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DIMENSIONS OF INTERSECTIONS OF THE SIERPINSKI CARPET WITH LINES OF RATIONAL SLOPES

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Abstract This paper computes the Box and Hausdorff dimensions of the intersections of the Sierpinski carpet with planar lines of rational slopes.

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1. Introduction

For eight points $(x_i, y_i) \in \{0,1,2\} \times \{0,1,2\} \setminus \{(1,1)\}$, let $\phi_i(x,y) = \frac{1}{3}(x,y) + \frac{1}{3}(x_i,y_i)$. Then the Sierpinski carpet F of $\mathbb{R}^2$ is the invariant set of $\{\phi_i\}_{i=1}^{8}$ with $\dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F = \log 8/\log 3$ [**4**].

Given $\theta \in [0,2\pi) \setminus \{\pi/2, 3\pi/2\}$, let $L_{\theta,a}$ be the line $y = (\tan\theta)x + a$, and the section $F_{\theta,a} = L_{\theta,a} \cap F$, the intersection of the Sierpinski carpet and the planar line. For any $\theta \neq \pi/2, 3\pi/2$, the interval J_θ is defined by

$$J_\theta = \begin{cases} [-\tan\theta, 1] & \text{if } \theta \in [0,\pi/2) \cup [\pi, 3\pi/2), \\ [0, 1-\tan\theta] & \text{if } \theta \in (\pi/2,\pi) \cup [3\pi/2, 2\pi). \end{cases}$$

Then for any $\theta \neq \pi/2, 3\pi/2$, we have $F_{\theta,a} \neq \varnothing \iff a \in J_\theta$.

In the paper, we focus on the intersections of the Sierpinski carpet with lines of rational slope. When both $\tan\theta$ and a are rational, [**6**] proved that $F_{\theta,a}$ has a graph-directed structure [**10**], and the corresponding dimension is obtained.

The intersections of some *special* planar sets with lines in a *fixed direction* are studied in [**3**], [**1**] and [**5**], among other publications. For example, [**3**] proved that $\dim_{\mathrm{H}}[L_{\pi/4,a} \cap (C \times C)] = \log 2/(3\log 3)$ for almost all $a \in [-1,1]$, where C is the Cantor ternary set.

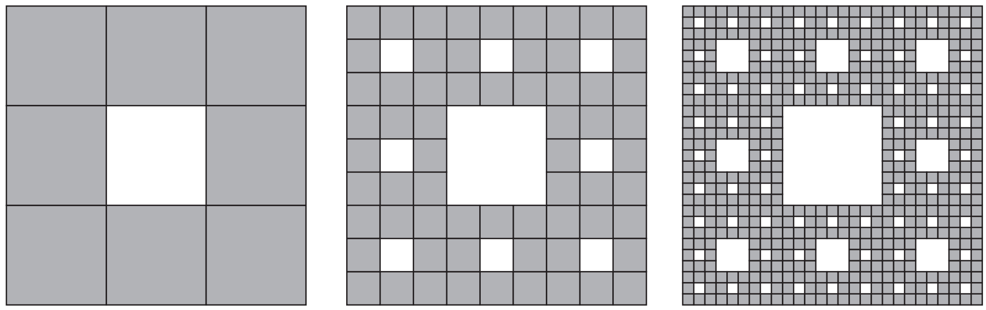

Figure 1. The steps for generating the Sierpinski carpet.

In [**1**] the dimensions of fibres $F_x = \{y \in [0,1] : (x,y) \in F\}$ for almost all $x \in [0,1]$ were discussed for some certain geometric constructions in the unit square $[0,1] \times [0,1]$. As shown in [**5**], we can calculate the typical value of the Hausdorff dimension of $L_{\pi/4,a} \cap F$ for almost all $a \in J_{\pi/4}$. In the literature listed above, $\tan\theta = 0$ or 1, but how about the general case for $\tan\theta \in \mathbb{Q}$? This question is the motivation for this paper.

The main result of paper is as follows.

Theorem 1.1. *Suppose F is the Sierpinski carpet in the plane and that $\tan\theta = M/N > 0$ is rational with $N, M \in \mathbb{N}$. Let A_0, A_1 and A_2 be $(M+N) \times (M+N)$ non-negative integer matrices defined by $A_t = (c^t_{pq})_{1\leqslant p,q\leqslant N+M}$ and $c^t_{pq} = \#\{i : x_iM - y_iN = 2M + 2 + q - 3p - t\}$. Then we have the following.*

(1) *If*

$$a = \frac{-M-1+k}{N} + \frac{1}{N}\biggl(\sum_{i=1}^{\infty} x_i 3^{-i}\biggr)$$

with $k \in \mathbb{N} \cap [1, N+M]$ and if $\{x_i\}_{i\geqslant 1} \in \{0,1,2\}^{\mathbb{N}}$ satisfies $3^n aN \notin \mathbb{Z}$ for all $n \in \mathbb{N}$, then

$$\overline{\dim}_{\mathrm{B}} F_{\theta,a} = \varlimsup_{n\to\infty} \frac{\log \|e_k A_{x_1} A_{x_2} \cdots A_{x_n}\|}{n \log 3},$$

where $e_k = (\delta_{k,1}, \delta_{k,2}, \dots, \delta_{k,N+M})$ is the kth natural basis of $\mathbb{R}^{N+M}$.

(2) *Denote by $\mathcal{L}$ the Lebesgue measure, then for $\mathcal{L}$-a.e. $a \in J_\theta$,*

$$\dim_{\mathrm{B}} F_{\theta,a} = \gamma/\log 3 \leqslant \log 8/\log 3 - 1,$$

where γ is the Lyapunov exponent for the symmetric independent random product of A_0, A_1, A_2, i.e.

$$\gamma = \lim_{n\to\infty} \frac{\log \|A_{x_1} A_{x_2} \cdots A_{x_n}\|}{n}$$

with x_n i.i.d. random variables assuming the values $\{0,1,2\}$ with equal probabilities.

(3) *For $\mathcal{L}$-a.e. $a \in J_\theta$, $\dim_{\mathrm{H}} F_{\theta,a} = \dim_{\mathrm{B}} F_{\theta,a}$.*

Remark 1.2. The results for $\theta \in (\pi/2, 2\pi)$ are the same.

Remark 1.3. The Marstrand theorem [**8**,**9**] concerns dimensions of sections for almost all θ, where θ is random.

Remark 1.4. By using the Hutchinson metric of fractal measures, we can compute the Lyapunov exponent in special cases and we obtain

$$\begin{aligned} &\text{when } \tan\theta = 1, && \text{for a.e. } a \in [-1,1], && \dim_{\mathrm{H}} F_{\theta,a} = 0.8858\dots, \\ &\text{when } \tan\theta = \tfrac{1}{2}, && \text{for a.e. } a \in [-\tfrac{1}{2},1], && \dim_{\mathrm{H}} F_{\theta,a} = 0.8914\dots, \\ &\text{when } \tan\theta = \tfrac{1}{3}, && \text{for a.e. } a \in [-\tfrac{1}{3},1], && \dim_{\mathrm{H}} F_{\theta,a} = 0.8926\dots; \end{aligned}$$

all of them are less than $\log 8/\log 3 - 1 = 0.8927\dots$.

