

THE CHARACTER OF CERTAIN CLOSED SETS

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Let X be a topological space and let $A \subset X$. The *character* of A in X is the minimal cardinal of a base for the neighborhoods of A in X . Previous studies have shown that the character of certain subsets of X (or of X^2) is related to compactness conditions on X . For example, in [12], Ginsburg proved that if the diagonal

$$\Delta_X (= \{ \langle x, x \rangle : x \in X \})$$

of a space X has countable character in X^2 , then X is metrizable and the set of nonisolated points of X is compact. In [2], Aull showed that if every closed subset of X has countable character, then the set of nonisolated points of X is countably compact. In [18], we noted that if every closed subset of X has countable character, then $MA + \neg CH$ (Martin's axiom with the negation of the continuum hypothesis) implies that X is paracompact.

In this paper we study the character of closed discrete sets, of zero-sets, and of nowhere dense zero-sets of X , and introduce associated cardinal functions. Much of this work comes from the author's Ph.D. thesis at Ohio University (done under the direction of R. L. Blair to whom the author is greatly indebted). The paper has two purposes: The first is to develop some of the results necessary for [19], in which we give topological characterizations of $2^\alpha = \alpha^+$ for every infinite cardinal α (see Section 5 below), and the second is to present results, of interest in themselves, relating the character of these special sets to other properties of the space. For example, we show that if X is normal, first countable, and has no isolated points, then X is pseudocompact if and only if every zero-set of X has countable character (4.5), and that, under suitable cardinality restrictions, if X is hereditarily normal and extremally disconnected and if every nowhere dense zero-set has countable character, then X is discrete (6.6). We also show that in the theorem from [2] cited above we need only assume that every closed discrete subset of X has countable character (2.19).

In addition, we consider particular neighborhood bases for zero-sets. We show, for example, that a Tychonoff space X is countably compact

Received June 3, 1982 and in revised form February 15, 1983.

(resp. pseudocompact) if and only if for all $f \in C(X)$, $\{f^{-1}\left(-\frac{1}{n}, \frac{1}{n}\right):n \in \mathbf{N}\}$ is a base for the neighborhoods (resp. for the cozero-set neighborhoods) of $Z(f)$ in X , and we show that if Z is a zero-set of a space X , then $\text{cl}_{\beta X}Z$ is a zero-set of βX if and only if Z has a countable cozero-set base for its cozero-set neighborhoods.

Section 1 contains definitions and notation while Section 2 is concerned with methods of calculating the character of certain sets. Sections 3-6 contain theorems about the relation between certain properties of a space and the character of special sets.

1. Definitions and preliminaries. In Sections 1-3 we assume that all topological spaces are regular and T_1 and in Sections 4-6 we assume that all spaces are Tychonoff. We denote the set of real numbers by \mathbf{R} , the set of positive real numbers by \mathbf{R}^+ , and the set of natural numbers by \mathbf{N} . If X is a space, we denote the set of all isolated points of X by $I(X)$.

Let X be a space and let $A \subset X$. We set

$$C(X) = \{f: X \rightarrow \mathbf{R}: f \text{ is continuous}\}$$

and

$$C^*(X) = \{f \in C(X): f \text{ is bounded}\}.$$

For $f \in C(X)$, the zero-set $Z(f)$ of f is $\{x \in X: f(x) = 0\}$. The complement in X of a zero-set of X is a cozero-set of X . We say that A is C -embedded (resp. C^* -embedded) in X if for all $f \in C(A)$ (resp. for all $f \in C^*(A)$), there exists $g \in C(X)$ such that $g|_A = f$. A is z -embedded in X if every zero-set of A is of the form $Z \cap A$ for some zero-set Z of X . For $f \in C(X)$, we set

$$\mathcal{B}_f = \{f^{-1}\left(-\frac{1}{n}, \frac{1}{n}\right): n \in \mathbf{N}\}.$$

