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ON THE SELECTION OF COMPACT SUBSETS OF POSITIVE MEASURE FROM ANALYTIC SETS OF POSITIVE MEASURE

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An important but seemingly difficult problem is to decide whether or not an analytic set A of positive h-measure, for some continuous Hausdorff function h, contains a compact subset C of positive h-measure, in every complete separable metric space Ω.

By extending some earlier work of R. O. Davies [**1**], M. Sion and D. Sjerve [**8**] proved that

(i) the selection of the set C is always possible in a σ-compact metric space Ω.

More recently Davies [**2**] has shown that it is always possible to select C

(ii) when $h(t) = t^s$, $t \geqq 0$, for some fixed positive number s,

(iii) when Ω is finite dimensional in the sense of [**4**],

(iv) when A has σ-finite h-measure, and

(v) when Ω is an ultra metric space.

The purpose of this article is to prove a common generalization (Theorem 1) of (i), (iii), (iv) and also to prove (Theorem 2) that if A is really large in that it has infinite generalized Hausdorff dimension, i.e., $\Lambda^h(A) = +\infty$ for all Hausdorff functions h (see P. R. Goodey [**3**]), then for each Hausdorff function h, A contains c disjoint compact subsets, each of non-σ-finite h-measure. This second theorem related to another unsolved problem of Hausdorff measure theory, namely: Does every compact (analytic) set of non-σ-finite h-measure contain c disjoint compact subsets each of non-σ-finite h-measure? (See C. A. Rogers [**6**, pp. 123–27].)

Definition. Let E be a subset of a complete separable metric space Ω and let h be a continuous Hausdorff function. We say that E is h-compact if $\Lambda_\delta{}^h(E)$ is finite for all positive numbers δ. We say that E is σ-h-compact if

$$E = \bigcup_{i=1}^{\infty} E_i$$

and each set E_i is h-compact.

THEOREM 1. *Let A be a σ-h-compact analytic subset of a complete separable metric space Ω and let A have positive h-measure. Then A contains a compact subset of positive h-measure.*

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THEOREM 2. *Let A be an analytic subset, of infinite generalized Hausdorff dimension, of a complete separable metric space Ω. Then, for each Hausdorff function h, A contains c disjoint compact subsets each of non-σ-finite h-measure.*

For the proofs of these theorems we shall draw freely on the techniques available in [**1**]–[**6**] and, in particular, those of [**2**]. We prove Theorem 1 by proving the increasing sets lemma (see [**6**, p. 90]), for an h-compact subset of a complete separable metric space Ω.

LEMMA 1. *Let E be an h-compact subset of a complete separable metric space Ω for some continuous Hausdorff function h and let δ, ϵ, η, $\delta < \eta$ be positive numbers. Let $E = \bigcup_{m=1}^{\infty} E_m$, where $E_1 \subset E_2 \subset \ldots$. Then, if*

$$(1) \qquad \Lambda_\delta^h(E_m) \leqq l < (1+\epsilon)^{-1}\Lambda_\eta^h(E), \qquad m = 1, 2, \ldots,$$

there exists a subset F of E such that

$$\Lambda_\delta^h(E_m \backslash F) \leqq l - (1+\epsilon)^{-1}\Lambda_\eta^h(F) < (1+\epsilon)^{-1}\Lambda_\eta^h(E\backslash F),$$

$m = 1, 2, \ldots,$ *and* $0 < \Lambda_\eta^h(F) < +\infty$.

Proof. We shall write $h(d(F)) = h(F)$ for all subsets F of Ω. Let $E = \bigcup_{r=1}^{\infty} F_n^r$, where $\delta \geqq d(F_n^1) \geqq d(F_n^2) \geqq \ldots$ and

$$\sum_{r=1}^{\infty} h(F_n^r) \to \lambda = \lim_{m\to\infty} \Lambda_\delta^h(E_m) \leqslant l \quad \text{as } n \to \infty.$$

By picking subsequences if necessary, we may suppose that $d(F_n^1) \to d_1 \geqq 0$ as $n \to \infty$. If $d_1 = 0$ then it follows that $\Lambda_{\delta^*}^h(E_m) \leqq l$ for all $\delta^* > 0$. Consequently $\Lambda^h(E_m) \leqq l$ and so $\Lambda^h(E) \leqq l$ which contradicts $\Lambda_\eta^h(E) > l$. So $d_1 > 0$. Also, as

$$\Lambda_\delta^h(E_n) \leqslant \Lambda_\delta^h(F_n^1) + \sum_{i=2}^{\infty} h(F_n^i) \leqslant \sum_{i=1}^{\infty} h(F_n^i)$$

we conclude, letting $n \to \infty$, that $\Lambda_\delta^h(F_n^1) \to h(d_1)$ as $n \to \infty$.

