Canad. Math. Bull. Vol. 14 (1), 1971

A NOTE ON COMBINATORIAL IDENTITIES FOR PARTIAL SUMS

BY
S. G. MOHANTY

1. **Introduction.** For a sequence $\sigma=(x_1,\ldots,x_n)$ of real numbers, let σ_i, and σ_j^* respectively denote the cyclic permutation $(x_i, x_{i+1},\ldots,x_{i-1})$ and the reverse cyclic permutation $(x_j, x_{j-1},\ldots,x_{j+1})$, and let $s_k=\sum_{j=1}^{k} x_j$. Also denote by $M_{rj}(\sigma)$ and $m_{rj}(\sigma)$ the rth largest and the rth smallest numbers respectively, among the first j partial sums $s_1, s_2,\ldots,s_j$ for $1\le r\le j\le n$. As usual, let the superscripts + and − respectively mean maximize and minimize with zero. In a paper of Harper [3], the main result which generalizes earlier results of Dwass [1] and Graham [2], is as follows:

THEOREM.

$$\sum_{i=1}^{n} [M_{rj}^{+}(\sigma_i)+m_{rj}^{-}(\sigma_i^*)] = (j-r+1)s_n. \tag{1}$$

The proof mainly depends on the following identities:

$$M_{rj}^{+}(\sigma_i)+m_{rj}^{-}(\sigma_{i+j-1}^*) = M_{r-1,j-1}^{+}(\sigma_i)+m_{r-1,j-1}^{-}(\sigma_{i+j-1}^*) \tag{2}$$

and

$$M_{1j}^{+}(\sigma_i)+m_{1j}^{-}(\sigma_{i+j-1}^*) = s_j. \tag{3}$$

In this note, we give a generalization of this theorem and interpret the result for a sequence of vectors in real Hilbert space.

2. **The main result.** Let

$$\sigma(u) = \sum_{i=1}^{b} x_i+\sum_{i=b+c+1}^{2b+c} x_i+\cdots+\sum_{i=(u-1)(b+c)+1}^{ub+(u-1)c} x_i.$$

Suppose $\max_{1\le u\le j}^{(r)}(y_u)$ and $\min_{1\le u\le j}^{(r)}(y_u)$ represent the rth largest and the rth smallest numbers respectively among $y_1, y_2,\ldots,y_j$. Then we define

$$M(r, b, c, j; \sigma) = \max{}_{1\le u\le j}^{(r)}[\sigma(u)]$$

$$m(r, b, c, j; \sigma) = \min{}_{1\le u\le j}^{(r)}[\sigma(u)]$$

where

$$jb+(j-1)c \le n \quad\text{and}\quad 1\le r\le j.$$

Note that $M(r, 1, 0, j; \sigma)=M_{rj}(\sigma)$ and $m(r, 1, 0, j; \sigma)=m_{rj}(\sigma)$.

Received by the editors May 6, 1970.

https://doi.org/10.4153/CMB-1971-012-8 Published online by Cambridge University Press

THEOREM 1.

$$(4)\qquad \sum_{i=1}^{n} [M^{+}(r, b, c, j;\, \sigma_i)+m^{-}(r, b, c, j;\, \sigma_i^{*})] = (j-r+1)bs_n.$$

Proof. The proof follows the same line of argument as in [3] and therefore consists of obtaining the generalized form of (2) and (3). The corresponding identities are

$$(5)\qquad \begin{aligned} &M^{+}(r, b, c, j;\, \sigma_i)+m^{-}(r, b, c, j;\, \sigma^{*}_{jb+(j-1)c+i-1}) \\ &\qquad = M^{+}(r-1, b, c, j-1;\, \sigma_i)+m^{-}(r-1, b, c, j-1;\, \sigma^{*}_{jb+(j-1)c+i-1}) \end{aligned}$$

and

$$(6)\qquad \begin{aligned} &M^{+}(1, b, c, j;\, \sigma_i)+m^{-}(1, b, c, j;\, \sigma^{*}_{jb+(j-1)c+i-1}) \\ &\qquad = \sum_{k=i}^{b+i-1} x_k+\sum_{k=b+c+i}^{2b+c+i-1} x_k+\cdots+\sum_{k=(j-1)(b+c)+i}^{jb+(j-1)c+i-1} x_k \end{aligned}$$

where $x_{n+u}\equiv x_u$.

Introducing

$$a_k = M(k, b, c, j-1;\, \sigma) \quad\text{and}\quad b_k = m(k, b, c, j-1;\, \sigma^{*}_{jb+(j-1)c})$$

and proceeding exactly the same way as in [3], (5) can be checked. A simple verification establishes (6). The left-hand side of (4), with the help of (5) and (6), reduces to

$$\sum_{i=1}^{n}\left[\sum_{k=i}^{b+i-1} x_k+\sum_{k=b+c+i}^{2b+c+i-1} x_k+\cdots+\sum_{k=(j-r)(b+c)+i}^{(j-r+1)b+(j-r)c+i-1} x_k\right] = (j-r+1)bs_n.$$

This completes the proof.

The generalized expressions for (6) in [3] are

$$(7)\qquad \sum_{i=1}^{n} [M(r, b, c, j;\, \sigma_i)+m(r, b, c, j;\, \sigma_i^{*})] = (j+1)bs_n$$

and

$$(8)\qquad \sum_{i=1}^{n} [\,|M(r, b, c, j;\, \sigma_i)|-|m(r, b, c, j;\, \sigma_i^{*})|\,] = (j-2r+1)bs_n.$$

3. **Concluding remarks.** The method used in the proof suggests that the above results should be true for a sequence of vectors instead of real numbers. Let H be an arbitrary Hilbert space over the reals and let $\sigma=(x_1, \ldots, x_n)$ be a sequence with $x_i \in H$. For each $i=1, 2, \ldots, n$, we can write $x_i=x_i'+x_i''$ where x_i' and x_i'' are, respectively, the perpendicular and projection of x_i on the one-dimensional subspace spanned by s_n. Furthermore, x_i'' can be written as $\lambda_i e$, where $e=s_n/\|s_n\|$. We say that vector x_i is larger or smaller than vector x_j in relation to the subspace spanned by s_n, according as $\lambda_i>\lambda_j$ or $\lambda_j>\lambda_i$. $M^{+}(r, b, c, j;\, \sigma_i)$ and $m^{-}(r, b, c, j;\, \sigma_i)$

https://doi.org/10.4153/CMB-1971-012-8 Published online by Cambridge University Press

for vectors are defined as before. Then, the above results are also valid for vectors in H. Note that we can take $e=-s_n/\|s_n\|$, without altering anything.

References

1. M. Dwass, *A fluctuation theorem for cyclic random variables*, Ann. Math. Statist. **33** (1962), 1450–1454.
2. R. L. Graham, *A combinatorial theorem for partial sums*, Ann. Math. Statist. **34** (1963), 1600–1602.
3. L. H. Harper, *A family of combinatorial identities*, Ann. Math. Statist. **37** (1966), 509–512.

McMaster University,
Hamilton, Ontario