = \pi(QVP) = \pi(Q)\pi(V)\pi(P) = \bar{q}\bar{v}\bar{p}$$

and hence

$$\pi(WW^*) = \bar{q} \quad \text{and} \quad \pi(W^*W) = \bar{p}.$$

We assume that $QVP = V$. Since $\pi(V^*V - P) = \bar{0}$,

$$a = V^*V - P \in \text{her}(P).$$

Now we can repeat the proof of Lemma (2.8) of [**28**].

2.7. *The proof of Theorem* (2.5).

Proof. ($\Leftarrow$) If $\bar{\tau}(Q) = \infty$ for some projection Q in $\pi^{-1}(\bar{q})$, then $\bar{\tau}(P) = \infty$ for any P in $\pi^{-1}(\bar{p})$ since

$$\bar{\tau}(P) \equiv \bar{\tau}(Q) \bmod[\tilde{\tau}(K_0(A))].$$

It follows that $P \sim 1 \sim Q$ and so $\bar{p} \sim \bar{1} \sim \bar{q}$. Assume that $\bar{\tau}(Q) < \infty$ for any projection Q in $\pi^{-1}(\bar{q})$. Then $\bar{\tau}(P) < \infty$ for any projection P in $\pi^{-1}(\bar{p})$ since

$$\bar{\tau}(P) \equiv \bar{\tau}(Q) \bmod[\tilde{\tau}(K_0(A))].$$

Hence $\bar{\tau}(P) - \bar{\tau}(Q)$ makes sense and it is equal to

$$\tilde{\tau}([e] - [f]) = \tau(e) - \tau(f)$$

for some projections e and f in $A \otimes K$. Then

$$\bar{\tau}(P) + \tau(f) = \bar{\tau}(Q) + \tau(e).$$

If $\tau(e) = \tau(f)$, then $\bar{\tau}(P) = \bar{\tau}(Q)$. Since $\bar{p} \neq \bar{0}$ and $\bar{q} \neq \bar{0}$, $P \notin A$ and $Q \notin A$. Proposition (2.2) implies $P \sim Q$ and so $\bar{p} \sim \bar{q}$.

If $\tau(e) \neq \tau(f)$, say $\tau(e) > \tau(f)$, then $f \sim f_1 < e$ by [**16**, 2.9]. Hence

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$$\overline{\tau}(P) = \overline{\tau}(Q) + \tau(e - f_1)$$

and so $\tau(e - f_1) < \overline{\tau}(P)$. It follows that $e - f_1 \sim p_0 < P$, $p_0 \in A$, again by Proposition (2.2). Therefore

$$\overline{\tau}(P - p_0) = \overline{\tau}(Q).$$

Since $P - p_0 \in M(A)\backslash A$ and $Q \in M(A)\backslash A$, Proposition (2.2) applies once more: $P - p_0 \sim Q$. Thus $\overline{p} \sim \overline{q}$.

($\Rightarrow$) If $\overline{\tau}(Q) = \infty$ for some projection Q in $\pi^{-1}(\overline{q})$, then $Q \sim 1$. Then $\overline{\tau}(Q_1) = \infty$ for any projection $Q_1 \in \pi^{-1}(\overline{q})$. Since $\overline{p} \sim \overline{q}$, $\overline{p} \sim \overline{1}$ and so $\overline{\tau}(P) = \infty$ for any projection $P \in \pi^{-1}(\overline{p})$. Hence the conclusion is true.

If $\overline{\tau}(Q) < \infty$ for some projection Q in $\pi^{-1}(\overline{q})$, then $\overline{\tau}(Q_1) < \infty$ for any projection Q_1 in $\pi^{-1}(\overline{q})$. Since $\overline{p} \sim \overline{q}$, $\overline{p} \not\sim \overline{1}$. Then $\overline{\tau}(P) < \infty$ for any projection $P \in \pi^{-1}(\overline{p})$. By Lemma (2.6), for any projection Q in $\pi^{-1}(\overline{q})$ and any projection P in $\pi^{-1}(\overline{p})$ we can choose projections P_1 in $\pi^{-1}(\overline{p})$ with $P_1 \leqq P$ and Q_1 in $\pi^{-1}(\overline{q})$ with $Q_1 \leqq Q$ such that $P_1 \sim Q_1$. Hence $P - P_1 \in A$, $Q - Q_1 \in$ A and $\overline{\tau}(P_1) = \overline{\tau}(Q_1)$. Then