By choosing subsequences if necessary we may suppose that

$$(1+\epsilon)^{-1/3}h(d_1) < \Lambda_\delta^h(F_n^1), \qquad n = 1, 2, \ldots,$$

and

$$d(F_n^1) < d_1 + \theta$$

where

$$0 < \theta < \min(\delta, \tfrac{1}{3}(\eta - \delta)) \quad \text{and} \quad h(d_1 + 3\theta) < (1+\epsilon)^{1/2}h(d_1).$$

As E is h-compact, $\Lambda_\theta^h(E) < +\infty$. So there exists a partition $\{G_i\}_{i=1}$ of E into sets of diameter less than θ such that

$$\sum_{i=1}^{\infty} h(G_i) < +\infty.$$

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Hence there exists N such that

$$\sum_{i=N+1}^{\infty} h(G_i) < ((1+\epsilon)^{-1/3} - (1+\epsilon)^{-1/2})h(d_1).$$

We may suppose, by choosing subsequences if necessary, that there exists a partition R, S of $\{1, \ldots, N\}$ such that

$$F_n^1 \cap G_i \neq \emptyset, \qquad i \in R$$
$$F_n^1 \cap G_i = \emptyset, \qquad i \in S,$$

$n = 1, 2, \ldots$. Let

$$F = \bigcup_{i \in R} G_i.$$

Then $d(F) < d_1 + 3\theta$ and consequently

$$h(F) < (1+\epsilon)^{1/2} h(d_1).$$

Hence

$$\Lambda_\eta^h(F) < (1+\epsilon)^{1/2} h(d_1).$$

Also, as

$$F_n^1 \subset F \cup \bigcup_{N+1}^{\infty} G_i,$$

$$\Lambda_\delta^h(F) \geqslant \Lambda_\delta^h(F_n^1) - \sum_{N+1}^{\infty} h(G_i) > (1+\epsilon)^{-1/2} h(d_1) > 0,$$

and consequently $\Lambda_\eta^h(F) > 0$. Further

$$E_m \backslash F \subset (E_m \backslash F_m^1) \cup \left\{ \bigcup_{N+1}^{\infty} G_i \right\}.$$

Now $\sum_{i=1}^{\infty} h(F_m^i) \to \lambda \leqq l$ as $m \to \infty$ and $h(F_m^1) > (1+\epsilon)^{-1/3} h(d_1)$. Consequently, for m sufficiently large, and hence always,

$$\begin{aligned}\Lambda_\delta^h(E_m \backslash F) &\leqslant \sum_{i=2}^{\infty} h(F_m^i) + \sum_{i=N+1}^{\infty} h(G_i) \\ &\leqslant l - (1+\epsilon)^{-1/3} h(d_1) + ((1+\epsilon)^{-1/3} - (1+\epsilon)^{-1/2}) h(d_1) \\ &\leqslant l - (1+\epsilon)^{-1} \Lambda_\eta^h(F),\end{aligned}$$

which proves the left hand side of (1). The right hand side of (1) follows immediately from the observation that

$$\Lambda_\eta^h(E) \leqq \Lambda_\eta^h(E \backslash F) + \Lambda_\eta^h(F).$$

LEMMA 2. *Let E be an h-compact subset of a complete separable metric space Ω*

for some Hausdorff function h, and let δ, η, $\delta < \eta$ *be positive numbers. Then if* $E = \bigcup_{m=1}^{\infty} E_m$, $E_1 \subset E_2 \ldots$

$$\text{(2)} \qquad \Lambda_\eta^h(E) \leqslant \lim_{m\to\infty} \Lambda_\delta^h(E_m) \leqslant \Lambda_\delta^h(E).$$

Remark. We may interpret Lemma 2 as proving the increasing sets lemma for h-compact sets. Although we shall not use the fact, it is perhaps worth noting that in view of [**2**, Theorem 3], the lemma also holds for σ-h-compact subsets, and with $\delta = \eta$.

Proof of Lemma 2. Only the left hand side of (2) is non-trivial. If

$$\lim_{m\to\infty} \Lambda_\delta^h(E_m) < \Lambda_\eta^h(E)$$

then there exists l, $\epsilon > 0$ such that

$$\lim_{m\to\infty} \Lambda_\delta^h(E_m) \leqslant l, \quad l(1+\epsilon) < \Lambda_\eta^h(E).$$

By Lemma 1, there exists F_1, $0 < \Lambda_\eta^h(F_1) < +\infty$ such that for all m

$$\text{(3)} \qquad \Lambda_\delta^h(E_m \backslash F_1) \leqq l - (1+\epsilon)^{-1}\Lambda_\eta^h(F_1) < (1+\epsilon)^{-1}\Lambda_\eta^h(E\backslash F_1).$$