The paper is organized as follows. Section 2 is gives some preliminary information about the box dimension. In § 3, we prove Theorem 1.1 (1). In § 4, Theorem 1.1 (2) is proved using our key lemma: Lemma 4.2 of ergodic type. Section 5 is devoted to the proof of Theorem 1.1 (3). In § 6, we describe the method mentioned in Remark 1.4. In the final section, we point out that our method can apply to fractals like the Sierpinski carpet.

2. Preliminaries

In this section, we do not need the condition that the slope $\tan\theta$ is rational. For $i = 1,\dots,8$, let $\phi_i(x,y) = \frac{1}{3}(x,y) + \frac{1}{3}(x_i,y_i)$, where $(x_i,y_i) \in \{0,1,2\} \times \{0,1,2\} \setminus \{(1,1)\}$.

Fix any $\theta \in [0,2\pi) \setminus \{\pi/2, 3\pi/3\}$, set $T_i(x) = 3x + x_i\tan\theta - y_i$, then $S_i = (T_i)^{-1} : J_\theta \to J_\theta$ are linear contractions satisfying

$$J_\theta = \bigcup_{i=1}^{8} S_i(J_\theta).$$

Let m_θ denote the normalized Lebesgue measure on J_θ, i.e. $m_\theta = \mathcal{L}/|J_\theta|$ with $m_\theta(J_\theta) = 1$. Write $\phi_{i_1\cdots i_n} = \phi_{i_1} \circ \cdots \circ \phi_{i_n}$.

Let $N_n(a) = \#\{i_1\cdots i_n : \phi_{i_1\cdots i_n}([0,1]\times[0,1]) \cap L_{\theta,a} \neq \varnothing\}$. Denote by $K_n(a)$ the number of 3-adic squares of side length 3^{-n} intersecting $F \cap L_{\theta,a}$.

Since $\phi_{i_1\cdots i_n}([0,1]\times[0,1]) \cap L_{\theta,a} \neq \varnothing$ implies $\phi_{i_1\cdots i_n}(F) \cap L_{\theta,a} \neq \varnothing$, we have $N_n(a) \leqslant K_n(a) \leqslant 9N_n(a)$. It follows from the definition of the box dimension that

$$\overline{\dim}_{\mathrm{B}}(F \cap L_{\theta,a}) = \varlimsup_{n\to\infty} \frac{\log K_n(a)}{n\log 3} = \varlimsup_{n\to\infty} \frac{\log N_n(a)}{n\log 3}, \tag{2.1}$$

$$\underline{\dim}_{\mathrm{B}}(F \cap L_{\theta,a}) = \varliminf_{n\to\infty} \frac{\log K_n(a)}{n\log 3} = \varliminf_{n\to\infty} \frac{\log N_n(a)}{n\log 3}. \tag{2.2}$$

We have the following lemma.

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Lemma 2.1.

$$N_n(a) = \sum_{i_1 \cdots i_n} 1_{S_{i_1 \cdots i_n}(J_\theta)}(a). \tag{2.3}$$

$$\int_{J_\theta} N_n(a)\,\mathrm{d}m_\theta(a) = (\tfrac{8}{3})^n. \tag{2.4}$$

Proof. By induction, it is easy to verify that

$$L_{\theta,a} \cap \phi_{i_1 \cdots i_n}([0,1] \times [0,1]) \neq \varnothing \quad \Longleftrightarrow \quad a \in S_{i_1 \cdots i_n}(J_\theta).$$

Therefore,

$$\begin{aligned} N_n(a) &= \#\{i_1 \cdots i_n : L_{\theta,a} \cap \phi_{i_1 \cdots i_n}([0,1] \times [0,1]) \neq \varnothing\} \\ &= \#\{i_1 \cdots i_n : a \in S_{i_1 \cdots i_n}(J_\theta)\} \\ &= \sum_{i_1 \cdots i_n} 1_{S_{i_1 \cdots i_n}(J_\theta)}(a). \end{aligned}$$

And thus,

$$\int N_n(a)\,\mathrm{d}m_\theta(a) = \frac{1}{|J_\theta|} \sum_{i_1 \cdots i_n} \int 1_{S_{i_1 \cdots i_n}(J_\theta)}(a)\,\mathrm{d}\mathcal{L}(a) = (\tfrac{8}{3})^n.$$

□

3. The upper box dimension

In this section, we prove Theorem 1.1 (1). Without loss of generality, we suppose that $\tan\theta = M/N > 0$ is rational, where $M, N \in \mathbb{N}$ with $(M, N) = 1$. Here $J_\theta = [-M/N, 1]$.

Let $D = \{a \in \mathbb{R} : 3^n(aN) \notin \mathbb{Z}$ for any integer $n \geqslant 0\}$.

Lemma 3.1. *$\mathbb{R} \setminus D$ is an enumerable set. If $a \in D$ and $n \in \{0\} \cup \mathbb{N}$, then $3^n a \in D$.*

For any $a \in D$, let $\Gamma_a = \{a + i/N \in J_\theta : i \in \mathbb{Z}\}$, then $\#\Gamma_a = (1 + (M/N))/(1/N) = (N + M)$ since $a \notin \mathbb{Z}/N$. Therefore, for any integer $n \geqslant 0$, we have $\#\Gamma_{3^n a} = (N + M)$ as $3^n a \in D$.

Given $a \in D$, we arrange the elements of Γ_a as follows:

$$\Gamma_a(1) < \Gamma_a(2) < \cdots < \Gamma_a(N + M),$$

where

$$\Gamma_a(i) \in \left(\frac{-M - 1 + i}{N}, \frac{-M + i}{N}\right) \mathrel{\hat=} I_i \quad (1 \leqslant i \leqslant N + M).$$

Notice that $\phi_j(F) \cap L_{\theta,a} \neq \varnothing \Leftrightarrow a \in S_j(J_\theta)$ and $\phi_j^{-1}(L_{\theta,a}) = L_{\theta,T_j(a)}$. Hence, for any $a \in J_\theta$, we have

$$\begin{aligned} F_{\theta,a} &= F \cap L_{\theta,a} \\ &= \left[\bigcup_{j=1}^{8} \phi_j(F)\right] \cap L_{\theta,a} \\ &= \bigcup_{j=1}^{8} [\phi_j(F) \cap L_{\theta,a}] \\ &= \bigcup_{j \text{ s.t. } a \in S_j(J_\theta)} \phi_j[F \cap \phi_j^{-1}(L_{\theta,a})] \\ &= \bigcup_{j \text{ s.t. } a \in S_j(J_\theta)} \phi_j(F \cap L_{\theta,T_j a}) \\ &= \bigcup_{j \text{ s.t. } a \in S_j(J_\theta)} \phi_j(F_{\theta,T_j a}). \end{aligned}$$

In particular, for any $a \in J_\theta$, as $a = T_i(S_i(a))$,

$$F_{\theta,S_i(a)} \supset \phi_i(F_{\theta,a}). \tag{$*$}$$

If $b \in \Gamma_a$ and $T_i(b) \in J_\theta$, then $T_i(b) \in \Gamma_{3a}$.