$$\begin{aligned}\overline{\tau}(Q) &= \overline{\tau}(Q_1) + \overline{\tau}(Q - Q_1) = \overline{\tau}(P_1) + \overline{\tau}(Q - Q_1)\\ &= \overline{\tau}(P) + [\overline{\tau}(Q - Q_1) - \overline{\tau}(P - P_1)].\end{aligned}$$

The conclusion follows, since

$$\overline{\tau}(Q - Q_1) - \overline{\tau}(P - P_1) \in \overline{\tau}(K_0(A)).$$

2.8. *Remark.* The conclusion of Theorem (2.5) is true for any simple σ-unital C^*-algebra with FS and a faithful real-valued homomorphism defined on $D(K(H_A))$. Such a generalization is obvious from the proof although we did not state the result in such a general setting.

3. A lifting of ideals from A to $M(A)/A$. In this section we always assume that A is a nonunital C^*-algebra so that the corona algebra $M(A)/A$ is not trivial. The ideal structure of $M(A)/A$ has been studied in [**1**], [**18**], [**20**], [**23**], [**27**], among other articles. A basic question is: How does the (closed) ideal structure of A relate to the (closed) ideal structure of $M(A)/A$? We will work on this problem in this section via a 'lifting of ideals' from A to $M(A)/A$. We denote the set of closed ideals of a C^*-algebra B by $I(B)$.

3.1. *A lifting of ideals.* If I is a closed ideal of A, I is naturally regarded as an ideal of $M(A)$ or of any C^*-subalgebra of $M(A)$ containing I. Let

$$M(A, I) = M(A) \cap I^{**},$$

where I^{**} is identified with a subset of A^{**}. $I^{**} = p_I A^{**}$, where p_I is the central open projection corresponding to I. Obviously,

$$M(A, I) = \{m \in M(A): \quad mA \subset I \text{ and } Am \subset I\}.$$

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It is easy to check that I is a closed ideal of $M(A, I)$, $M(A, I)$ is a closed ideal of $M(A)$ and

$$A \cap M(A, I) = I.$$

Generally speaking, $M(A, I)$ may not be contained in A.

Let $L(I) = A + M(A, I)$ for each closed ideal I of A; then $L(I)$ is a closed ideal of $M(A)$ containing A. $L(I) \neq A$ if and only if $M(A, I) \not\subset A$ if and only if $\pi(L(I))$ is a nonzero closed ideal of $M(A)/A$. For the canonical image of $L(I)$ in $M(A)/A$,

$$\begin{aligned}\pi(L(I)) &= [A + M(A, I)]/A \cong M(A, I)/[A \cap M(A, I)] \\ &\cong M(A, I)/I\end{aligned}$$

(see [**15**, 1.8.4]). Then we have the following diagram:

$$I(A) \xrightarrow{L} I(M(A)) \xrightarrow{\pi} I(M(A)/A)$$

$$I \mapsto A + M(A, I) \mapsto [A + M(A, I)]/A \cong M(A, I)/I.$$

We will refer to this construction as *the lifting of ideals*. The lifting of ideals has the following elementary properties:

3.2. PROPOSITION. *L is order-preserving and the composition $\pi \circ L$ is order- and orthogonality-preserving.*

Proof. It is clear from the definition that L is order-preserving. It is obvious that $I \perp J$ if and only if $M(A, I) \perp M(A, J)$. For any $a, b \in A$, $m \in M(A, I)$ and $n \in M(A, J)$, we have

$$(a + m)(b + n) = ab + an + mb + mn = ab + an + mb \in A.$$

Thus $I \perp J$ implies $\pi \circ L(I) \perp \pi \circ L(J)$.

We say that an ideal of A is *nontrivial* if it is neither $\{0\}$ nor A. It is possible that the lifting $L(I)$ of I may be equal to $M(A)$ or be contained in A. Equivalently it is possible that $M(A, I)/I$ may be $\{\bar{0}\}$ or $M(A)/A$ even for a nontrivial closed ideal I of A.