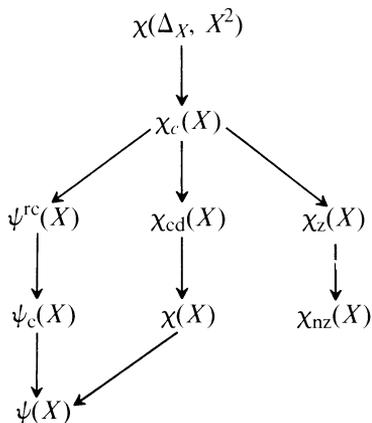
A collection \mathcal{B} is a base for the neighborhoods (resp. a base for the cozero-set neighborhoods) of A in X if each $B \in \mathcal{B}$ is a neighborhood of A in X and if whenever U is a neighborhood (resp. cozero-set neighborhood) of A in X , then there exists $B \in \mathcal{B}$ such that $B \subset U$. The character of A in X (denoted by $\chi(A, X)$) is $\omega \cdot \min\{|\mathcal{B}|: \mathcal{B} \text{ is a base for the neighborhoods of } A \text{ in } X\}$ and the pseudocharacter of A in X (denoted by $\psi(A, X)$) is $\omega \cdot \min\{|\mathcal{U}|: \mathcal{U} \text{ is a collection of open subsets of } X \text{ with } A = \bigcap \mathcal{U}\}$. (We write $\chi(x, X)$ or $\psi(x, X)$ instead of $\chi(\{x\}, X)$ or $\psi(\{x\}, X)$.)

The *character* of X (denoted by $\chi(X)$) is $\sup\{\chi(x, X):x \in X\}$ and the *pseudocharacter* of X (denoted by $\psi(X)$) is $\sup\{\psi(x, X):x \in X\}$. The *closed* (resp. *closed discrete*, resp. *zero-set*, resp. *nowhere dense zero-set*) *character* of X (denoted by $\chi_c(X)$ (resp. $\chi_{cd}(X)$, resp. $\chi_z(X)$, resp. $\chi_{nz}(X)$)) is $\sup\{\chi(A, X):A \text{ is a closed subset (resp. closed discrete subset, resp. zero-set, resp. nowhere dense zero-set) of } X\}$. The *closed pseudocharacter* of X (denoted by $\psi_c(X)$) is $\sup\{\psi(A, X):A \text{ is a closed subset of } X\}$.

A subset A of X is *regular closed* in X if $A = \text{cl int } A$. If A is closed in X , the *regular closed pseudocharacter* of A in X (denoted by $\psi^{rc}(A, X)$) is $\omega \cdot \min\{|\mathcal{Q}|:\mathcal{Q} \text{ is a collection of regular closed neighborhoods of } A \text{ in } X \text{ such that } A = \bigcap \mathcal{Q}\}$. The *regular closed pseudocharacter* of X (denoted by $\psi^{rc}(X)$) is $\sup\{\psi^{rc}(A, X):A \text{ is a closed subset of } X\}$.

2. Calculating the character of sets. The diagram in 2.1 represents inequalities holding among the cardinal functions defined in the previous section, with the arrows pointing to the smaller function. All are obvious with the possible exception of the topmost (a proof of which can be found in [12]). Examples can be found in 2.22 and 2.26 to show that no arrows can be added to the diagram.

2.1. PROPOSITION. *Let X be a space. The following inequalities hold:*



The next proposition shows that it is pointless to define the “nowhere dense closed character” of X . The routine proof is omitted, as are those of many of the simple propositions in this section.

2.2. PROPOSITION.

$$\chi_c(X) = \sup\{\chi(F, X):F \text{ is a closed nowhere dense subset of } X\}.$$

2.3. PROPOSITION. *If F is a finite subset of X , then*

$$\chi(F, X) = \max\{\chi(x, X):x \in F\}.$$

Hence, if X is countably compact, then $\chi_{cd}(X) = \chi(X)$.

The following proposition is an easy consequence of 2.2 and 2.3.

2.4. PROPOSITION. *If $|X - I(X)| < \omega$, then $\chi_c(X) = \chi(X)$.*

We turn next to a sequence of propositions which show that under certain circumstances we can use the pseudocharacter (or regular closed pseudocharacter) of a set to calculate its character. First we need some definitions.