We may repeat this process until the inequalities similar to (3) cease to be true, producing disjoint subsets $\{F_\alpha\}_{\alpha<\beta}$, α, β countable ordinals, of E such that

$$0 < \Lambda_\eta^h(F_\alpha) < +\infty$$

and

$$\text{(4)} \qquad \Lambda_\delta^h(E_m \backslash \bigcup_{\alpha<\beta} F_\alpha) \leqslant l - (1+\epsilon)^{-1} \sum_{\alpha<\beta} \Lambda_\eta^h(F_\alpha) < (1+\epsilon)^{-1} \Lambda_\eta^h(E\backslash \bigcup_{\alpha<\beta} F_\alpha).$$

Since $\sum_{\alpha<\beta} \Lambda_\eta^h(F_\alpha) \leqq l(1+\epsilon)$, and $\Lambda_\eta^h(F_\alpha) > 0$ for $\alpha < \beta$, it follows that the process must terminate at some countable limit ordinal β_0. As the left hand side of (4) will still be true at β_0, it follows that

$$(1+\epsilon)^{-1}\, l\, \Lambda_\eta^h(E\backslash \bigcup_{\alpha<\beta_0} F_\alpha) \leqslant l - (1+\epsilon)^{-1} \sum_{\alpha<\beta_0} \Lambda_\eta^h(F_\alpha).$$

But then

$$(1+\epsilon)^{-1}\Lambda_\eta^h(E) \leqslant (1+\epsilon)^{-1}\Lambda_\eta^h(E\backslash \bigcup_{\alpha<\beta_0} F_\alpha) + (1+\epsilon)^{-1} \sum_{\alpha<\beta_0} \Lambda_\eta^h(F_\alpha) \leqslant l$$

which contradicts $(1+\epsilon)^{-1}\Lambda_\eta^h(E) > l$.

So we conclude that the left hand side of (2) is true which completes the proof of Lemma 2.

Proof of Theorem 1. If A is a σ-h-compact analytic subset of Ω, we first show that A is representable as

$$A = \bigcup_{m=1}^{\infty} A_m,$$

where $A_m \subset A_{m+1}$, $m = 1, 2, \ldots$ and each set A_m is an h-compact analytic subset of Ω.

Now $A = \bigcup_{m=1}^{\infty} E_m$, where $E_m \subset E_{m+1}$, $m = 1, 2, \ldots$ and each E_m is an h-compact subset of Ω. As $\Lambda_\delta{}^h$ is G_δ-regular, and each Borel set in a complete separable metric space is analytic, we can choose an analytic subset $A_m{}^n$ of A such that $E_m \subset A_m{}^n$ and

$$\Lambda_{1/n}{}^h(E_m) = \Lambda_{1/n}{}^h(A_m{}^n).$$

Then, if

$$A_m = \bigcup_{k \leqslant m} \bigcap_{n=1}^{\infty} A_k{}^n$$

$A_m \subset A_{m+1}$, $m = 1, 2, \ldots$, $\bigcup_{m=1}^{\infty} A_m = A$ and each A_m is h-compact and analytic.

Now, if A has positive h-measure then there exists m such that A_m has positive h-measure. By Lemma 2, the increasing sets lemma holds for A_m. Consequently, by standard arguments, see for example C. A. Rogers [**6**, Theorem 48], A_m, and hence A, contains a compact subset of positive h-measure. This completes the proof of Theorem 1.

LEMMA 3. *Let E be a subset of a complete separable metric space Ω and let $E = \bigcup_{n=1}^{\infty} E_m$, where $E_1 \subset E_2 \subset \ldots$. Then if $\delta > 0$ and h is a Hausdorff function such that*

$$\Lambda^h(E) = +\infty$$

and

$$0 \leqslant \lim_{n\to\infty} \Lambda_\delta{}^h(E_n) \leqslant l < +\infty$$

then there exists a subset W of E such that for all n

$$\Lambda_\delta{}^h(E_n \backslash W) \leqq l - h(d)/2$$

where $0 < d \leqq d(W) < 6d \leqq 6\delta$.

Proof. For each n we write

$$E_n = \bigcup_{r=1}^{\infty} V_n{}^r$$

where

$$\delta \geqslant d(V_n{}^1) \geqslant d(V_n{}^2) \geqslant \ldots,$$

and

$$\lim_{n\to\infty} \sum_{r=1}^{\infty} h(V_n{}^r) \leqslant l.$$

Also, by choosing subsequences if necessary, we may suppose that $d(V_n{}^1) \to d$ as $n \to \infty$. If $d = 0$ then, as in Lemma 1, it follows that $\Lambda^h(E) \leqq l$ which contradicts $\Lambda^h(E) = +\infty$.

For any two non-empty subsets A, B of Ω write

$$\chi(A, B) = \inf_{\bar{a}\in A, \bar{b}\in B} \rho(\bar{a}, \bar{b})$$

where ρ is the metric of Ω.