We say that I is *liftable* if $M(A, I)/I$ is neither $\{\bar{0}\}$ nor $M(A)/A$. We will study this lifting of ideals from various perspectives. First of all, we note that the lifting of ideals is not well-behaved in general.

3.3. *Examples.*

Example (1). Let Ω_0 be the locally compact Hausdorff space consisting of the countable ordinals, which is not σ-compact. Its Stone-Čech compactification $\beta(\Omega_0)$ coincides with the one-point compactification of Ω_0. It was proved in [**1**] that for any unital C^*-algebra B,

$$M(C_0(\Omega_0) \otimes B) = C(\beta(\Omega_0)) \otimes B$$

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and so

$$M(C_0(\Omega_0 \otimes B)/C_0(\Omega_0) \otimes B) \cong \mathbf{C} \otimes B \cong B$$

since

$$C(\beta(\Omega_0))/C(\Omega_0) \cong \mathbf{C}.$$

It is wellknown that $C_0(\Omega_0) \otimes B$ has infinitely many nontrivial closed ideals for any simple unital C^*-algebra B, but none of these ideals is liftable. Also this example tells us that $M(A)/A$ can be any unital C^*-algebra.

As a special case, if $B = \mathbf{C}$, then

$$M(C_0(\Omega_0))/C_0(\Omega_0) \cong \mathbf{C}.$$

This is an example of a finite-dimensional corona algebra. It was proved in [**1**] that $M(A)/A$ is nonseparable if A is a nonunital σ-unital C^*-algebra.

Example (2). Let $A = K \oplus B$, where B is any unital C^*-algebra. Then A may be 'nice' (for example, separable AF, liminal, nonsimple, with Hausdorff spectrum and so on). A has at least two nontrivial closed ideals, $K \oplus 0$ and $0 \oplus B$; but none of them is liftable, since $M(A)/A$ is the Calkin algebra, which is simple.

Example (3). If A is any stable C^*-algebra, then by Theorem (3.1) of [**27**] the lifting of ideals is a lattice isomorphism (not onto in general). So the lifting of ideals is well-behaved in this case.

From one point of view as follows, we have the following necessary and sufficient condition for a nontrivial closed ideal to be liftable.

3.4. PROPOSITION. *If A is a σ-unital nonunital C^*-algebra and I is a nontrivial closed ideal of A, then I is liftable if and only if the natural map from $M(A)/A$ to $M(A/I)/(A/I)$ is not a *-isomorphism and A/I is nonunital.*

Proof. Since I is a closed ideal of A, we have the following exact sequence:

$$0 \to I \xrightarrow{i} A \xrightarrow{\lambda} A/I \to 0.$$

Since A is σ-unital, A/I is σ-unital (perhaps unital). By Theorem (10) of [**26**],

$$0 \to M(A, I) \xrightarrow{\tilde{i}} M(A) \xrightarrow{\tilde{\lambda}} M(A/I) \to 0$$

is exact, $\tilde{\lambda}$ induces the following exact sequence:

$$0 \to M(A, I)/I \xrightarrow{\bar{i}} M(A)/A \xrightarrow{\bar{\lambda}} M(A/I)/(A/I) \to 0.$$

Therefore we obtain the following commutative diagram:

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$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I & \xrightarrow{\ i\ } & A & \xrightarrow{\ \lambda\ } & A/I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M(A, I) & \xrightarrow{\ \tilde{i}\ } & M(A) & \xrightarrow{\ \tilde{\lambda}\ } & M(A/I) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M(A, I)/I & \xrightarrow{\ \bar{i}\ } & M(A)/A & \xrightarrow{\ \bar{\lambda}\ } & M(A/I)/(A/I) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

In the diagram every row and every column is exact; see Theorem (23) of [**26**]. Thus the following are equivalent:

(1) $M(A, I)/I = \{\bar{0}\}$;

(2) $M(A)/A \cong M(A, I)/(A/I)$ (excision property);

(3) $M(A)/I \cong M(A/I)$.

Also the following conditions are equivalent:

(1′) $M(A, I)/I \cong M(A)/A$;

(2′) A/I is unital.