A space X is a P_α -space if the intersection of $< \alpha$ open subsets of X is open in X . (Thus a P_{ω_1} -space is a P -space.) X is $[\alpha, \kappa]$ -compact if every open cover \mathcal{U} of X with $|\mathcal{U}| \leq \kappa$ has a subcover U' with $|U'| < \alpha$.

We also need a couple of lemmas.

2.5. LEMMA. *Let F be a closed subset of the P_α -space X and let $\kappa = \psi^{rc}(F, X)$. If $\kappa \leq \alpha$, then either $\kappa = \omega$, or $\kappa = \alpha$ and α is a regular cardinal. In either case there exists a decreasing sequence $\langle U_\xi:\xi < \kappa \rangle$ of open subsets of X such that*

$$F = \bigcap_{\xi < \kappa} U_\xi = \bigcap_{\xi < \kappa} \text{cl } U_\xi.$$

2.6. LEMMA. *Let F be a closed subset of X and let κ be a regular cardinal such that*

$$F = \bigcap_{\xi < \kappa} U_\xi = \bigcap_{\xi < \kappa} \text{cl } U_\xi$$

where $\langle U_\xi:\xi < \kappa \rangle$ is a decreasing sequence of open subsets of X . If X is $[\kappa, \kappa]$ -compact, then $\{U_\xi:\xi < \kappa\}$ is a base for the neighborhoods of F in X .

Proof. Let $F \subset U$ where U is an open subset of X . Then

$$\{U\} \cup \{X - \text{cl } U_\xi:\xi < \kappa\}$$

is an open cover of X of cardinality κ and hence there exists $I \subset \kappa$ such that $|I| < \kappa$ and $\{U\} \cup \{X - \text{cl } U_\xi:\xi \in I\}$ covers X . Let $\gamma = \sup I$. Then $\gamma < \kappa$ and $U_\gamma \subset U$.

2.7. COROLLARY. *If X is countably compact, then for all $f \in C(X)$, \mathcal{B}_f is a base for the neighborhoods of $Z(f)$ in X and hence $\chi_2(X) = \omega$.*

In Section 4 we prove the converse of 2.7 (see 4.4). The next result follows immediately from 2.5 and 2.6.

2.8. THEOREM. Let F be a closed subset of the P_α -space X and let $\kappa = \psi^{rc}(F, X)$. If $\kappa \leq \alpha$ and if X is $[\kappa, \kappa]$ -compact, then $\chi(F, X) = \kappa$.

If κ and α are cardinals with α infinite, then

$$\kappa^\alpha = \sum \{\kappa^\beta : \beta < \alpha\}.$$

If \mathcal{U} is a collection of sets, then

$$[\mathcal{U}]^{<\alpha} = \{\mathcal{A} \subset \mathcal{U} : |\mathcal{A}| < \alpha\}.$$

It is known that $\kappa^{\aleph^+} = \kappa^\alpha$, that $|[\mathcal{U}]^{<\alpha}| \leq |\mathcal{U}|^\alpha$ and that if $\kappa \geq \omega$, then $\kappa^\omega = \kappa$. (See [6, 1.22].)

2.9. THEOREM. Let F be a closed subset of the P_α -space X and let κ be a cardinal such that $\psi^{rc}(F, X) \leq \kappa$. If X is $[\text{cf}(\alpha), \kappa^\alpha]$ -compact, then

$$\chi(F, X) \leq \kappa^\alpha.$$

Proof. We may write $F = \bigcap_{\xi < \kappa} \text{cl } U_\xi$, where each U_ξ is an open neighborhood of F in X . Let

$$\mathcal{U} = \{U_\xi : \xi < \kappa\}$$

and let

$$\mathcal{B} = \{\bigcap \mathcal{F} : \mathcal{F} \in [\mathcal{U}]^{<\alpha}\}.$$

Each $B \in \mathcal{B}$ is open in X and $|\mathcal{B}| \leq \kappa^\alpha$. By a proof similar to that of 2.6, \mathcal{B} is a base for the neighborhoods of F in X .

2.10. COROLLARY. If X is a Lindelöf P -space, then $\chi(F, X) \leq \psi(F, X)^\omega$ for every closed subset F of X , and hence $\chi_c(X) \leq \psi_c(X)^\omega$.