By Ramsey's theorem either there exists an infinite subsequence N such that

$$\chi(V_n^1, V_m^1) \geqq 2d, \qquad n, m \in N, n \neq m$$

or

$$\chi(V_n^1, V_m^1) \leqq 2d, \qquad n, m \in N.$$

If the former is true then clearly it follows that

$$\lim_{n\to\infty} \Lambda_\delta^h(E_n) = +\infty,$$

which is not true.

Consequently we must have

$$\chi(V_n^1, V_m^1) \leqq 2d, \qquad n, m \in N.$$

As in Lemma 1, $\Lambda_\delta^h(V_n^1) \to h(d)$ as $n \to \infty$. Hence, as $\Lambda_\delta^h(V_n^1) \leqq h(V_n^1)$ if $d(V_n^1) \leqq \delta$, we can suppose that

$$h(V_n^1) \geqq h(d)/2, \quad d(V_n^1) < 2d, \quad n \in N.$$

Let $W = \bigcup_{n\in N} V_n^1$. Then $0 < d \leqq d(W) < 6d$ and

$$\begin{aligned}\Lambda_\delta^h(E_n \backslash W) &\leqslant \sum_{r=2}^{\infty} h(V_n^r)\\ &= \sum_{r=1}^{\infty} h(V_n^r) - h(V_n^1)\\ &\leqslant l - \tfrac{1}{2}h(d), \quad n = 1, 2, \ldots,\end{aligned}$$

which completes the proof of Lemma 3.

LEMMA 4. *Let E be a subset of a complete separable metric and let $E = \bigcup_{n=1}^{\infty} E_n$, where $E_1 \subset E_2 \subset \ldots$. Then if $\delta > 0$ and h is a Hausdorff function such that*

$$0 \leqslant \lim_{n\to\infty} \Lambda_\delta^h(E_n) \leqslant l < +\infty,$$

there exists a cover $\{W_r\}_{r=1}^{\infty}$ of E by sets W_r, with $d(W_r) \leqq 6\delta$ and $d(W_r) \to 0$ as $r \to \infty$.

Proof. If $\Lambda^h(E)$ is finite then, given $\delta > 0$, there exists a covering $\{W_r^1\}_{r=1}^{\infty}$ of E such that

$$\sum_{r=1}^{\infty} h(W_r^1) < +\infty,$$

and $d(W_r^1) \leqq \delta < 6\delta$, $r = 1, 2, \ldots$. Then $d(W_r^1) \to 0$ as $r \to \infty$ and consequently $\{W_r^1\}_{r=1}^{\infty}$ satisfy Lemma 4.

Otherwise $\Lambda^h(E) = +\infty$. So, by Lemma 3, there exists a subset $W_1{}^2$ of E such that

$$\Lambda_\delta{}^h(E_n \backslash W_1{}^2) \leqq l - h(d_1)/2 \quad \text{for all } n,$$

where $0 < d_1 \leqq d(W_1{}^2) < 6d_1 \leqq 6\delta$.

If $\Lambda^h(E\backslash W_1{}^2) = +\infty$ we may repeat this process. Consequently we choose a possibly transfinite sequence of disjoint sets $W_\alpha{}^2$, $\alpha < \beta$ such that

$$0 \leqslant \Lambda_\delta{}^h(E_n \backslash \bigcup_{\alpha<\beta} W_\alpha{}^2) \leqslant l - \sum_{\alpha<\beta} h(d_\alpha), \quad n = 1, 2, \ldots$$

and $0 < d_\alpha \leqq d(W_\alpha{}^2) \leqq 6d_\alpha \leqq 6\delta$.

As $l < +\infty$, this process must terminate at some countable ordinal β_0 and then it must be that

$$0 \leqslant \Lambda_\delta{}^h(E_n \backslash \bigcup_{\alpha<\beta_0} W_\alpha{}^2) \leqslant l - \sum_{\alpha<\beta_0} h(d_\alpha), \quad n = 1, 2, \ldots$$

but $\Lambda^h(E\backslash \bigcup_{\alpha<\beta_0} W_\alpha{}^2) < +\infty$. So we may choose a partition $E = \bigcup_{k=1}^\infty G_k$ of $E\backslash\bigcup_{\alpha<\beta_0} W_\alpha{}^2$ with $d(G_k) \leqq \delta$, $k = 1, 2, \ldots$ and $\sum_{k=1}^\infty h(G_k) < +\infty$. Re-writing

$$\{W_\alpha{}^2\}_{\alpha<\beta_0} \cup \{G_k\}_{k=1}^\infty \quad \text{as} \quad \{W_r\}_{r=1}^\infty$$

we see that

$$\sum_{r=1}^\infty h(W_r) < +\infty$$

and consequently $d(W_r) \to 0$ as $r \to \infty$.