(In (2), (3), and (1′), we mean that the natural maps are isomorphisms.)

3.5. THEOREM. *If A is a separable nonunital C^*-algebra with an approximate identity consisting of projections, then a nontrivial closed ideal I of A is liftable if and only if A/I is nonunital and $I \not\subset pAp$ for any projection $p \in A$.*

Proof. (Necessity). A/I is nonunital by Proposition (3.4). If there is a projection p in A such that $I \subset pAp$, then $px = xp = x$ for any x in I. Consequently $py = yp = y$ for any y in I^{**}. Since $M(A, I) \subset I^{**}$, $mp = pm = m$ for any m in $M(A, I)$. Since p is in A, $m = mp$ is in I and so $M(A, I) \subset I$. Hence I is not liftable.

(Sufficiency). Let $\{e_n\}$ be a sequential increasing approximate identity of A (the existence of $\{e_n\}$ is guaranteed by Proposition (1.2) of [**28**]).

If $(e_n - e_1)I(e_n - e_1) = \{0\}$ for all n, then $(e_n - e_1)I = \{0\}$ for all n and hence $(1 - e_1)I = \{0\}$. Thus

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$$I \subset e_1 I e_1 \subset e_1 A e_1,$$

which contradicts the hypothesis. Hence there exists n_1 such that

$$(e_{n_1} - e_1)I(e_{n_1} - e_1) \neq \{0\}.$$

Proceeding in this way we can find a subsequence of $\{e_n\}$ such that

$$(e_{n_1} - e_{n_{i-1}})I(e_{n_i} - e_{n_{i-1}}) \neq \{0\} \quad \text{for all } i \geqq 1.$$

Changing notation we assume that

$$(e_n - e_{n-1})I(e_n - e_{n-1}) \neq \{0\} \quad \forall n \geqq 1.$$

Choose a_n in $(e_n - e_{n-1})I(e_n - e_{n-1})$ for each $n \geqq 1$ such that $\|a_n\| = 1$. It is clear that

$$a_n a_m = a_m a_n = 0 \quad \text{for } n \neq m.$$

Let

$$\psi((t_i)) = \sum_{i=1}^{\infty} t_i a_i$$

for $(t_i) \in l^{\infty}$. It is clear that $\psi((t_i)) \in A^{**}$. We claim that $\psi((t_i))$ is in $M(A, I)$. In fact, for each $a \in A$ we have

$$(ae_n)\psi((t_i)) = a \sum_{i=1}^{n} t_i a_i \in I \quad \text{for all } n \geqq 1.$$

$$\|a\psi((t_i)) - (ae_n)\psi((t_i))\| \leqq \|a - ae_n\| \, \|\psi((t_i))\| \to 0$$
$$\text{as } n \to \infty$$

since $e_n \nearrow 1$ in the strict topology. Hence $a\psi((t_i)) \in I$. Similarly, $\psi((t_i))a \in I$. Therefore

$$\psi((t_i)) \in M(A, I).$$

We have defined a map between Banach spaces,

$$\psi: l^{\infty} \to M(A, I).$$

Clearly ψ is an isometric map. Therefore $M(A, I)$ cannot be separable and so $M(A, I) \not\subset A$, since A is separable. In view of Proposition (3.4), the proof is complete.

If we take $A = C_0(\Omega_0) \otimes M_n$ as in Example (1) of (3.3), then none of the ideals of A is liftable. On the other hand, every ideal of $A \otimes K$ is liftable by Theorem (3.1) of [**27**]. It is well known that the spectrum $\hat{A}$ of A is homeomorphic to the spectrum of $A \otimes K$. This tells us that the spectrum of a C^*-algebra does not determine the liftability of ideals despite the fact

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that the spectrum is closely related to the ideal structure of the C^*-algebra. Nevertheless, the spectrum sometimes gives some information on the lifting of ideals.

We denote by $C^b(\hat{A})$ the set of all bounded continuous complex-valued functions on $\hat{A}$ and let

$$C_0^b(\hat{I}) = \{f \in C^b(\hat{A}): f = 0 \text{ outside } \hat{I}\}$$

if I is a closed ideal of A. The center of a C^*-algebra B is denoted by $Z(B)$. By the Dauns-Hofmann theorem,

$$C_0^b(\hat{I}) \subset C^b(\hat{I}) \cong Z(M(I))$$

(see [**21**]).