We know that, for any $a \in J_\theta$, $F_{\theta,a}$ is composed of some reduced (with ratio $\frac{1}{3}$) copies of $F_{\theta,b}$ for some $b \in \Gamma_{3a}$. We record the number of copies with the following matrix: given $a \in J_\theta$, let $A(a) = (c_{pq})_{1 \leqslant p,q \leqslant N+M}$ be a non-negative integer matrix defined by

$$c_{pq} = \#\{i : T_i(\Gamma_a(p)) = \Gamma_{3a}(q)\},$$

where c_{pq} is just the number of the reduced copies of $F_{\theta,\Gamma_{3a}(q)}$ that are contained in $F_{\theta,\Gamma_a(p)}$, since

$$F_{\theta,\Gamma_a(p)} = \bigcup_{i \text{ s.t. } \Gamma_a(p) \in S_i(J_\theta)} \phi_i(F_{\theta,T_i(\Gamma_a(p))})$$

and $T_i(\Gamma_a(p)) = \Gamma_{3a}(q)$ implies $\Gamma_a(p) \in S_i(J_\theta)$.

Remark 3.2. We can see that for any $a \in J_\theta$, $F_{\theta,a}$ is a multi-type Moran set with generating matrices $\{A(3^n a)\}_{n \geqslant 0}$ and constant ratio $\frac{1}{3}$. Please refer to [**7**] for the definition of multi-type Moran sets. If $\{A(3^n a)\}_{n \geqslant 0}$ is an ultimately periodic sequence, e.g. when $a \in D$ is rational as in [**6**], then $F_{\theta,a}$ can be characterized as a graph-directed set.

For any $a \in D \cap J_\theta$, let $i_0(a)$ be the integer satisfying

$$\Gamma_a(i_0(a)) = a.$$

Then

$$N_n(a) = \|e_{i_0(a)} A(a) A(3a) \cdots A(3^{n-1}a)\|_1, \tag{3.1}$$

where the norm of the row vector is given by

$$\|(x_1, \dots, x_{N+M})\|_1 = \sum_i |x_i|.$$

For any $x \in \mathbb{R}$, let $x(\operatorname{mod}(1/N))$ denote the unique value $x' \in [0, 1/N)$ with $x \equiv x' \mod(1/N)$. Here $x \equiv y \mod(1/N)$ means that $N(x-y)$ is an integer.

Given any $a \in D \cap J_\theta$, we write

$$a = \frac{-M-1+k}{N} + \frac{1}{N}\left(\sum_{i=1}^{\infty} x_i 3^{-i}\right)$$

with $k \in \mathbb{N} \cap [1, N+M]$ and $\{x_i\}_{i \geqslant 1} \in \{0,1,2\}^{\mathbb{N}}$.

We have the following properties.

(1) Suppose $x, y \in D$, then

$$x \equiv y \mod(1/N) \implies \Gamma_x = \Gamma_y \text{ and } A(x) = A(y). \tag{3.2}$$

Furthermore, given $j \in \{0,1,2\}$,

$$A(x) \text{ is constant on } \left\{x \in D : x(\operatorname{mod}(1/N)) \in \left(\frac{j}{3N}, \frac{j+1}{3N}\right)\right\},$$

since, for any $(x_i, y_i) \in \{0,1,2\} \times \{0,1,2\} \setminus \{(1,1)\}$ and $\eta \in \mathbb{Z}$,

$$3\left(\frac{\eta}{3N}, \frac{\eta+1}{3N}\right) + x_i\frac{M}{N} - y_i = \left(\frac{k}{N}, \frac{k-1}{N}\right) \quad \text{for some } k \in \mathbb{Z}.$$

Denote by A_j the above constant matrix. In fact, define intervals

$$I_q = \left(\frac{-M-1+q}{N}, \frac{-M+q}{N}\right) \quad \text{and} \quad J_p^t = \left(\frac{-M-1+p}{N} + \frac{t}{3N}, \frac{-M-1+p}{N} + \frac{t+1}{3N}\right)$$

for $p, q \in \mathbb{N} \cap [1, N+M]$ and $t \in \{0,1,2\}$. Then $A_t = (c_{pq}^t)_{1 \leqslant p,q \leqslant N+M}$ with

$$c_{pq}^t = \#\{i : T_i(J_p^t) = I_q\},$$

i.e. $c_{pq}^t = \#\{i : x_i M - y_i N = 2M + 2 + q - 3p - t\}$. That means that, for each j, the matrix A_j is the same one in Theorem 1.1.

Hence, for

$$b = \frac{M'}{N} + \frac{1}{N}\left(\sum_{i=1}^{\infty} x_i 3^{-i}\right) \in D$$

with $M' \in \mathbb{Z}$ and $\{x_i\}_i \in \{0,1,2\}^{\mathbb{N}}$, we have

$$A(b) = A_{x_1}. \tag{3.3}$$

(2) Let $\mu = N \cdot \mathcal{L}|_{[0,1/N)}$ and let $T(x) = 3x \mod(1/N)$. Then

$$(\mathbb{R}/\operatorname{mod}(1/N), T, \mu) \text{ is ergodic}, \tag{3.4}$$

since $(\mathbb{R}/\operatorname{mod}(1/N), T, \mu) \backsimeq (\mathbb{R}/\operatorname{mod}(1), x \to 3x(\operatorname{mod} 1), \mathcal{L})$.

(3) For $x \in D/\bmod(1/N)$, let

$$A_n(x) = A(x)A(3x)\cdots A(3^{n-1}x) = A(x)A(Tx)\cdots A(T^{n-1}x). \tag{3.5}$$

Then $\{A_n(\cdot)\}_n$ are measurable functions defined on $D/\bmod(1/N)$, a subset of full measure contained in $(\mathbb{R}/\bmod(1/N),\mu)$.

We will prove Theorem 1.1 (1). For

$$a = \frac{-M-1+k}{N} + \frac{1}{N}\biggl(\sum_{i=1}^{\infty} x_i 3^{-i}\biggr),$$

it follows from (2.1), (3.1) and (3.3) that

$$\begin{aligned}\overline{\dim}_{\mathrm{B}} F_{\theta,a} &= \varlimsup_{n\to\infty} \frac{\log N_n(a)}{n\log 3} \\ &= \varlimsup_{n\to\infty} \frac{\log \|e_{i_0(a)}A(a)A(3a)\cdots A(3^{n-1}a)\|_1}{n\log 3} \\ &= \varlimsup_{n\to\infty} \frac{\log \|e_k A_{x_1}A_{x_2}\cdots A_{x_n}\|_1}{n\log 3},\end{aligned}$$

where $i_0(a) = k$ and $A(3^j a) = A_{x_{j+1}}$ for any $j \in \{0\}\cup\mathbb{N}$.