Further, $d(W_r) \leqq 6\delta$, $r = 1, 2, \ldots$ and $E \subset \bigcup_{r=1}^\infty W_r$, which completes the proof of Lemma 4.

Following Davies [**2**] we define

$$\Phi_\delta{}^h(E) = \inf\left[\lim_{n\to\infty} \Lambda_\delta{}^h(E_n)\right],$$

the infimum being taken over all increasing sequences of sets with union E. Let

$$\Phi^h(E) = \lim_{\delta\to 0} \Phi_\delta{}^h(E).$$

Now $\Phi^h(E)$ is a Borel regular metric outer measure on the subsets E of a complete separable metric space Ω. Further $\Phi_\delta{}^h(E) \leqq \Lambda_\delta{}^h(E)$ for all subsets E of Ω.

LEMMA 5. *Let E be a subset of a complete separable metric space Ω and suppose that E has infinite generalized Hausdorff dimension, i.e. $\Lambda^h(E) = +\infty$ for all Hausdorff functions h. Then there exists $\delta_1 > 0$, such that*

$$\Phi_{\delta_1}{}^h(E) = +\infty$$

for all Hausdorff functions h.

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Proof. We say that a set E has a fine repeated cover if there exists a sequence $\{U_i\}_{i=1}^{\infty}$ of sets such that

$$E \subset \bigcup_{j=i}^{\infty} U_j, \quad i = 1, 2, \ldots,$$

and $d(U_j) \to 0$ as $j \to \infty$. Then, by a result of P. R. Goodey [**3**] E has infinite generalized Hausdorff dimension if and only if E does not have a fine repeated cover.

If Lemma 5 is false then there exists a sequence of Hausdorff functions $\{h_i\}_{i=1}^{\infty}$ and a sequence of positive numbers $\{\delta_i\}_{i=1}^{\infty}$, $\delta_i \to 0$ as $i \to \infty$ such that

$$\Phi_{\delta_1}{}^{h_i}(E) < +\infty.$$

Consequently, there exists a sequence $\{E_n{}^i\}_{n=1}^{\infty}$ with $E_1{}^i \subset E_2{}^i \subset \ldots$, $E = \bigcup_{n=1}^{\infty} E_n{}^i$ and

$$\lim_{n\to\infty} \Lambda_{\delta_i}{}^{h_i}(E_n{}^i) < +\infty.$$

By Lemma 4, there exists a cover $\{W_r{}^i\}_{r=1}^{\infty}$ of E such that $d(W_r{}^i) \leqq 6\delta_i$ and $d(W_r{}^i) \to 0$ as $r \to \infty$, $i = 1, 2, \ldots$. Rearranging

$$\{\{W_r{}^i\}_{r=1}^{\infty}\}_{i=1}^{\infty}$$

as a single sequence, we obtain a fine repeated cover of E. This contradicts E having infinite generalized Hausdorff dimension and completes the proof of Lemma 5.

Let A be an analytic subset of a complete separable metric space Ω, and let $\mathbf{I}$ be the set of irrationals $\mathbf{i} = (i_1, \ldots, i_n, \ldots)$ in $[0, 1]$ expressed as continued fractions. Then, by definition, there exists a relatively closed subset $\mathbf{I}_0$ of $\mathbf{I}$ and a continuous function F on $\mathbf{I}_0$ such that

$$A = \bigcup_{\mathbf{i}\in\mathbf{I}_0} F(\mathbf{i}) \equiv F(\mathbf{I}_0).$$

It is usual, if $\mathbf{i} = (i_1, \ldots, i_n, \ldots) \in \mathbf{I}$ to write $\mathbf{i}/n = (i_1, \ldots, i_n)$, $n = 1, 2, \ldots$ and

$$F(\mathbf{i}/n) = \bigcup_{\mathbf{j}/n=\mathbf{i}/n} F(j).$$

LEMMA 6. *Let A be an analytic subset of a complete separable metric space with $A = F(\mathbf{I}_0)$ as above. Suppose also that A has infinite generalized Hausdorff dimension. Then, for each Hausdorff function h there exists a collection $\{C_\alpha\}_{\alpha<\omega_1}$, where ω_1 is the first uncountable ordinal, of disjoint compact subsets of A, each of positive h-measure and with $C_\alpha = F(\mathbf{I}^\alpha)$ for some compact subset $\mathbf{I}^\alpha$ of $\mathbf{I}_0$.*

Proof. By Lemma 5, there exists $\delta_1 > 0$ such that

$$\Phi_{\delta_1}{}^h(A) = +\infty.$$

By [**2**, Theorem 6], A contains a compact subset C_1 of positive Φ^h-measure and, *a fortiori*, C_1 has positive h-measure. Further there exists a compact subset $\mathbf{I}_1$ of $\mathbf{I}_0$ such that $C_1 = F(\mathbf{I}_1)$.