3.6. Lemma. *If A is any C^*-algebra and I is a closed ideal of A, then $C_0^b(\hat{I})M(I)$ can be injectively mapped to $M(A, I)$, where*

$$C_0^b(\hat{I})M(I) = \{fm: f \in C_0^b(\hat{I}) \text{ and } m \in M(I)\}.$$

Proof. By the Dauns-Hofmann theorem (see [**21**]),

$$C_0^b(\hat{I}) \subset C^b(\hat{A}) \cong Z(M(A)).$$

For any $\pi \in \hat{A}\backslash\hat{I}$ and any $a \in A$,

$$\pi(fa) = f(\pi)\pi(a) = 0,$$

since $f(\pi) = 0$. Hence

$$af = fa \in I = \cap\{\ker \pi: \pi \in \hat{A}\backslash\hat{I}\}$$

(see [**15**, Chapter 3]). Thus $C_0^b(\hat{I}) \subset M(A, I)$. It is easy to check that

$$M(A, I)M(I)M(A, I) \subset M(A, I).$$

It follows that $M(A, I)$ is a hereditary C^*-subalgebra of $M(I)$. Then $C_0^b(\hat{I}) \subset M(A, I)$ implies that

$$C_0^b(\hat{I})M(I) \subset M(A, I).$$

3.7. *Remarks.* (1) If $\hat{A}$ is Hausdorff, then $C_0^b(\hat{I}) \neq \{0\}$ if I is a nonzero closed ideal of A. But if $\hat{A}$ is not Hausdorff, then $C_0^b(\hat{I}) = \{0\}$ is possible. For example, if A has a faithful irreducible representation, then $\hat{A}$ has a dense point (see [**15**, 3.9.1]). Since every nonempty open subset of $\hat{A}$ contains this point, $C^b(\hat{A}) \cong C$. If $I \neq A$, then $C_0^b(\hat{I}) = \{0\}$.

(2) Two easy consequences of the above lemma are as follows:

(i) If A is a σ-unital C^*-algebra and I is a nontrivial ideal of A such that $Z(I) = \{0\}$ and $C_0^b(\hat{I}) \neq \{0\}$, then I is liftable whenever A/I is nonunital.

In fact, $C_0^b(\hat{I}) \subset M(A, I) \cap Z(M(I))$ implies $M(A, I) \not\subset I$. Then Proposition (3.4) applies.

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(ii) If A is a σ-unital C^*-algebra with $Z(A) = \{0\}$, then a closed ideal I of A is liftable whenever $C_0^b(\hat{I}) \neq \{0\}$ and A/I is nonunital. (The proof is similar to (i).)

Note that A stable implies $Z(A) = \{0\}$.

3.8. THEOREM. *If A is a nonunital separable C^*-algebra with Hausdorff spectrum and I is a nontrivial closed ideal of A such that $\hat{I}$ is not compact, then I is liftable whenever A/I is nonunital. In addition there exist always uncountably many liftable closed ideals of A if $\hat{A}$ is not compact.*

Proof. First we recall that if $\hat{A}$ is Hausdorff, then $\hat{A}$ is locally compact (see [**15**, 3.3.8]). Thus $\hat{A}$ is completely regular. Since $\hat{A}$ is Hausdorff, $\hat{I}$ is Hausdorff. For any $\pi_1 \neq \pi_2 \in \hat{I}$ there exist two disjoint open subsets U_1 and U_2 of $\hat{I}$ such that $\pi_1 \in U_1$ and $\pi_2 \in U_2$. We can find two open subsets V_1 and V_2 of $\hat{I}$ such that $\pi_1 \in V_1 \subset \bar{V}_1 \subset U_1$ and $\pi_2 \in V_2 \subset \bar{V}_2 \subset U_2$ and $\bar{V}_1$ and $\bar{V}_2$ are compact. It is clear that $\bar{V}_1 \cup \bar{V}_2 \neq \hat{I}$ since $\hat{I}$ is not compact. Let