Replacing $\|\cdot\|_1$ by any norm $\|\cdot\|$ of $\mathbb{R}^{N+M}$, we get

$$\overline{\dim}_{\mathrm{B}} F_{\theta,a} = \varlimsup_{n\to\infty} \frac{\log \|e_k A_{x_1}A_{x_2}\cdots A_{x_n}\|}{n\log 3}. \tag{3.6}$$

This completes the proof of Theorem 1.1 (1).

4. A typical value of the box dimension

For any real matrix $A_{k\times k} = (a_{ij})_{1\leqslant i,j\leqslant k}$, let $\|A\| = \sum_{ij}|a_{ij}|$. Then for any matrices $A_{k\times k}$, $B_{k\times k}$, we have $\|AB\| \leqslant \|A\|\cdot\|B\|$.

Lemma 4.1. *For $\mathcal{L}$ almost all $a \in J_\theta$,*

$$\varlimsup_{n\to\infty} \frac{\log N_n(a)}{n} \leqslant \lim_{n\to\infty} \frac{\log \|A_n(a)\|}{n} = \gamma \leqslant \log(\tfrac{8}{3}).$$

Here γ is the Lyapunov exponent with respect to A_0, A_1 and A_2.

Proof. Now $A_n(\cdot)$ is defined on $D/\bmod(1/N)$, a set of full measure contained in $(\mathbb{R}/\bmod(1/N),\mu)$. Moreover,

$$A_{n+m}(a) = A_n(a)A_m(3^n a) = A_n(a)A_m(T^n a)$$

for $a \in D/\bmod(1/N)$. Therefore, we have

$$\log\|A_{n+m}(a)\| \leqslant \log\|A_n(a)\| + \log\|A_m(T^n a)\|.$$

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It follows from the subadditive ergodic theorem [**11**] that there exists a constant γ such that

$$\gamma = \lim_{n\to\infty} \frac{\log \|A_n(a)\|}{n} = \lim_{n\to\infty} \frac{1}{n} \int \log \|A_n(a)\| \, \mathrm{d}\mu(a)$$

for μ almost all $a \in D/\operatorname{mod}(1/N)$. This means that γ is the corresponding Lyapunov exponent with respect to A_0, A_1 and A_2, which is independent of the given matrix norm.

By the definition of μ and (3.2), for $\mathcal{L}$ almost all $a \in J_\theta$,

$$\gamma = \lim_{n\to\infty} \frac{\log \|A_n(a)\|}{n} = \lim_{n\to\infty} \frac{1}{n} \int_{J_\theta} \log \|A_n(a)\| \, \mathrm{d}m_\theta(a). \tag{4.1}$$

By the definition of $A_n(a)$,

$$\|A_n(a)\| = \sum_{b\in\Gamma_a} N_n(b). \tag{4.2}$$

As $\log(x)$ is convex, $\int \log f(x) \leqslant \log \int f(x)$. By using this inequality, (2.4), (4.1) and (4.2), we have

$$\begin{aligned}
\gamma &= \lim_{n\to\infty} \frac{1}{n} \int_{J_\theta} \log \|A_n(a)\| \, \mathrm{d}m_\theta(a) \\
&\leqslant \lim_{n\to\infty} \frac{1}{n} \log \left[\int_{J_\theta} \|A_n(a)\| \, \mathrm{d}m_\theta(a) \right] \\
&= \lim_{n\to\infty} \frac{1}{n} \log \left[\int_{J_\theta} \sum_{b\in\Gamma_a} N_n(b) \, \mathrm{d}m_\theta(a) \right] \\
&= \lim_{n\to\infty} \frac{1}{n} \log[(N+M)(\tfrac{8}{3})^n] \\
&= \log(\tfrac{8}{3}).
\end{aligned}$$

In addition, for $\mathcal{L}$ almost all $a \in J$,

$$\overline{\lim_{n\to\infty}} \frac{\log N_n(a)}{n} \leqslant \lim_{n\to\infty} \frac{\log \sum_{b\in\Gamma_a} N_n(b)}{n} = \lim_{n\to\infty} \frac{\log \|A_n(a)\|}{n} = \gamma.$$

□

Lemma 4.2. *Suppose that $B \subset J_\theta$ is a $\mathcal{L}$-measurable set with $m_\theta(B) > 0$. If $\bigcup_i S_i(B) \subset B$, then $m_\theta(B) = 1$.*

Proof. Suppose that $m_\theta(B) < 1$, then $0 < m_\theta(B^c) < 1$ and $\mathcal{L}(J_\theta \cap B^c) > 0$.

As $\bigcup_i S_i(B) \subset B$, we have

$$S^{-1}_{i_1\cdots i_k}(B^c) \subset B^c \quad \text{for any } i_1 \cdots i_k. \tag{4.3}$$

Since $\mathcal{L}(J_\theta \cap B^c) > 0$, we can take a Lebesgue point $x_0 \in J_\theta \cap B^c$ of density 1, which implies that for any $\delta > 0$ there exists $\varepsilon_0 > 0$ such that

$$\mathcal{L}[I \cap (J_\theta \cap B^c)]/|I| \geqslant 1 - \delta \tag{4.4}$$

whenever $x_0 \in I$ with $|I| \leqslant \varepsilon_0$.

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Given a sufficiently large integer k, as $J_\theta = \bigcup_{i_1\cdots i_k} S_{i_1\cdots i_k}(J_\theta)$, we can select an interval $I = S_{j_1\cdots j_k}(J_\theta) \subset J_\theta$ such that $x_0 \in I$ with

$$|I| = 3^{-k}|J_\theta| < \varepsilon_0,$$

then, by (4.4),

$$\mathcal{L}(I \cap B^{\mathrm{c}})/|I| = \mathcal{L}(I \cap J_\theta \cap B^{\mathrm{c}})/|I| \geqslant 1 - \delta.$$

By using (4.3), we have

$$(S_{j_1\cdots j_k})^{-1}(I \cap B^{\mathrm{c}}) = (S_{j_1\cdots j_k})^{-1}(S_{j_1\cdots j_k}(J_\theta) \cap B^{\mathrm{c}}) \subset J_\theta \cap B^{\mathrm{c}}.$$

Hence,

$$\frac{\mathcal{L}(J_\theta \cap B^{\mathrm{c}})}{|J_\theta|} \geqslant \frac{\mathcal{L}[(S_{j_1\cdots j_k})^{-1}(I \cap B^{\mathrm{c}})]}{\mathcal{L}[(S_{j_1\cdots j_k})^{-1}(I)]} = \frac{3^n \mathcal{L}(I \cap B^{\mathrm{c}})}{3^n|I|} \geqslant (1-\delta).$$

This implies that

$$m_\theta(B^{\mathrm{c}}) \geqslant 1 - \delta.$$

Letting $\delta \to 0$, we have $m_\theta(B^{\mathrm{c}}) = 1$. This yields a contradiction. □

Proposition 4.3. *For $\mathcal{L}$ almost all $a \in J_\theta$,*

$$\overline{\dim}_{\mathrm{B}} F_{\theta,a} = \gamma/\log 3,$$

where γ is the Lyapunov exponent with respect to A_0, A_1 and A_2.