Suppose now that β is a countable ordinal and that disjoint compact subsets C_α, $\alpha < \beta$ of A have been defined and corresponding disjoint compact subsets $\mathbf{I}_\alpha$, $\alpha < \beta$ of $\mathbf{I}_0$ have been defined so that

$$C_\alpha = F(\mathbf{I}_\alpha), \qquad \alpha < \beta$$

and $\Lambda^h(C_\alpha) > 0$, $\alpha < \beta$.

Then $\bigcup_{\alpha<\beta} C_\alpha$ is a σ-compact subset of Ω and consequently, by [**6**, Theorem 33], there exists a Hausdorff function g such that

$$\Lambda^g\Big(\bigcup_{\alpha<\beta} C_\alpha\Big) = 0.$$

Consequently $A\backslash\bigcup_{\alpha<\beta} C_\alpha$ is an analytic set and has infinite generalized Hausdorff dimension. So, by Lemma 5, there exists $\delta_\beta > 0$ such that

$$\Phi_{\delta_\beta}\Big(A\backslash \bigcup_{\alpha<\beta} C_\alpha\Big) = +\infty.$$

Let $\mathbf{I}_\alpha^* = \{\mathbf{i} : \mathbf{i} \in \mathbf{I}_0, F(\mathbf{i}) \in C_\alpha\}$, $\alpha < \beta$. Then $\mathbf{I}_\alpha^*$ is a relatively closed subset of $\mathbf{I}_0$ containing $\mathbf{I}_\alpha$. So

$$I_0\backslash \bigcup_{\alpha<\beta} I_\alpha^*$$

is a G_δ-subset of $\mathbf{I}_0$. Consequently

$$I_0\backslash \bigcup_{\alpha<\beta} I_\alpha^*$$

is the continuous one-one image, under f_β, of a closed subset $\mathbf{I}_\beta$ of $\mathbf{I}$. Again we may pick a compact subset $\mathbf{I}_\beta'$ of $\mathbf{I}_\beta$ such that if

$$C_\beta = F(f_\beta(\mathbf{I}_\beta')),$$

then C_β is a compact subset of A with $\Lambda^h(C_\beta) > 0$. We write $\mathbf{I}_\beta = f_\beta(I_\beta')$ which is a compact subset of $\mathbf{I}_0$ disjoint from $\bigcup_{\alpha<\beta} \mathbf{I}_\alpha$.

Lemma 6 now follows by transfinite induction.

LEMMA 7. *Let* $\mathbf{I}_0$ *be a relatively closed subset of the irrationals* $\mathbf{I}$ *in* $[0, 1]$. *Let* $\mathscr{I}_0$ *denote the space of all compact subsets of* $\mathbf{I}_0$ *with the Hausdorff metric. Then* $\mathscr{I}_0$ *is an analytic set.*

Proof. There exists in $[0, 1] \times [0, 1]$ a closed subset U which is universal, for the closed sets of $[0, 1]$ (see, for example, W. Sierpiński [**7**, pp. 252–255]) i.e. if

$$U^x = \{y : (x, y) \in U\}$$

then, for $0 \leqq x \leqq 1$, every set U^x is congruent to a closed subset of $[0, 1]$ and, given any closed subset V of $[0, 1]$ there exists x, $0 \leqq x \leqq 1$, such that V is congruent to U^x.

Now consider

$$V = U \backslash \{[0, 1] \times \mathbf{I}_0\}.$$

For $0 \leqq x \leqq 1$, V^x is an F_σ-set. Consequently, by Kunugui's theorem (see [**5**]), $\text{proj}_X V$ is a Borel set, where X denotes the 1st coordinate axis.

So

$$W = [0, 1] \backslash \text{proj}_X V$$

is a Borel set. Now $\mathbf{x} \in W$ if and only if $U^x \subset \mathbf{I}_0$. Let f be the continuous one-one map of a relatively closed subset $\mathbf{I}_1$ of $\mathbf{I}$ onto W. If $\mathbf{i} \in \mathbf{I}_1$ let

$$g(\mathbf{i}) = \{C \in J_0 : C \text{ is congruent to } U^{f(i)}\}.$$

As U is compact, g is a continuous mapping from $\mathbf{I}_1$ onto $\mathscr{J}_0$ and consequently $\mathscr{J}_0$ is an analytic set.

Remark. By choosing U more carefully, i.e. so that $U^{x_1} \neq U^{x_2}$ if $\bar{x}_1 \neq \bar{x}_2$ we can ensure that g is one-one and consequently deduce that $\mathscr{J}_0$ is a Borel set.