$$\pi_3 \in \hat{I}\backslash(\bar{V}_1 \cup \bar{V}_2);$$

then we can find an open subset V_3 of $\hat{I}$ such that

$$\pi_3 \in V_3,\ \bar{V}_3 \cap (\bar{V}_1 \cup \bar{V}_2) = \emptyset$$

and $\bar{V}_3$ is compact. By repeating this procedure, we can find a sequence $\{\pi_i\} \subset \hat{I}$ and a sequence of disjoint open subsets $\{V_i\}$ of $\hat{I}$ such that

$$\pi_i \in V_i, \quad \bar{V}_i \cap \bar{V}_j = \emptyset \quad (i \neq j)$$

and $\bar{V}_i$ is compact. Let

$$C_0^b(V_i) = \{f \in C_0^b(\hat{A}): f = 0 \text{ outside bar } V_i\}$$

and D be the l^∞-direct sum of the $C_0^b(V_i)$'s. Then $D \subset C_0^b(\hat{I})$ and D is nonseparable. By Lemma (3.6)

$$D \subset C_0^b(\hat{I}) \subset M(A, I)$$

and so $M(A, I)$ is nonseparable. Since A is separable, $M(A, I) \not\subset A$. On the other hand, A/I is nonunital by hypothesis. It follows from Proposition (3.4) that I is liftable. It is clear that there exist uncountably many distinct subsets of $\hat{I}$ which are not compact, denoted by $\{U_\lambda: \lambda \in \Lambda\}$. Let I_λ be the closed ideal of A such that $\hat{I}_\lambda = U_\lambda$. It suffices to show that A/I_λ is nonunital for the second conclusion to be true. But this is clear since the canonical map $A/I_\lambda \rightarrow A/I$ is onto and A/I is nonunital.

3.9. COROLLARY. *If A is a separable nonunital C^*-algebra with Hausdorff spectrum and I is a closed ideal of A, then I is liftable whenever both $\hat{I}$ and $\hat{A}\backslash\hat{I}$ are not compact.*

Proof. Since $\hat{A}\backslash\hat{I}$ is not compact and the spectrum of A/I is $\hat{A}\backslash\hat{I}$, A/I is not unital (see [**15**, Chapter 3]). Theorem (3.8) applies.

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3.10. THEOREM. *If A is a separable nonunital C^*-algebra with noncompact Hausdorff spectrum $\hat{A}$, then A has uncountably many distinct chains of closed liftable ideals. Consequently $M(A)/A$ has uncountably many distinct closed ideals.*

Proof. Claim 1. There exists a nontrivial closed ideal I of A such that $\hat{I}$ and $\hat{A}\backslash\hat{I}$ are both not compact.

Since A is separable, $\hat{A}$ is second countable (see [**15**, 3.3.4]). Let $\pi_n \in \hat{A}$ be such that $\pi_n \to \infty$ in the one point compactification of $\hat{A}$. We can find a sequence of open subsets of $\hat{A}$, say $\{U_n\}$, such that $\pi_n \in U_n$ and $\bar{U}_n \cap \bar{U}_m = \emptyset$ if $m \neq n$. Let

$$U = \bigcup_{n=1}^{\infty} U_{2n-1}.$$

Then we can find a closed ideal I_U of A such that $\hat{I}_U = U$. It is clear that both $\hat{I}_U$ and $\hat{A}\backslash\hat{I}_U$ are not compact. I_U is as desired.

Claim 2. There exist uncountably many chains of liftable closed ideals in A.

Let I_1 be any liftable ideal of A such that $\hat{I}_1$ is not compact. In a way similar to the procedure in Claim 1, we can find a closed ideal $I_2 \subset I_1$ such that both $\hat{I}_2$ and $\hat{I}_1\backslash(\hat{I}_2)^-$ are not relatively compact. Hence $\hat{I}_2$ and $\hat{A}\backslash\hat{I}_2$ are not compact, and I_2 is liftable by Corollary (3.9). Clearly

$$A + M(A, I_2) \subset A + M(A, I_1).$$

We claim that this inclusion is strict. Let $V = \hat{I}_1\backslash(\hat{I}_2)^-$; then V is an open subset of $\hat{A}$ which is not compact. Hence $C_0^b(V)$ is nonseparable and is contained in $M(A, I_1)\backslash M(A, I_2)$ by the proof of Theorem (3.8). Thus

$$A + M(A, I_2) \neq A + M(A, I_1).$$

Proceeding in this way, we obtain a chain of distinct liftable closed ideals of A. Since there exist uncountably many possibilities for I_1, we can find uncountably many chains of liftable closed ideals of A.