Proof. By (2.1) and Lemma 4.1, $\overline{\dim}_{\mathrm{B}} F_{\theta,a} \leqslant \gamma/\log 3$ for $\mathcal{L}$ almost all a. So we need to prove only that $\overline{\dim}_{\mathrm{B}} F_{\theta,a} \geqslant \gamma/\log 3$ for $\mathcal{L}$ almost all a. By (4.2) and Lemma 4.1, for $\mathcal{L}$ almost all $a \in J_\theta$, we have

$$\max_{b\in\Gamma_a} \overline{\dim}_{\mathrm{B}} F_{\theta,b} = \varlimsup_{n\to\infty} \frac{\log\|A_n(a)\|}{n\log 3} = \frac{\gamma}{\log 3}. \tag{4.5}$$

Let

$$B = \{a \in J_\theta : \overline{\dim}_{\mathrm{B}} F_{\theta,a} \geqslant \gamma/\log 3\},$$

which is an $\mathcal{L}$-measurable set. It follows from (4.5) that there is a set K of zero Lebesgue measure such that

$$J_\theta \setminus K \subset \bigcup_{i=-(N+M)}^{N+M} \left(B + \frac{i}{N}\right),$$

where $B + x = \{b + x : b \in B\}$. Hence

$$0 < \mathcal{L}(J_\theta) \leqslant \sum_{i=-(N+M)}^{N+M} \mathcal{L}\left(B + \frac{i}{N}\right) = (2N + 2M + 1)\mathcal{L}(B),$$

which implies that

$$m_\theta(B) = \mathcal{L}(B)/|J_\theta| > 0.$$

By using $(*)$ in § 3, we have $\overline{\dim}_{\mathrm{B}} F_{\theta,S_i(a)} \geqslant \overline{\dim}_{\mathrm{B}} F_{\theta,a}$, i.e.

$$\bigcup_i S_i(B) \subset B.$$

It follows from Lemma 4.2 that $m_\theta(B) = 1$. □

Proposition 4.4. *For $\mathcal{L}$ almost all $a \in [-\tan\theta, 1]$,*

$$\dim_{\mathrm{B}} F_{\theta,a} = \gamma/\log 3.$$

Proof. Notice that $A_{n+m}(a) = A_n(a)A_m(T^n a)$, and $(\mathbb{R}/\operatorname{mod}(1/N), T, \mu)$ is ergodic. Then it follows from the multiplicative ergodic theorem [**11**] that for each e_i $(1 \leqslant i \leqslant N+M)$, and for μ-almost all (i.e. $\mathcal{L}$-almost all) $a \in [0, 1/N) \cap D$,

$$\lim_{n\to\infty} \frac{\log \|e_i A_n(a)\|}{n} = \lambda(a,i), \tag{4.6}$$

where $\lambda(a,i)$ depends on a and e_i. It follows from (2.1), (2.2), (3.1), (4.6) and Proposition 4.3 that, for $\mathcal{L}$ almost all a,

$$\dim_{\mathrm{B}} F_{\theta,a} = \gamma/\log 3.$$

□

5. Equality of the Hausdorff and box dimensions

In this section we will prove Theorem 1.1 (3), i.e. for a fixed rational slope $\tan\theta$ and almost all $a \in J_\theta$, the Hausdorff dimension and the upper box dimension of the section $F_{\theta,a}$ are equal. To prove this, we shall use Proposition 2.6 in [**5**] provided by Ledrappier.

Let T_3 denote the endomorphism $T_3 x = 3x (\operatorname{mod} 1)$ of the one-dimensional torus $\mathbb{T} = \mathbb{R}/(\operatorname{mod} 1)$, and let S be a continuous transformation of a metric space Y. Assume that $\Lambda \subset \mathbb{T}\times Y$ is compact and invariant under the map $T_3 \times S$, and that ν is an S-invariant probability measure on Y. Then for ν-a.e. y, we have

$$\dim_{\mathrm{H}}[\pi^{-1}(y)] = \dim_{\mathrm{B}}[\pi^{-1}(y)],$$

where $\pi : \Lambda \to Y$ is the projection onto the second coordinate.

Proof of Theorem 1.1 (3). At first, we will show that, for almost all $a \in [-M/N, 1]$,

$$\max_{b\in\Gamma_a} \dim_{\mathrm{B}} F_{\theta,b} = \max_{b\in\Gamma_a} \dim_{\mathrm{H}} F_{\theta,b}. \tag{5.1}$$

In fact, fix $a \in [-M/N, 1]$, we have

$$\begin{aligned}\bigcup_{b\in\Gamma_a} F_{\theta,b} &= \bigcup_{i\in\mathbb{Z}} [F \cap \{(x,y) : y = (M/N)x + i/N + a\}] \\ &= F \cap \{(x,y) : Ny - Mx \equiv aN \pmod 1\}.\end{aligned}$$

Let $T_3(x) = 3x(\operatorname{mod} 1)$ be a map on the one-dimensional torus $\mathbb{T}$, and let $T = T_3 \times T_3$ and $P = (x, (Ny - Mx)(\operatorname{mod} 1))$ be the maps on the two-dimensional torus $\mathbb{T}^2$. Then the Sierpinski carpet F and its image $P(F)$ are invariant under T. Here P is a bi-Lipschitz endomorphism when restricted to a subset

$$\{(x, y) \in \mathbb{T}^2 : y \in [j/N, (j+1)/N)\}$$

for each integer $j \in [0, N-1]$. Therefore, let 'dim' stand for any one of $\dim_{\mathrm{H}}$, $\overline{\dim}_{\mathrm{B}}$ and $\underline{\dim}_{\mathrm{B}}$,

$$\dim P\biggl(\bigcup_{b \in \Gamma_a} F_{\theta,b}\biggr) = \dim\biggl(\bigcup_{b \in \Gamma_a} F_{\theta,b}\biggr). \tag{5.2}$$

We now prepare to apply the stated result from [**5**]. Let $S = T_3$ on $Y = \mathbb{T}$ equipped with a normalized Lebesgue measure ν, then $T = T_3 \times T_3 = T_3 \times S$, $P(F) \subset \mathbb{T} \times Y = \mathbb{T}^2$ is compact and invariant under T, and

$$\pi^{-1}[Na(\operatorname{mod} 1)] = P\biggl(\bigcup_{b \in \Gamma_a} F_{\theta,b}\biggr).$$

Applying the result from [**5**], we have, for almost all a,

$$\dim_{\mathrm{H}} P\biggl(\bigcup_{b \in \Gamma_a} F_{\theta,b}\biggr) = \dim_{\mathrm{B}} P\biggl(\bigcup_{b \in \Gamma_a} F_{\theta,b}\biggr). \tag{5.3}$$

Since the low box dimension lies between the Hausdorff and upper box dimensions, by using (5.2) and (5.3), we obtain (5.1). Hence, for $\mathcal{L}$ almost all $a \in J_\theta$,

$$\max_{b \in \Gamma_a} \dim_{\mathrm{H}} F_{\theta,b} = \dim_{\mathrm{B}} F_{\theta,a} = \gamma / \log 3.$$

By an argument analogous to Proposition 4.3, we have

$$\dim_{\mathrm{H}} F_{\theta,a} = \dim_{\mathrm{B}} F_{\theta,a} = \gamma / \log 3 \quad \text{for } \mathcal{L} \text{ almost all } a \in J_\theta.$$

□

6. Computation of the Lyapunov exponent and dimension

In this section, we give a method mentioned in Remark 1.4 following Theorem 1.1. Suppose $\tan\theta = N/M > 0$. Let $n = N + M$. Then $\{A_i\}_{i=0}^{2}$ are non-negative integer $n \times n$ matrices.