Proof of Theorem 2. Consider the compact subsets $\{C_\alpha\}_{\alpha<\omega_1}$ of A and compact subsets $\{\mathbf{I}_\alpha\}_{\alpha<\omega_1}$ of $\mathbf{I}_0$ as in Lemma 6. As there are uncountably many C_α, we may suppose, by choosing a subcollection if necessary, that there exist $\delta, \eta > 0$ such that

$$\Lambda_\delta^h(C_\alpha) \geqq \eta, \qquad 0 \leqq \alpha < \omega_1.$$

Now let $\mathscr{J}_0$ denote the compact subsets of $\mathbf{I}_0$ with the Hausdorff metric ρ_1. By Lemma 7, $\mathscr{J}_0$ is an analytic subset of the complete separable metric space formed by the non-empty closed subsets of [0, 1]. We write

$$\mathscr{J}_0 = f(\mathbf{I}_1)$$

where f is a continuous function from a relatively closed subset $\mathbf{I}_1$ of $\mathbf{I}$ onto $\mathscr{J}_0$. Let $A = F(\mathbf{I}_0)$ where F is a continuous function from a relatively closed subset $\mathbf{I}_0$ of $\mathbf{I}$ onto A. Then, if J is a compact subset of $\mathbf{I}_0$,

$$G(J) = \{F(\mathbf{j}) : \mathbf{j} \in J\}$$

is a compact subset of A. We next show that the map G is continuous from $\mathscr{J}_0$ into the space $\mathscr{C}$ of compact subsets of A with Hausdorff metric ρ_2.

Let $J \in \mathscr{J}_0$ and, for $\epsilon > 0$, let

$$N_\epsilon = \{G(J^*) : J^* \in \mathscr{J}_0, \rho_2(G(J), G(J^*)) < \epsilon\}.$$

If $\mathbf{j} \in J$ then there exists $\theta(\mathbf{j}) > 0$ such that if $\mathbf{i} \in \mathbf{I}_0$ and $|\mathbf{i} - \mathbf{j}| < \theta(\mathbf{j})$ then

$$\rho(F(\mathbf{i}), F(\mathbf{j})) < \epsilon/2$$

where ρ is the metric on Ω. As J is compact we may choose $\theta(\mathbf{j}) = \theta > 0$, independent of $\mathbf{j}$ in J. So, if $M_\theta = \{J^* : \rho_1(J, J^*) < \theta\} \subset \mathscr{J}_0$, $G(J^*) \in N_\epsilon$

whenever $J^* \in M_\theta$. So G is continuous and consequently the map

$$g(\mathbf{i}) = Gf(\mathbf{i}), \mathbf{i} \in \mathbf{I}_1$$

is a continuous map from $\mathbf{I}$ to a subset $\mathscr{B}$ of $\mathscr{C}$.

Further, for each set C_α there exists $\mathbf{i}_\alpha \in \mathbf{I}_1$ such that

$$C_\alpha = F(\mathbf{i}_\alpha).$$

From $\mathbf{I}_1$, we remove all the intervals $[\mathbf{i}/n]$ where

$$[\mathbf{i}/n] = \{\mathbf{j} \in \mathbf{I} : \mathbf{j}/n = \mathbf{i}/n\},$$

such that $g(\mathbf{i}/n)$ does not contain uncountably many members of the collection $\{C_\alpha\}_{\alpha<\omega_1}$. If $\mathbf{I}_2$ is the remaining subset of $\mathbf{I}_1$ then $\mathbf{I}_2$ is a relatively closed uncountable subset of $\mathbf{I}$ and if $\mathbf{i} \in \mathbf{I}_2$ then $g(\mathbf{i}/n)$ contains uncountably many members of $\{C_\alpha\}_{\alpha<\omega_1}$, $n = 1, 2, \ldots$.

Now $g(\mathbf{I}_2)$ is an uncountable analytic subset of $\mathscr{C}$ and we next show that

(5) $\quad \Lambda_\delta^h(g(\mathbf{i})) \geqq \eta,\ i \in \mathbf{I}_2.$

For suppose that $\Lambda_\delta^h(g(\mathbf{i})) < \eta$. Then there exists a cover $\{G_j\}_{j=1}^k$ of $g(\mathbf{i})$ by open sets of diameter less than or equal to δ such that

$$\sum_{j=1}^{k} h(G_j) < \eta.$$

As g is continuous, there exists n such that $g^*(\mathbf{i}/n) \subset \bigcup_{j=1}^k G_j$, where

$$g^*(\mathbf{i}/n) = \bigcup_{\mathbf{i}^*/n=\mathbf{i}/n} g(\mathbf{i}^*).$$

By construction, there exists α and $\mathbf{i}_\alpha \in \mathbf{I}_2$ such that $C_\alpha = g(\mathbf{i}_\alpha)$ and $\mathbf{i}_\alpha/n = \mathbf{i}/n$. So $C_\alpha \subset \bigcup_{j=1}^k G_j$, and hence

$$\Lambda_\delta^h(C_\alpha) \leqslant \sum_{j=1}^{k} h(G_j) < \eta,$$

which contradicts $\Lambda_\delta^h(C_\alpha) \geqq \eta$ and thus establishes (5). As $g(\mathbf{I}_2)$ is an uncountable analytic set, it follows (see for example W. Sierpiński [7, p. 290]), that $g(\mathbf{I}_2)$ contains c "points". Thus, using (5), it follows that A contains c distinct compact sets, each of positive h-measure, but we cannot, without further argument, ensure that these sets are pairwise disjoint.