2.11. COROLLARY. If F is a closed subset of a normal $[\omega, \psi(F, X)]$ -compact space X , then

$$\chi(F, X) = \psi(F, X).$$

Hence if X is $[\omega, \psi_c(X)]$ -compact, then

$$\chi_c(X) = \psi_c(X).$$

2.12. COROLLARY. ([2, Theorem 5]). If X is perfectly normal and countably compact, then $\chi_c(X) = \omega$.

2.13. COROLLARY. ([1, Chap. II, Theorem 4]). If F is closed in the compact space X , then $\psi(F, X) = \chi(F, X)$.

Before stating the next theorem, from which Corollaries 2.12 and 2.13 also follow, we need two definitions. If X is a Tychonoff space, then

$$\beta_\alpha X = \{p \in \beta X: \text{for every } \mathcal{F}' \subset p, \text{ with } |\mathcal{F}'| < \alpha, \cap \mathcal{F}' \neq \emptyset\},$$

and a Tychonoff space X is α -pseudocompact if $\beta_{\alpha+} X = \beta X$ [13]. Equivalently, X is a α -pseudocompact if every cozero-set cover of X of cardinality $\leq \alpha$ has a finite subcover (this is an easy dual of [13, 2.2]).

2.14. THEOREM. *If X is normal and α -pseudocompact and if F is a closed subset of X with $\psi(F, X) \leq \alpha$, then $\chi(F, X) = \psi(F, X)$.*

Proof. Let $F = \bigcap_{\xi < \kappa} U_\xi$, where each U_ξ is open in X and $\kappa < \alpha$. For all $\xi < \kappa$, F has a zero-set neighborhood $Z_\xi \subset U_\xi$. Let \mathcal{B} be the set of all finite intersections of members of $\{Z_\xi: \xi < \kappa\}$. Then $|\mathcal{B}| \leq \kappa$, each $B \in \mathcal{B}$ is a neighborhood of F , and $F = \bigcap \mathcal{B}$. Since X is normal and α -pseudocompact, \mathcal{B} is a base for the neighborhoods of F in X .

2.15. Remarks. (a) The inequality in 2.10 can be strict: The ordinal space ω is a Lindelöf P -space, and for all $F \subset \omega$,

$$\omega = \psi(F, X) = \chi(F, X) \neq \psi(F, X)^\omega = 2^\omega.$$

Equality can also hold: Let X be the set of ordinals $2^\omega + 1$ with basic neighborhoods of 2^ω of the form $X - B$ where B is a countable subset of 2^ω and with all other points isolated. Then X is a Lindelöf P -space and

$$2^\omega = \psi(2^\omega, X) \leq \chi(2^\omega, X) \leq \psi(2^\omega, X)^\omega = 2^\omega.$$

(b) In contrast to 2.7, pseudocompact spaces need not have countable zero-set character (see 2.26 (b)), but zero-sets in a pseudocompact space do have countable bases for their cozero-set neighborhoods (see 4.2).

We turn next to a discussion of the function χ_{cd} . A pairwise disjoint collection of open subsets of X is called a *cellular family* in X . A subset A of X is *cellularly embedded* in X if every cellular family in A is the restriction to A of a cellular family in X .

2.16. PROPOSITION. *If D is a discrete subset of a P_α -space X and if $|D| \leq \alpha$, then D is cellularly embedded in X .*

We will use the following known result several times. A proof can be found in [18, 2.1].

2.17. PROPOSITION. *If $D \subset X - I(X)$ is discrete and cellularly embedded in X , then $\chi(D, X) > |D|$.*

2.18. THEOREM. *If X is a P_α -space and if α is a regular cardinal, then $\chi_{cd}(X) \leq \alpha$ if and only if $\chi(X) \leq \alpha$ and every closed discrete subset of $X - I(X)$ has cardinality $< \alpha$.*

Proof. Assume first that $\chi_{cd}(X) \leq \alpha$. By 2.1, $\chi(X) \leq \alpha$. Let D be a closed discrete subset of $X - I(X)$ with $|D| = \alpha$. Since X is a P_α -space, D is cellularly embedded in X by 2.16 and hence, by 2.17, $\chi(D, X) > \alpha$, contradicting the assumption.

