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ON THE HOLLAND–WALSH CHARACTERIZATION OF BLOCH FUNCTIONS

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Abstract It is proved that the Bloch norm of an arbitrary C^1-function defined on the unit ball $\mathbb{B}_n \subset \mathbb{R}^n$ is *equal* to

$$\sup_{x,y\in\mathbb{B}_n,\, x\neq y} (1-|x|^2)^{1/2}(1-|y|^2)^{1/2}\frac{|f(x)-f(y)|}{|x-y|}.$$

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Let $\mathbb{B}_n$ denote the unit ball in $\mathbb{R}^n$, where $n \geqslant 2$. For a complex-valued function $f \in C^1(\mathbb{B}_n)$, let $\|f\|_{\mathfrak{B}}$ denote the Bloch norm of f,

$$\|f\|_{\mathfrak{B}} = \sup_{x\in\mathbb{B}_n} (1-|x|^2)|\mathrm{d}f(x)|,$$

where $|\mathrm{d}f(x)|$ denotes the norm of the derivative $\mathrm{d}f(x)$ treated as a linear operator from $\mathbb{R}^n$ to $\mathbb{C} = \mathbb{R}^2$. If f is real-valued, then $|\mathrm{d}f(x)| = |\nabla f(x)|$, where ∇f denotes the gradient of f:

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right), \quad x = (x_1, \dots, x_n).$$

If f is holomorphic in the unit disc $\mathbb{D} = \mathbb{B}_2$, then $|\mathrm{d}f(x)| = |f'(x)|$, where f' denotes the ordinary derivative. Our starting point here is the following theorem of Holland and Walsh [**1**].

Theorem 1. *For a function f holomorphic in $\mathbb{D}$, we have*

$$\|f\|_{\mathfrak{B}} \asymp \sup_{x,y\in\mathbb{D},\, x\neq y} (1-|x|^2)^{1/2}(1-|y|^2)^{1/2}\frac{|f(x)-f(y)|}{|x-y|}. \tag{1}$$

Here we write $A \asymp B$ to denote that A/B lies between two positive constants. In (1), the $C-1$ and C_2 are independent of f. Recently, Ren and Kähler extended (1) to the case of harmonic [**3**] and hyperbolically harmonic [**2**] functions on $\mathbb{B}_n$. In this note we show that (1) holds for an arbitrary C^1-function f on $\mathbb{B}_n$ and, moreover, that '$\asymp$' can be replaced by '$=$'.

Theorem 2. *For an arbitrary function $f \in C^1(\mathbb{B}_n)$, $n \geqslant 2$, we have*

$$\|f\|_{\mathfrak{B}} = \sup_{x,y\in\mathbb{B}_n,\, x\neq y} (1-|x|^2)^{1/2}(1-|y|^2)^{1/2}\frac{|f(x)-f(y)|}{|x-y|}. \tag{2}$$

Proof. Denote the quantity on the right-hand side of (2) by $\|f\|_1$. Assuming that $\|f\|_1 \leqslant 1$ we have

$$\frac{|f(x)-f(y)|}{|x-y|} \leqslant \frac{1}{(1-|x|^2)^{1/2}(1-|y|^2)^{1/2}}, \quad x,y \in \mathbb{B}_n. \tag{3}$$

Now we use the formula

$$|\mathrm{d}f(x)| = \limsup_{y\to x} \frac{|f(x)-f(y)|}{|x-y|}$$

to conclude that

$$|\mathrm{d}f(x)| \leqslant (1-|x|^2)^{-1}, \quad x \in \mathbb{B}_n, \tag{4}$$

i.e. that $\|f\|_{\mathfrak{B}} \leqslant 1$.

In the other direction, assume that $\|f\|_{\mathfrak{B}} \leqslant 1$. We want to prove that this implies (3). In proving this we can suppose, after a suitable rotation, that x and y lie in $\mathbb{R}^2 = \{(x_1, x_2, 0, \dots, 0) : x_1, x_2 \in \mathbb{R}\}$. Now let g be the restriction of f to $\mathbb{R}^2 = \mathbb{C}$. Then, by (4),

$$|\mathrm{d}g(x)| \leqslant (1-|x|^2)^{-1}, \quad x \in \mathbb{D}, \tag{5}$$

whence, by integration,

$$|g(x)-g(0)| \leqslant \tfrac{1}{2}\log\frac{1+|x|}{1-|x|}, \quad x \in \mathbb{D}. \tag{6}$$

Now we use the simple inequality

$$\tfrac{1}{2}\log\frac{1+t}{1-t} \leqslant t(1-t^2)^{-1/2}, \quad 0 \leqslant t < 1, \tag{7}$$

to deduce from (6) that

$$|g(x)-g(0)| \leqslant |x|(1-|x|^2)^{-1/2}. \tag{8}$$

Finally, let

$$\varphi_a(x) = \frac{a-x}{1-\bar{a}x}, \quad a, x \in \mathbb{D} \text{ (complex notation)}.$$

We know that φ_a is a conformal automorphism of the unit disc, that $\varphi_a(\varphi_a(x)) = x$, and that

$$1-|\varphi_a(x)|^2 = (1-|x|^2)|\varphi_a'(x)| = \frac{(1-|x|^2)(1-|a|^2)}{|1-\bar{a}x|^2}.$$

This and (5) imply that

$$\begin{aligned}|\mathrm{d}(g\circ\varphi_a)(x)| &= |(\mathrm{d}g)(\varphi_a(x))|\,|\varphi_a'(x)| \\ &\leqslant (1-|\varphi_a(x)|^2)^{-1}|\varphi_a'(x)| \\ &= (1-|x|^2)^{-1}.\end{aligned}$$

Thus $g\circ\varphi_a$ satisfies (5) so we can apply (8) to $g\circ\varphi_a$ to get

$$|g(\varphi_a(x)) - g(\varphi_a(0))| \leqslant |x|(1-|x|^2)^{-1/2}.$$

Hence

$$\begin{aligned}|f(y)-f(a)| &= |g(y)-g(a)| \\ &\leqslant |\varphi_a(y)|(1-|\varphi_a(y)|^2)^{-1/2} \\ &= |a-y|(1-|a|^2)^{-1/2}(1-|y|^2)^{-1/2},\end{aligned}$$

i.e. $\|f\|_1 \leqslant 1$, which was to be proved. □

Remark 3. Inequality (7) is a direct consequence of the formulae

$$\tfrac{1}{2}\log\frac{1+t}{1-t} = t + \sum_{n=1}^{\infty}\frac{1}{2n+1}t^{2n+1}$$

and

$$t(1-t^2)^{-1/2} = t + \sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}t^{2n+1}.$$

Remark 4. The above proof shows that Theorem 2 remains valid if we assume that f is a C^1-function from the unit ball of a Hilbert space and with values in a Banach space.

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