One may wonder when $M(A, I) = M(I)$. We have the following easy characterizations:

3.11. PROPOSITION. *If A is any C^*-algebra and I is a closed ideal of A, then the following are equivalent*:

(i) $M(I) = M(A, I)$.

(ii) $P_I \in Z(M(A))$.

(iii) *$\hat{I}$ is clopen in $\hat{A}$.*

(iv) $A = I \oplus J$ *for some ideal J.*

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Proof. (i) $\Rightarrow$ (ii). P_I is the central open projection corresponding to I. (ii) $\Rightarrow$ (iii). Since P_I corresponds to $\chi_{\hat{I}}$, the characteristic function of $\hat{I}$, under the isomorphism of $Z(M(A))$ with $C^b(\hat{A})$, we have that $\chi_{\hat{I}}$ is continuous. Therefore $\hat{I}$ is clopen (iii) $\Leftrightarrow$ (iv). Since $\hat{I}$ is clopen, $\hat{A}\backslash\hat{I}$ is clopen. There is a closed ideal J of A such that $\hat{A}\backslash\hat{I} = \hat{J}$. Hence $I \cap J = \{0\}$ (see [**15**, Chapter 3]). $\hat{A} = \hat{I} \cup \hat{J}$ implies $A = I \oplus J$. (iv) $\Rightarrow$ (i) is trivial.

4. Covering elements and non-separability of $M(A)/A$.

4.1. *Definition.* Assume that A is a nonunital C^*-algebra. An element m in $M(A)_+$ is said to be a *covering element* of A if $(mA)^- = A$ and $C^*(m)$ is not unital, where $C^*(m)$ is the C^*-subalgebra of $M(A)$ generated by m.

It is obvious that every strictly positive element of A is a covering element of A. Hence every σ-unital C^*-algebra contains covering elements. Generally speaking, a covering element is in $M(A)\backslash A$ if A is not σ-unital. There is a non-σ-unital C^*-algebra having a covering element.

4.2. *Lemma – Example.*

(1) Lemma. *If A is nonunital and $m \in M(A)_{s.a.}$, then $C^*(m)$ contains the identity of $M(A)$ if and only if $0 \notin \sigma(m)$.*

Proof. The proof consists of elementary application of the operator calculus. We leave it to the reader.

(2) *Example.* Let A be the set of all compact operators on a non-separable Hilbert space H. Then A is not σ-unital but has a covering element in $M(A) \cong L(H)$.

The following easy result is a rather modest generalization of Theorem (2.7) in [**1**].

4.3. Theorem. *If A is a nonunital C^*-algebra with a covering element m, then $M(A)/A$ is nonseparable.*

Proof. It is well known that $C^*(m) \cong C_0(\sigma(m))$. Since m is a covering element of A, $\sigma(m)\backslash\{0\}$ is not compact. It follows that

$$M(C^*(m)) \cong C^b(\sigma(m)\backslash\{0\})$$

is nonseparable. For any $f_m \in M(C^*(m))$ we have $f_m m \in C^*(m)$ and so $f_m mA \subset A$. Since $(mA)^- = A$, $f_m A \subset A$ and hence $f_m \in M(A)$. So

$$M(C^*(m)) \subset M(A).$$

We claim that

$$A \cap M(C^*(m)) = A \cap C^*(m).$$

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In fact, if $a \in A \cap M(C^*(m))$, then $am \in A \cap C^*(m)$. Let

$$b_i = (i^{-1} + m)^{-1}m \quad \text{for each } i \geqq 1.$$

Then $\{b_i\}$ is an approximate identity of the hereditary C^*-subalgebra H_m of $M(A)$ generated by m and hence $ab_i \rightarrow a$. Since $ab_i \in A \cap C^*(m)$, $a \in A \cap C^*(m)$ by the fact that $A \subset H_m$.