Conversely, assume that $\chi(X) \leq \alpha$ and that every closed discrete subset of $X - I(X)$ has cardinality $< \alpha$. Let D be a closed discrete subset of X . We may write

$$D - I(X) = \{x_\xi : \xi < \lambda\}$$

where $\lambda < \alpha$. For each $\xi < \lambda$, let $\{B_{\xi\gamma} : \gamma < \chi(X)\}$ be an open neighborhood base for x_ξ , and for each $\xi < \lambda$ and each $\gamma < \chi(X)$, let

$$G_{\xi\gamma} = \bigcap_{\delta < \gamma} B_{\xi\delta}.$$

Since X is a P_α -space, each $G_{\xi\gamma}$ is open in X and hence $\{G_{\xi\gamma} : \gamma < \chi(X)\}$ is a decreasing neighborhood base for x_ξ in X . For each $\gamma < \chi(X)$, let

$$H_\gamma = (\bigcup_{\xi < \lambda} G_{\xi\gamma}) \cup (D \cap I(X)).$$

It is easy to see that $\{H_\gamma : \gamma < \chi(X)\}$ is a base for the neighborhoods of D in X and hence $\chi_{cd}(X) \leq \chi(X) \leq \alpha$.

2.19. COROLLARY. $\chi_{cd}(X) = \omega$ if and only if $\chi(X) = \omega$ and $X - I(X)$ is countably compact.

A space X is \aleph_1 -compact if every closed discrete subset of X is countable.

2.20. COROLLARY. *If X is a P -space, then $\chi_{cd}(X) \leq \omega_1$ if and only if $\chi(X) \leq \omega_1$ and $X - I(X)$ is \aleph_1 -compact.*

2.21. COROLLARY. *If $\chi_{cd}(X) = \omega$ and $|I(X)| < \omega$, then $\chi_z(X) = \omega$.*

Proof. The result follows from 2.7 and 2.19.

The following examples are devoted to showing that no arrows can be added to the diagram of 2.1.

2.22. Examples. (a) There exists a space X such that

$$\psi^{rc}(X) \cdot \chi_z(X) \cdot \chi_{cd}(X) < \chi_c(X).$$

Let $Y = (\omega_1 + 1) \times (\omega_1 + 1)$ and let $X = Y - \{ \langle \omega_1, \omega_1 \rangle \}$. X is countably compact and hence $\chi_z(X) = \omega$ by 2.7. By 2.3,

$$\chi_{cd}(X) = \chi(X) = \omega_1.$$

Since Y is normal,

$$\psi^{rc}(Y) = \psi_c(Y) \leq |Y| = \omega_1$$

and hence $\psi^{rc}(X) \leq \omega_1$. Then

$$\psi^{rc}(X) \cdot \chi_z(X) \cdot \chi_{cd}(X) = \omega_1,$$

but if $F = [0, \omega_1) \times \{\omega_1\}$, then

$$\omega_1 < \chi(F, X) \leq \chi_c(X).$$

(b) Let X be the Michael line (that is, the reals with all irrational points isolated and with all rational points having their usual neighborhoods). $\chi(X) = \omega$, but since \mathbf{N} is both a nowhere dense zero-set and a closed discrete set of nonisolated points of X ,

$$\min\{\chi_{nz}(X), \chi_{cd}(X)\} \geq \chi(\mathbf{N}, X) > \omega$$

by 2.17. The rationals are a closed set but not a G_δ , and therefore we have

$$\omega = \chi(X) < \min\{\chi_{nz}(X), \chi_{cd}(X), \psi_c(X)\}.$$

(c) Let $X = \omega \cup \{p\}$ where $p \in \beta\omega - \omega$. X is countable and normal and hence $\psi^{rc}(X) = \omega$. Since $\{p\}$ is a nowhere dense zero-set and since X is not first countable at p ,