For $x = (x_1, \dots, x_n)^{\mathrm{T}}$, let $\|x\|_1 = \sum_i |x_i|$. We define $\Delta_{n-1} = \{(x_1, \dots, x_n)^{\mathrm{T}} \mid x_i \geqslant 0,\ \text{for all } i \text{ and } \|x\|_1 = 1\}$. Let $\mathcal{L}$ denote the Lebesgue measure on $[0, 1]$. For almost all $t \in [0, 1]$, write $t = \sum_{i=1}^{\infty} t_i 3^{-i}$ with $\{t_i\}_{i=1}^{\infty} \in \{0, 1, 2\}^{\mathbb{N}}$.

Lemma 6.1. *For $i \in \{0, 1, 2\}$, $\|A_i x\|_1 \geqslant 1$ for all $x \in \Delta_{n-1}$.*

Proof. Given any $x = (x_1, \dots, x_n)^{\mathrm{T}} \in \Delta_{n-1}$,

$$\begin{aligned}\|A_i x\|_1 &= \sum_i \|A_i(0, \dots, x_i, \dots, 0)^{\mathrm{T}}\|_1 \\ &= \sum_i x_i \|A_i(0, \dots, 1, \dots, 0)^{\mathrm{T}}\|_1,\end{aligned}$$

we need only to show that $\|A_i(0, \dots, 1, \dots, 0)^{\mathrm{T}}\|_1 \geqslant 1$.

It suffices to prove that, for any $c \in J \cap D$, every column vector of $A(c)$ is non-zero, i.e. for any $b \in \mathit{\Gamma}_{3c}$ there exists some i such that

$$S_i(b) \in \mathit{\Gamma}_c.$$

Suppose that $b = 3c + (i/N)$. Notice that $\{S_i : J_\theta \to J_\theta\}_i$ are contractions satisfying

$$S_i(x) = \tfrac{1}{3}x + \tfrac{1}{3}y_i - \tfrac{1}{3}x_i \tan\theta = \tfrac{1}{3}x + \tfrac{1}{3}y_i - \frac{M}{3N}x_i$$

with $(x_i, y_i) \in \{0, 1, 2\} \times \{0, 1, 2\} \setminus \{(1, 1)\}$.

We will distinguish three cases.

(1) When $M \equiv 1$ or $M \equiv 2 \pmod 3$, we can always select $k \in \{0, 1, 2\}$ such that

$$i - kM \equiv 0 \pmod 3.$$

By using the self-mapping $x \to (\tfrac{1}{3})x - k(M/3N)$ on J_θ, we have

$$(\tfrac{1}{3})b - k\frac{M}{3N} = c + \frac{(i - kM)/3}{N} \in \mathit{\Gamma}_c.$$

(2) When $N \equiv 1$ or $N \equiv 2 \pmod 3$, we can always select $k \in \{0, 1, 2\}$ such that

$$i + kN \equiv 0 \pmod 3.$$

By using the self-mapping $x \to (\tfrac{1}{3})x + k\tfrac{1}{3}$ on J_θ, we have

$$(\tfrac{1}{3})b + k\tfrac{1}{3} = c + \frac{(i + kN)/3}{N} \in \mathit{\Gamma}_c.$$

(3) If $M, N \equiv 0 \pmod 3$, then $1 = (M, N) \geqslant 3$. This yields a contradiction.

□

By Lemma 6.1, given $i \in \{0, 1, 2\}$, a mapping $\bar{A}_i : \Delta_{n-1} \to \Delta_{n-1}$ is defined by

$$\bar{A}_i(x) = \frac{A_i x}{\|A_i x\|_1} \in \Delta_{n-1} \quad \text{for any } x \in \Delta_{n-1}.$$

As shown in [**5**, p. 615], a probability measure ν on Δ_{n-1} is called a stationary measure for the random product if

$$\nu = \int_{[0,1]} \bar{A}_{t_1} \nu \, \mathrm{d}\mathcal{L}(t), \tag{6.1}$$

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where t_1 is the first term of the 3-adic expansion of t. By using the notion in [**4**], the above formula is

$$\nu = \tfrac{1}{3}[\nu \circ (\bar{A}_0)^{-1} + \nu \circ (\bar{A}_1)^{-1} + \nu \circ (\bar{A}_2)^{-1}]. \tag{6.2}$$

If the stationary measure is unique, the results of [**2**] show that

$$\gamma = \frac{1}{3} \int_{\Delta_{n-1}} (\log \|A_0 x\|_1 + \log \|A_1 x\|_1 + \log \|A_2 x\|_1)\, \mathrm{d}\nu(x). \tag{6.3}$$

6.1. Hutchinson's metric

Hutchinson's metric is a kind of metric for measures [**4**]. Let δ be a metric on Δ_{n-1} such that (Δ_{n-1}, δ) is a compact space. Then the diameter $\mathrm{diam}_\delta(\Delta_{n-1})$ of Δ_{n-1} is finite. For any Borel probability measure ν_1, ν_2 on (Δ_{n-1}, δ), Hutchinson's metric is defined by

$$d_{\mathrm{H}}(\nu_1, \nu_2) = \sup_{\mathrm{Lip}(f) \leqslant 1} \left| \int f\, \mathrm{d}\nu_1 - \int f\, \mathrm{d}\nu_2 \right|, \tag{6.4}$$

where

$$\mathrm{Lip}(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\delta(x, y)}$$

for any function $f : \Delta_{n-1} \to \mathbb{R}$.