Since

$$\pi(M(C^*(m))) = [A + M(C^*(m))]/A$$

$$\cong M(C^*(m))/[A \cap M(C^*(m))] = M(C^*(m))/[A \cap C^*(m)],$$

and since $A \cap C^*(m)$ is separable, $\pi(M(C^*(m)))$ is nonseparable.

For any state f on A let $\tilde{f}$ be the unique extension of f to $M(A)$. For any nondegenerate representation π of A on a Hilbert space H let $\tilde{\pi}$ denote the unique nondegenerate extension of π to $M(A)$ on the same underline Hilbert space.

4.4. PROPOSITION. *If A is a C^*-algebra and $m_0 \in M(A)$ is such that $(m_0A)^- = A$, then*

$$\{\text{span}[\tilde{\pi}(m_0)H]\}^- = H$$

for any nondegenerate representation π of A.

Proof. If $\{\text{span}[\tilde{\pi}(m_0)H\}^- \neq H$, then there exists

$$0 \neq \xi \perp \{\text{span}[\tilde{\pi}(m_0)H]\}.$$

Since π is nondegenerate, there exists $a \in A$ such that $\pi(a)\xi \neq 0$. Define

$$\rho(b) = \langle \pi(a)\xi, \xi \rangle \quad \text{for each } b \in A.$$

Then ρ is a positive form on A and $\rho \neq 0$ since $\rho(a^*a) \neq 0$. We may assume that ρ is a state. Then

$$\tilde{\rho}(m) = \langle \tilde{\pi}(m)\xi, \xi \rangle, \quad \forall m \in M(A).$$

By the choice of ξ, we have $\tilde{\rho}(m_0) = 0$ and $\tilde{\rho}(m_0^2) = 0$. By Schwarz's inequality we have

$$|\tilde{\rho}(m_0b)|^2 \leqq \rho(b^*b)\tilde{\rho}(m_0^2) = 0 \quad \text{for all } b \text{ in } A.$$

It follows that $(m_0A)^- = A$ is contained in the kernel of $\tilde{\rho}$ and so $\rho = 0$. This is a contradiction.

4.5. PROPOSITION. *If A is a C^*-algebra and $m_0 \in M(A)$, then the following statements are equivalent*:

(1) $(m_0A)^- = A$.

(2) $\tilde{f}(m_0) > 0$ *for any state f of A.*

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(3) $\tilde{f}(m_0) > 0$ *for any pure state* f *of* A.

(4) *Any closed left ideal of* $M(A)$ *containing* m_0 *includes* A.

Proof. (1) $\Rightarrow$ (2). If $\tilde{f}(m_0) = 0$, let

$$\tilde{f}(m) = \langle \tilde{\pi}_f(m)\xi_f, \xi_f \rangle$$

for any $m \in M(A)$ by the GNS construction (see [**15**, Chapter 2]).

$$\tilde{f}(m_0^2) = \|\tilde{\pi}_f(m_0)\xi_f\|^2 \leqq \|\pi_f(m_0)^{1/2})\|^2\|\tilde{\pi}_f(m_0^{1/2})\xi_f\|^2 = 0.$$

By Schwarz's inequality, we have

$$|f(m_0a)|^2 \leqq |\tilde{f}(m_0a)|^2 \leqq f(a^*a)\tilde{f}(m_0^2) = 0, \quad \text{for all } a \in A.$$

Then f would be zero everywhere.

(2) $\Rightarrow$ (1). If $(m_0A)^- \neq A$, then $(m_0A)^-$ is a proper closed right ideal of A. It follows that there is a pure state f of A such that

$$f(a_1m_0^2a_2) = 0 \quad \text{for all } a_1, a_2 \in A.$$

Let a_λ be an approximate identity of A. Then

$$f(a_\lambda m_0^2 a_\lambda) = 0 \quad \text{for all } \lambda.$$

It follows that $\tilde{f}(m_0^2) = 0$. By Schwarz's inequality again we have $\tilde{f} = 0$ and so $f = 0$. That is a contradiction.

(3) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4) are trivial (see [**15**, Chapter 2]).

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