$$\psi^{rc}(X) < \min\{\chi(X), \chi_{nz}(X)\}.$$

(d) Let $X = \omega_1$. By 2.7, $\chi_z(X) = \omega$ and by 2.3, $\chi_{cd}(X) = \omega$. Since the set of limit ordinals is not a G_δ ,

$$\chi_{cd}(X) \cdot \chi_z(X) < \psi_c(X).$$

Next let $X = \omega_1 + 1$. In this case

$$\chi_z(X) = \omega < \psi(X).$$

(e) We use ideas from the tangent disk space and from the Arens-Fort space (see [17, p. 54 and p. 100]) to construct a space X such that

$$\psi_c(X) < \min\{\psi^{rc}(X), \chi(X), \chi_{nz}(X)\}.$$

Let $X = L \cup P$ where $L = \mathbf{R}^+$ and where P consists of all points of the first quadrant of the plane with rational coordinates. Let the points of P

be isolated while basic neighborhoods of $x \in L$ are constructed as follows: Let $\{A_k:k \in \omega\}$ be a partition of ω into pairwise disjoint infinite sets. For $x \in L$, pick $\langle q_{xn}:n \in \omega \rangle$, a sequence of positive rationals, such that in the usual topology on \mathbf{R} , $q_{xn} \rightarrow x$. Let

$$B_{nx} = \{q_{xj}:j \geq n\} \cup \{x\}$$

and let basic neighborhoods of x be of the form $B_{xn} - A$ where for some $J \subset \omega$ with $|J| < \omega$,

$$A = \cup_{j \in J} \left\{ \left\langle q_{xm}, \frac{1}{m} \right\rangle : m \in A_j \right\} \cup \cup_{j \notin J} F_j,$$

and where $F_j \subset \left\{ \left\langle q_{xm}, \frac{1}{m} \right\rangle : m \in A_j \right\}$ and $|F_j| < \omega$ (that is, A includes all but finitely many of the points of $\left\{ \left\langle q_{xm}, \frac{1}{m} \right\rangle : m \in A_j \right\}$ for all but finitely many $j \in \omega$). As in the Arens-Fort space, X is not first countable at $x \in L$. For each $x \in L$, $\{x\}$ is a nowhere dense zero-set of X (let $f(x) = 0, f\left(\left\langle q_{xn}, \frac{1}{n} \right\rangle\right) = \frac{1}{n}$, and $f(y) = 1$ if $y \notin B_{1x}$). Since P is dense and countable and L is closed discrete and uncountable, X is not normal, and hence $\omega < \psi^{rc}(X)$ ([2, Corollary 4]). Every open set is an F_σ , and hence

$$\omega = \psi_c(X) < \min\{\psi^{rc}(X), \chi(X), \chi_{nz}(X)\}.$$

(f) Finally, we give an example of a space X for which

$$\omega = \chi_{cd}(X) < \chi_{nz}(X) = 2^\omega.$$

The same example shows that the hypothesis on isolated points cannot be omitted from 2.21. Let $X = Y \cup P$ where $Y = [0, 1] \times \{0\}$ and $P = [0, 1] \times \mathbf{R}^+$. For all $y \in [0, 1]$, let $A_y = \{ \langle y, x \rangle : x \in \mathbf{R}^+ \}$ and let the collection

$$\{X \cap B_{1/n}(\langle y, 0 \rangle) - A_y : n \in \mathbf{N}\}$$

(where $B_{1/n}(\langle y, 0 \rangle)$ is the usual $1/n$ -ball around $\langle y, 0 \rangle$ in \mathbf{R}^2) be a local base at $\langle y, 0 \rangle$. Let points of P be isolated. Since $X - I(X) = Y$ is compact and $\chi(X) = \omega, \chi_{cd}(X) = \omega$ by 2.19. Y is a nowhere dense zero-set of X (Y is clearly a G_δ in the paracompact space X). We show that $\chi(Y, X) = 2^\omega$. Let $\{U_\xi : \xi < \kappa\}$ be a base for the neighborhoods of Y in X and assume that $\kappa < 2^\omega$. For all $y \in [0, 1]$, there exists $\xi_y < \kappa$ such that $U_{\xi_y} \subset X - A_y$. There is $\xi < \kappa$ such that $\xi = \xi_y$ for uncountably many $y \in [0, 1]$. Then $\{y : \xi_y = \xi\}$ is discrete in $[0, 1]$, a contradiction.