Notice that, for any ν_1, ν_2 on (Δ_{n-1}, δ),

$$d_{\mathrm{H}}(\nu_1, \nu_2) \leqslant 2 \sup_{x,y \in \Delta_{n-1}} (\delta(x, y)) = 2\, \mathrm{diam}_\delta(\Delta_{n-1}). \tag{6.5}$$

Let $\mathcal{M}^1$ be the collection of Borel probability measures on (Δ_{n-1}, δ). Then the metric space $(\mathcal{M}^1, d_{\mathrm{H}})$ is compact. Define $\mathcal{F} : \mathcal{M}^1 \to \mathcal{M}^1$ by

$$\mathcal{F}\nu = \frac{1}{3} \sum_{i=0}^{2} \nu \circ (\bar{A}_i)^{-1}. \tag{6.6}$$

We assume that

(1) there is a constant τ such that

$$\|x - y\|_1 \leqslant \tau \cdot \delta(x, y) \quad \text{for all } x, y \in \Delta_{n-1}; \tag{6.7}$$

(2) $\mathcal{F}$ is contractive, i.e. there is a constant $c \in (0, 1)$ such that, for any $\nu_1, \nu_2 \in \mathcal{M}^1$,

$$d_{\mathrm{H}}(\mathcal{F}\nu_1, \mathcal{F}\nu_2) \leqslant c d_{\mathrm{H}}(\nu_1, \nu_2). \tag{6.8}$$

So, by [**4**] there is a unique stationary measure, denoted by ν.

Here, for any i,

$$\log \|A_i x\|_1 : (\Delta_{n-1}, \delta) \to \mathbb{R} \text{ is Lipschitz} \tag{6.9}$$

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due to Lemma 6.1 and (6.7). In fact, suppose that $A_i = (a_{pq})_{1\leqslant p,q\leqslant n}$ and let $\|A_i\|^* = \max_q(\sum_p a_{pq})$, then, by Lemma 6.1, for any $x, y \in \Delta_{n-1}$, we have

$$\begin{aligned}|\log\|A_i x\|_1 - \log\|A_i y\|_1| &\leqslant |\|A_i x\|_1 - \|A_i y\|_1| \\ &\leqslant \max_q\biggl(\sum_p a_{pq}\biggr)\|x-y\|_1 \\ &\leqslant (\tau\|A_i\|^*)\cdot\delta(x,y).\end{aligned}$$

Under the assumption above, we can estimate the Lyapunov exponent in the following way.

Let ν_0 be an atom measure supported on any point of Δ_{n-1} and let $\nu_1 = \mathcal{F}\nu_0, \dots, \nu_{k+1} = \mathcal{F}\nu_k$ by induction, then

$$\begin{aligned}d_{\mathrm{H}}(\nu,\nu_k) &= d_{\mathrm{H}}(\mathcal{F}\nu,\mathcal{F}\nu_{k-1}) \\ &\leqslant c d_{\mathrm{H}}(\nu,\nu_{k-1}) \\ &\;\;\vdots \\ &\leqslant c^k d_{\mathrm{H}}(\nu,\nu_0) \leqslant 2c^k \operatorname{diam}_\delta(\Delta_{n-1}).\end{aligned}$$

Let $f(x) = \frac{1}{3}\sum_i \log\|xA_i\|_1$ be a function on Δ_{n-1}, and let

$$\gamma_k = \int f(x)\,\mathrm{d}\nu_k(x).$$

Therefore, we have the following estimate:

$$\begin{aligned}|\gamma-\gamma_k| &= \biggl|\int f\,\mathrm{d}\nu - \int f\,\mathrm{d}\nu_k\biggr| \\ &\leqslant \mathrm{Lip}(f) d_{\mathrm{H}}(\nu,\nu_k) \leqslant c^k\Bigl[2\tau\Bigl(\max_{0\leqslant i\leqslant 2}\|A_i\|^*\Bigr)\operatorname{diam}_\delta(\Delta_{n-1})\Bigr].\end{aligned}$$

That is,

$$|\gamma-\gamma_k| \leqslant c^k\Bigl[2\tau\Bigl(\max_{0\leqslant i\leqslant 2}\|A_i\|^*\Bigr)\operatorname{diam}_\delta(\Delta_{n-1})\Bigr]. \tag{6.10}$$

6.2. Example

We mainly compute the Lyapunov exponent in the case of $\tan\theta = \frac{1}{2}$. We equip Δ_2 with a metric defined by

$$\delta(x,y) = \max\{|x_1-y_1|, |x_3-y_3|\}$$

for any $x = (x_1,x_2,x_3)^{\mathrm{T}}$, $y = (y_1,y_2,y_3)^{\mathrm{T}} \in \Delta_2$. Here δ is a metric satisfying $\delta(x,y) \leqslant \|x-y\|_1 \leqslant 4\delta(x,y)$ and $\operatorname{diam}_\delta(\Delta_2) = 1$. We note that (Δ_2,δ) is compact.

In the case of $\tan\theta = \frac{1}{2}$, we have

$$A_0 = \begin{bmatrix}1&0&0\\0&2&1\\1&1&2\end{bmatrix},\qquad A_1 = \begin{bmatrix}1&1&0\\2&0&2\\0&1&1\end{bmatrix},\qquad A_2 = \begin{bmatrix}2&1&1\\1&2&0\\0&0&1\end{bmatrix},$$

with $\max_{0\leqslant i\leqslant 2}\|A_i\|^* = 3$. Take any $x=(x_1,x_2,x_3)^{\mathrm{T}}$, $y=(y_1,y_2,y_3)^{\mathrm{T}}\in\Delta_2$; we then obtain

$$\delta(\bar{A}_0(x),\bar{A}_0(y)) = \max\left\{\left|\frac{x_1}{3-x_1}-\frac{y_1}{3-y_1}\right|,\left|\frac{1+x_3}{3-x_1}-\frac{1+y_3}{3-y_1}\right|\right\},$$
$$\delta(\bar{A}_1(x),\bar{A}_1(x)) = \max\left\{\left|\frac{1-x_3}{3-x_2}-\frac{1-y_3}{3-y_2}\right|,\left|\frac{1-x_1}{3-x_2}-\frac{1-y_1}{3-y_2}\right|\right\},$$
$$\delta(\bar{A}_2(x),\bar{A}_2(x)) = \max\left\{\left|\frac{1+x_1}{3-x_3}-\frac{1+y_1}{3-y_3}\right|,\left|\frac{x_3}{3-x_3}-\frac{y_3}{3-y_3}\right|\right\}.$$

We have

$$\left|\frac{x_1}{3-x_1}-\frac{y_1}{3-y_1}\right| = \frac{3(x_1-y_1)}{(3-x_1)(3-y_1)} \leqslant \tfrac{3}{4}|x_1-y_1|,$$

$$\begin{aligned}\left|\frac{1+x_3}{3-x_1}-\frac{1+y_3}{3-y_1}\right| &= \left|\frac{(1+y_3)(x_1-y_1)+(3-y_1)(x_3-y_3)}{(3-x_1)(3-y_1)}\right| \\ &\leqslant \frac{1+y_3}{(3-x_1)(3-y_1)}|x_1-y_1| + \frac{1}{3-x_1}|x_3-y_3| \\ &\qquad\qquad \left(\text{as } \frac{1+y_3}{3-y_1}\leqslant\frac{2-y_1}{3-y_1}\leqslant\frac{2}{3}\right) \\ &\leqslant \tfrac{1}{3}|x_1-y_1| + \tfrac{1}{2}|x_3-y_3|,\end{aligned}$$

$$\begin{aligned}\left|\frac{1-x_3}{3-x_2}-\frac{1-y_3}{3-y_2}\right| &= \left|\frac{(3-y_1)(y_3-x_3)+(1-y_3)(y_1-x_1)}{(2+x_1+x_3)(2+y_1+y_3)}\right| \\ &\leqslant \tfrac{3}{4}|x_3-y_3| + \tfrac{1}{4}|x_1-y_1|.\end{aligned}$$

