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COUNTEREXAMPLES TO TWO PROBLEMS ON ONE-RELATOR GROUPS

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In [2] G. Baumslag presents a list of twenty-three unsolved problems on one-relator groups. We give counterexamples to two of them.

Problem 5 asks whether a maximal locally free subgroup of a one-relator group always has finite "rank" (G has "rank" k if each finitely generated subgroup of G is contained in a k-generator subgroup of G).[(1)]

Consider the following two properties for a group G:

(P1) Every finitely generated non-trivial normal subgroup of G has finite index in G;

(P2) Every maximal locally free subgroup of G has finite "rank".

Let C be the class of finitely generated, torsion-free, infinite cyclic extensions of free groups, and let Φ be the class of free groups.

THEOREM. *If $G \in C-\Phi$ and G satisfies* (P1) *then G does not satisfy* (P2).

Proof. If $G \in C-\Phi$ satisfies (P1) then it is an infinite cyclic extension of a free group $F_\infty = \langle x_1, x_2, \ldots \rangle$. If F_∞ were not a maximal locally free subgroup of G then $G/F_\infty \cong Z$ forces G to be a finite extension of a locally free group H, but clearly then H must be free. By Stallings [7] we must now have G free, a contradiction. Hence, F_∞ is maximal locally free and if it had finite rank k we would have $F_{k+1} = \langle x_1, \ldots, x_{k+1} \rangle$ contained in a k generator subgroup K of F_∞. By p. 103, problem 22 of [6], K must have F_{k+1} as a free factor, hence by Grushko's Theorem K has $\geqslant k+1$ generators contradicting the assumption that K has finite rank k.

COROLLARY. *Any surface group of genus $\geqslant 2$ is a counterexample to problem* 5 *of* [2].

Proof. Let $G = \langle a_1, b_1, \ldots, a_n, b_n; \prod_{i=1}^{n} [a_i, b_i] = 1 \rangle$ be a surface group of genus $n \geqslant 2$; By [3] we know that G satisfies (P1). If $N = (b_1, \ldots, b_n, a_2, \ldots, a_n)^G$ then G/N is infinite cyclic and by [5] N is free. Hence $G \in C-\Phi$ and satisfies (P1).

Problem 9 asks whether Auslander and Lyndon's theorem that $N < F$ and $|F/N| = \infty$ implies F/N' has trivial center for F free of rank ≥ 2 (see [1]) may be

(1) Professor Baumslag has informed the author that Problem 5 is to be reformulated as follows: Is every maximal locally free, freely indecomposable subgroup of a one-relator group of finite rank?

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generalized to one-relator groups having $\geqslant 3$ generators. We construct a counterexample:

Let $G=\langle a, b, c; [a, b]=[c, b]\rangle$ and let $N=(b, c)^G$—then $|G/N|=\infty$ and $[c, b]$ $(=[a, b])$ is in N'. Thus, bN' must be in the center of G/N' and all we need do is show $bN' \neq N'$. It suffices to show $b \notin G'$. Let $\alpha: G \to Z$ map b to a generator and both a and c to the identity element: Clearly α determines a homomorphism $\bar{\alpha}$ and $b \notin \ker \bar{\alpha}$. Since the image of $\bar{\alpha}$ is abelian we have $G' \leqslant \ker \bar{\alpha}$, hence G is a counterexample as claimed.

We note that $G=\langle a, b, c; aba^{-1}cb^{-1}c^{-1}\rangle$ and, thus, is a free product of a free group and a Fuchsian group by Lemma 1 of [4]. G is, in fact, isomorphic to $Z*(Z\times Z)$. In the above construction we may take $G=\langle a, b, c, \ldots t; [a, b]=W\rangle$, where W is any commutator in the free group on $b, c, \ldots, t$, and $N=(b, c, \ldots, t)^G$. Since W need involve only a subset of $\{b, c, \ldots, t\}$ we see, for example, that any free product of a (possibly trivial) free group and a surface group of genus $\geqslant 1$ provides a counterexample to problem 9 provided it has $\geqslant 3$ generators in its natural presentation.

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