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ELEMENTARY REMARKS ON MULTIPLY MONOTONIC FUNCTIONS AND SEQUENCES([1])

BY
M. E. MULDOON

1. **Introduction and statement of results.** A function $f(x)$ is said to be completely monotonic on $(0, \infty)$ if

$$(1) \qquad (-1)^n f^{(n)}(x) \geq 0 \quad (0 < x < \infty; \quad n = 0, 1, 2, \ldots).$$

Familiar examples of such functions are given by $f(x) = \exp(-\alpha x)$ and $f(x) = (x+\beta)^{-\alpha}$, where $\alpha \geq 0, \beta \geq 0$. A discussion of completely monotonic functions is given in [5, Ch. IV].

J. Dubourdieu [1, p. 98] showed that if $f(x)$ satisfies (1), then we must necessarily have

$$(-1)^n f^{(n)}(x) > 0 \quad (0 < x < \infty; \quad n = 0, 1, 2, \ldots),$$

unless $f(x)$ is constant. This fact was rediscovered, and proved in a more elementary way, by Lee Lorch and Peter Szego [3, pp. 71–72]. An equivalent result on completely monotonic sequences was proved by Lee Lorch and Leo Moser [2]. They showed that if

$$(-1)^n \Delta^n x_k \geq 0 \quad (n, k = 0, 1, 2, \ldots),$$

then in fact we must have

$$(-1)^n \Delta^n x_k > 0 \quad (n, k = 0, 1, 2, \ldots)$$

unless $x_1 = x_2 = \cdots$, i.e., unless the sequence $\{x_0, x_1, x_2, \ldots\}$ is constant from the second term on. (Here, and in what follows, we use the notation $\Delta^0 x_k = x_k$, $\Delta x_k = x_{k+1} - x_k$, etc.)

In this note we prove some corresponding results for N-times monotonic functions and sequences, using the methods of [2] and [3, pp. 71–72]. Recalling some work of I. J. Schoenberg [4] and R. E. Williamson [6] we give the following:

DEFINITION. A function $f(x)$ is said to be N-times monotonic on $(0, \infty)$, where N is an integer ≥ 2, if

$$(-1)^n f^{(n)}(x) \geq 0 \quad (0 < x < \infty; \quad n = 0, 1, \ldots, N-2)$$

and if $(-1)^{N-2} f^{(N-2)}(x)$ is nonincreasing and convex on $(0, \infty)$. For $N = 1$, the

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N-times monotonic functions are those which are non-negative and nonincreasing, and for $N=0$, they are simply the non-negative functions.

Williamson [6, p. 190] gives the example

$$f(x) = \begin{cases} (1-x)^{N-1}, & 0 < x < 1, \\ 0, & x \geq 1 \end{cases}$$

of a function which is N-times monotonic on $(0,\infty)$ $(N \geq 1)$, but is not r-times monotonic for any $r > N$. Another example of such a function is given by

$$f(x) = \int_x^\infty (t-x)^{N-1} t^{-N-1} (\sin t)^2 \, dt. \tag{2}$$

We have the following:

THEOREM 1. *Let $f(x)$ be N-times monotonic on $(0,\infty)$, where N is a non-negative integer, and suppose that $f(x)$ is not eventually constant (i.e., there do not exist numbers b and c, such that $f(x)=c$ for $x \geq b$). Then*

$$(-1)^n f^{(n)}(x) > 0 \quad (0 < x < \infty; \quad N = 0, 1, \ldots, N-2), \tag{3}$$

and $(-1)^{N-2} f^{(N-2)}(x)$ is strictly decreasing on $(0,\infty)$. If $f^{(N-1)}(x)$ exists, we have $(-1)^{N-1} f^{(N-1)}(x) > 0$ on $(0,\infty)$ and if $f^{(N)}(x)$ exists, there are points in every neighbourhood of $+\infty$ at which $(-1)^N f^{(N)}(x) > 0$.

The corresponding result for N-times monotonic sequences is

THEOREM 2. *Suppose that for a sequence $\{x_0, x_1, x_2, \ldots\}$, we have*

$$(-1)^n \Delta^n x_k \geq 0 \quad (n = 0, 1, \ldots, N; \quad k = 0, 1, 2, \ldots), \tag{4}$$

and that x_k is not eventually constant. Then

$$(-1)^n \Delta^n x_k > 0 \quad (n = 0, 1, \ldots, N-1; k = 0, 1, 2, \ldots), \tag{5}$$

and $(-1)^N \Delta^N x_k > 0$ for infinitely many values of k.

2. **Proofs of the theorems.** Let $N \geq 2$ and let f satisfy the hypotheses of Theorem 1. Suppose that (3) does not hold. Then there exists an integer $m=0, 1, \ldots, N-2$ and a number $\xi (0 < \xi < \infty)$ such that $f^{(m)}(\xi)=0$. Since $(-1)^m f^{(m)}(\xi)$ is nonincreasing we find $f^{(m)}(x)=0$ on $[\xi,\infty)$. This means that f is a polynomial of degree at most m on $[\xi,\infty)$. However, since f is non-negative and nonincreasing on $[\xi,\infty)$, it must be bounded there and so is constant on $[\xi,\infty)$. This contradiction shows that (3) holds.