Hence, $\mathrm{Lip}(\bar{A}_0)\leqslant\frac{5}{6}$, $\mathrm{Lip}(\bar{A}_1)\leqslant 1$ and $\mathrm{Lip}(\bar{A}_2)\leqslant\frac{5}{6}$. So in the case of $\tan\theta=\frac{1}{2}$, $\mathcal{F}$ is contractive with ratio not greater than $(\frac{5}{6}+1+\frac{5}{6})/3=\frac{8}{9}$. It follows from (6.10) that, for any k,

$$|\gamma-\gamma_k|\leqslant 24(\tfrac{8}{9})^k,$$

which implies that we can compute this Lyapunov exponent to any accuracy. By numerical computation, we obtain $\gamma=0.9793\dots$ and, for a.e. $b\in J_\theta$, $\dim_{\mathrm{H}} F_{\theta,b}=\dim_{\mathrm{B}} F_{\theta,b}=\gamma/\log 3=0.8914\dots$. This typical value is strictly less than $\log 8/\log 3-1=0.8927\dots$.

Remark 6.2. For $\tan\theta=1$,

$$A_0=\begin{bmatrix}1&0\\2&2\end{bmatrix},\qquad A_1=\begin{bmatrix}2&1\\1&2\end{bmatrix},\qquad A_2=\begin{bmatrix}2&2\\0&1\end{bmatrix}.$$

Let $\delta(x,y)=|x_1-y_1|$. Then

$$\mathrm{Lip}(\bar{A}_0)\leqslant\tfrac{1}{2},\qquad \mathrm{Lip}(\bar{A}_1)\leqslant\tfrac{1}{3},\qquad \mathrm{Lip}(\bar{A}_2)\leqslant\tfrac{1}{2},$$

and thus $c\leqslant(\frac{1}{2}+\frac{1}{3}+\frac{1}{2})/3=\frac{4}{9}$. We have the error estimation

$$|\gamma-\gamma_k|\leqslant(12)(\tfrac{4}{9})^k.$$

We list some typical values of dimensions of sections as follows:

$$\begin{aligned}
\tan\theta &= 0 : 0.8927\ldots \quad (= \log 8/\log 3 - 1),\\
\tan\theta &= 1 : 0.8858\ldots,\\
\tan\theta &= \tfrac{1}{3} : 0.8926\ldots,\\
\tan\theta &= \tfrac{1}{4} : 0.8917\ldots.
\end{aligned}$$

According to the above numerical result, we pose the following conjecture.

Conjecture 6.3. *If* $\tan\theta \in \mathbb{Q}$ *and* d_θ *is the typical value of* $\dim_{\mathrm{H}} F_{\theta,a}$ *for almost all* $a \in J_\theta$*, then* $d_\theta < (\log 8/\log 3) - 1$.

7. Fractals like the Sierpinski carpet

In fact, we can deal with the fractals like the Sierpinski carpet.

Given an integer $m \geqslant 2$, let $\{\psi_i\}_{i=1}^k$ be a family of different similitudes of $\mathbb{R}^2$ such that $\psi_i(x,y) = (x,y)/m + (c_i,d_i)/m$, where $c_i, d_i \in \mathbb{Z}\cap[0,m-1]$. Suppose that $E = \bigcup_{i=1}^k \psi_i(E)$ ($\subset [0,1]^2$) is the self-similar set.

Fix θ and let $\{\tau_j : \mathbb{R} \to \mathbb{R}\}_{j=1}^k$ be linear mappings such that

$$\psi_j^{-1}(L_{\theta,a}) = L_{\theta,\tau_j(a)} \quad \text{for all } j.$$

Write $\varsigma_j = \tau_j^{-1}$, $E_{\theta,a} = E \cap L_{\theta,a}$ and $\mathcal{J}_\theta = \{a : L_{\theta,a} \cap E \neq \varnothing\}$. Here $\varsigma_j|_{\mathcal{J}_\theta} : \mathcal{J}_\theta \to \mathcal{J}_\theta$, since we have

$$E_{\theta,\varsigma_j(a)} \supset \psi_j(E_{\theta,a}), \tag{$**$}$$

which is like formula ($*$).

Suppose that $\tan\theta = M/N > 0$ is rational with $N, M \in \mathbb{N}$.

We make the *assumption* that $\mathcal{J}_\theta$ *is an interval*. For example, if the boundary of $[0,1]^2$ is contained in E, then $\mathcal{J}_\theta$ is an interval for each θ. In fact, when the boundary $\partial[0,1]^2$ is contained in $\bigcup_{i=1}^k \psi_i([0,1]^2)$, e.g. the Sierpinski carpet, we have $\partial[0,1]^2 \subset E$.

As in § 3, there exist matrices $\mathcal{A}_0, \ldots, \mathcal{A}_{m-1}$, which are $(N+M)\times(N+M)$ non-negative integer matrices. Because of the assumption, we can prove a lemma of ergodic type similar to Lemma 4.2. By using this lemma and formula ($**$), we can prove results similar to Propositions 4.3 and 4.4. As in § 5, we also establish the equality for Hausdorff and box dimensions.

Consequently, a result like Theorem 1.1, for the fractals like the Sierpinski carpet, can be established when $\mathcal{J}_\theta$ *is an interval*.

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References

1. I. Benjamini and Y. Peres, On the Hausdorff dimension of fibres, *Israel J. Math.* **74** (1991), 267–279.

2. H. Furstenberg, Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation, *Math. Systems Theory* **1** (1967), 1–49.
3. J. Hawkes, Some algebraic properties of small sets, *Q. J. Math.* **26** (1975), 195–201.
4. J. E. Hutchinson, Fractals and self similarity, *Indiana Univ. Math. J.* **30** (1981), 713–747.
5. R. Kenyon and Y. Peres, Intersecting random translates of invariant Cantor sets, *Invent. Math.* **104** (1991), 601–629.
6. J. F. Li, On the dimensions of intersections of Sierpinski carpet with lines, Master's thesis, Wuhan University (1997).
7. Q. H. Liu and Z. Y. Wen, On dimensions of multitype Moran sets, *Math. Proc. Camb. Phil. Soc.* **139**(3) (2005), 541–553.
8. J. M. Marstrand, Some fundamental geometrical properties of plane sets of fractional dimensions, *Proc. Lond. Math. Soc.* **4** (1954), 257–302.
9. P. Mattila, *Geometry of sets and measures in Euclidean spaces* (Cambridge University Press, 1995).
10. R. D. Mauldin and S. C. Williams, Hausdorff dimension in graph directed constructions, *Trans. Am. Math. Soc.* **309**(1–2) (1988), 811–839.
11. P. Walters, *An introduction to ergodic theory* (Springer, 1982).