Consider $\mathbf{i}(1)$, $\mathbf{i}(2) \in \mathbf{I}_2$, where $g(\mathbf{i}(1)) = C_1$, $g(\mathbf{i}(2)) = C_2$. Because $C_1 \cap C_2 = \emptyset$, there exists disjoint open sets G_1, G_2 with $C_1 \subset G_1$, $C_2 \subset G_2$. As g is continuous, there exists a positive integer n_1 such that

$$g(\mathbf{i}(1)/n_1) \subset G_1, \qquad g(\mathbf{i}(2)/n_1) \subset G_2.$$

Suppose now we have defined, for some positive integer k, positive integers

(6) $\quad n_1 < n_2 < \ldots < n_k,$

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and points

(7) $\mathbf{i}(h_1, \ldots, h_k) \in \mathbf{I}_2, \quad h_j = 1 \text{ or } 2, 1 \leqq j \leqq k$

such that

(8) $\mathbf{i}(h_1, \ldots, h_j)/n_j = \mathbf{i}(h_1, \ldots, h_k)/n_j, \qquad 1 \leqq j \leqq k$

and

(9) $g(\mathbf{i}(h_1, \ldots, h_k)/n_k \cap g(\mathbf{i}(h_1', \ldots, h_k')/n_k) = \emptyset$

if $(h_1, \ldots, h_k) \neq (h_1', \ldots, h_k')$. By the construction of $\mathbf{I}_2$ there exist $\mathbf{i}(\alpha)$, $\mathbf{i}(\beta)$, $\alpha \neq \beta$ such that $\mathbf{i}(\alpha) = \mathbf{i}(h_1, \ldots, h_k)/n_k$, $\mathbf{i}(\beta) = \mathbf{i}(h_1, \ldots, h_k)/n_k$, and $g(\mathbf{i}(\alpha)) = C_\alpha$, $g(\mathbf{i}(\beta)) = C_\beta$. In particular, therefore $\mathbf{i}(\alpha) \neq \mathbf{i}(\beta)$. So there exists $n_{k+1} > n_k$ such that

$$g(\mathbf{i}(\alpha)|n_{k+1}) \cap g(\mathbf{i}(\beta)|n_{k+1}) = \emptyset.$$

We define

$$\mathbf{i}(h_1, \ldots, h_k, 1) = \mathbf{i}(\alpha)$$
$$\mathbf{i}(h_1, \ldots, h_k, 2) = \mathbf{i}(\beta).$$

With these definitions (6)–(9) are satisfied for k replaced by $k + 1$. By induction we suppose that a system has been defined to satisfy (6)–(9) for $k = 1, 2, \ldots$.

If $\mathscr{H}$ is the set of infinite sequences of one's and two's and

$\mathbf{h} = (h_1, \ldots, h_k, \ldots) \in \mathscr{H}$ we define $\mathbf{i}(\mathbf{h})$ by

$$\mathbf{i}(\mathbf{h})/n_k = \mathbf{i}(h_1, \ldots, h_k)/n_k, \qquad k = 1, 2, \ldots .$$

Properties (7) and (8) ensure that $\mathbf{i}(\mathbf{h})$ is well-defined, and, as $\mathbf{I}_2$ is relatively closed, $\mathbf{i}(\mathbf{h}) \in I_2, \mathbf{h} \in \mathscr{H}$. Further, if

$$\mathbf{h} = (h_1, \ldots, h_k, \ldots), \qquad \mathbf{h}' = (h_1', \ldots, h_k', \ldots)$$

are in $\mathscr{H}$ and $\mathbf{h} \neq \mathbf{h}'$ then there exists k such that

$$(h_1, \ldots, h_k) \neq (h_1', \ldots, h_k').$$

So, by (9),

(10) $g(\mathbf{i}(\mathbf{h})) \cap g(\mathbf{i}(\mathbf{h}')) = \emptyset.$

Consequently, combining (5) and (10), the collection

$$\{g(\mathbf{i}(\mathbf{h}))\}_{\mathbf{h} \in \mathscr{H}}$$

form c pairwise disjoint compact subsets of A, each of positive h-measure.

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