If $(-1)^{N-2} f^{(N-2)}(x)$ is not strictly decreasing on $(0,\infty)$ we have $f^{(N-2)}(x_1) = f^{(N-2)}(x_2)$ for some x_1, x_2 with $0 < x_1 < x_2 < \infty$. Then the convexity and nonincreasing character of $(-1)^{N-2} f^{(N-2)}(x)$ shows that $f^{(N-2)}(x)$ is constant on (x_1,∞). This implies that f is a polynomial on (x_1,∞) and we obtain the same contradiction as before.

If $f^{(N-1)}(x)$ exists and if $f^{(N-1)}(\xi)=0$, the convexity and nonincreasing character of $(-1)^{N-2}f^{(N-2)}(x)$ shows that $f^{(N-1)}(x)=0$ for $x \geq \xi$, which leads again to a contradiction.

Finally, if $f^{(N)}(x)$ exists and is eventually zero we arrive at the same contradiction. Thus there are points in every neighbourhood of $+\infty$ at which $f^{(N)}(x) \neq 0$. The convexity of $(-1)^{N-2}f^{(N-2)}(x)$ shows that at these points, we have in fact $(-1)^N f^{(N)}(x) > 0$.

Minor changes are required in the above proof in the cases $N=0$ and $N=1$.

The proof of Theorem 2 is similar to that of Theorem 1, the concept of polynomial sequence replacing that of polynomial. The proof that (4) implies (5) is really contained, though not explicitly stated, in the proof given by Lorch and Moser for the case $N=\infty$.

3. **Additional remarks.** The remark of Dubourdieu follows from Theorem 1 on using the fact that a completely monotonic function on $(0, \infty)$, being analytic [5, p. 146], is identically constant if it is eventually constant. Similarly the result of Lorch and Moser is a consequence of Theorem 2. It follows from the work of Lorch and Moser [2, p. 172] that a completely monotonic sequence which is eventually constant must be constant from the second term on.

Williamson [6, p. 191, Theorem 1] showed that f is N-times monotonic on $(0, \infty)$ $(N \geq 1)$, if and only if it is representable in the form

$$f(x) = \int_0^{1/x} (1-xt)^{N-1}\, d\varphi(t), \quad 0 < x < \infty, \tag{6}$$

where $\varphi(t)$ is nondecreasing and bounded below.

An alternative proof of Theorem 1 may be based on this representation on making the observation that the N-times monotonic functions which are not eventually constant are precisely those for which the function $\varphi(t)$ in the representation (6) has points of increase in the open interval $(0, \varepsilon)$ for every $\varepsilon > 0$. In case $f(x)$ is N-times monotonic and $f^{(N)}(x)$ exists it follows from a formula of Williamson [6, p. 192 (1.2)] (or may be proved directly, using (6)) that $\varphi'(t)$ exists and that

$$(-1)^N f^{(N)}(x) = (N-1)!\, x^{-N-1}\varphi'(1/x), \quad 0 < x < \infty.$$

This last equation shows that under the hypotheses of Theorem 1, $f^{(N)}(x)$ may vanish on an arbitrarily large interval $(0, a)$, $a > 0$, and may have zeros in every neighbourhood of $+\infty$. This last possibility is exemplified by the function in equation (2), for which we have

$$(-1)^n f^{(n)}(x) = (N-1)(N-2)\ldots(N-n) \int_x^\infty (t-x)^{N-n-1} t^{-N-1} (\sin t)^2\, dt$$

for $n=0, 1, \ldots, N-1$, $0 < x < \infty$, and

$$(-1)^N f^{(N)}(x) = (N-1)!\, x^{-N-1} (\sin x)^2, \quad 0 < x < \infty.$$

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References

1. J. Dubourdieu, *Sur un théorème de M. S. Bernstein relatif à la transformation de Laplace-Stieltjes*, Compositio Math. **7** (1939–1940), 96–111.

2. Lee Lorch and Leo Moser, *A remark on completely monotonic sequences with an application to summability*, Canad. Math. Bull. **6** (1963), 171–173.

3. Lee Lorch and Peter Szego, *Higher monotonicity properties of certain Sturm-Liouville functions*, Acta Math. **109** (1963), 55–73.

4. I. J. Schoenberg, *On integral representations of completely monotone and related functions* (abstract), Bull. Amer. Math. Soc. **47** (1941), p. 208.

5. D. V. Widder, *The Laplace transform*, Princeton Univ. Press, Princeton, N.J., 1941.

6. R. E. Williamson, *Multiply monotone functions and their Laplace transforms*, Duke Math. J. **23** (1956), 189–207.

York University,
Downsview, Ontario