We give in 2.26 (a) an example of a space X for which $\chi_{nz}(X) < \chi_z(X)$. (Thus there is no analogue of 2.2 for $\chi_z(X)$.) First, however, we need the following definitions and preliminary results: A space X is an *almost- P -space* if every nonempty zero-set of X has nonempty interior. Clearly if X is an almost- P -space, then $\chi_{nz}(X) = \omega$.

2.23. PROPOSITION. *If W is a z -embedded union of zero-sets of an almost- P -space X , then W is an almost- P -space.*

2.24. COROLLARY. *If W is a cozero-set of an almost- P -space, then W is an almost- P -space.*

2.25. PROPOSITION. *If $\{Z_\alpha : \alpha \in I\}$ is a family of zero-sets of a space X and if there exists a discrete family $\{U_\alpha : \alpha \in I\}$ of cozero-sets of X such that $Z_\alpha \subset U_\alpha$ for all $\alpha \in I$, then $\cup_{\alpha \in I} Z_\alpha$ is a zero-set of X .*

2.26. Examples. (a) To obtain a space X such that $\chi_{nz}(X) < \chi_z(X)$, let $Y = \beta\mathbf{R}^+ - \mathbf{R}^+$, and let X be a proper cozero-set of Y . By [10, 3.1], Y is an almost- P -space, and hence so is X by 2.24. Clearly then, $\chi_{nz}(X) = \omega$.

By [11, 6.10], Y is connected and hence X is not compact. Since X is a Lindelöf F_σ -set in the normal space Y , X is not pseudocompact and therefore there exists an infinite discrete family $\{U_n : n \in \omega\}$ of nonempty cozero-sets of X , and for all $n \in \omega$, there exists a nonempty zero-set Z_n of Y , and hence of X , such that $Z_n \subset U_n$. Then $Z = \cup_{n \in \omega} Z_n$ is a zero-set of X by 2.25.

We show next that $\chi(Z, X) > \omega$. Suppose $\{G_n : n \in \omega\}$ is any countable collection of open sets in X containing Z . Since each Z_n is closed and each $U_n \cap G_n$ is open in the connected space Y ,

$$Z_n \not\subseteq U_n \cap G_n.$$

Pick $y_n \in (U_n \cap G_n) - Z_n$. Then $\cup_{n \in \omega} (U_n \cap G_n - \{y_n\})$ is a neighborhood of Z that contains no G_n .

(b) The next example shows that a pseudocompact space can have uncountable (nowhere dense) zero-set character (cf. 2.7 and 4.2). In [8, 17. 1 (c)], van Douwen proves that if

$$\Phi(\mathbf{R}) = \{p \in \beta\mathbf{R} - \mathbf{R} : p \notin \text{cl}_{\beta\mathbf{R}} A$$

for all closed discrete subsets A of $\mathbf{R}\}$

and if $Y = \mathbf{R} \cup \Phi(\mathbf{R})$, then Y is pseudocompact. The set \mathbf{N} is a zero-set of \mathbf{R} and \mathbf{R} is z -embedded in Y , and hence there exists a nowhere dense zero-set Z of Y such that $\mathbf{N} = Z \cap \mathbf{R}$. We show that $\chi(Z, Y) > \omega$. Let $\{G_n : n \in \mathbf{N}\}$ be any countable collection of open subsets of Y containing Z . For each $n \in \mathbf{N}$, pick

$$y_n \in \left(G_n \cap \left(n - \frac{1}{2}, n + \frac{1}{2} \right) \right) - \mathbf{N}.$$

For each $y \in Z - \mathbf{R}$,

$$y \notin \text{cl}_{\beta\mathbf{R}} \{y_n : n \in \mathbf{N}\}$$

and hence y has a neighborhood U_y in Y such that

$$U_y \cap \{y_n : n \in \mathbf{N}\} = \emptyset.$$

Then

$$\cup_{y \in Z - \mathbf{R}} U_y \cup \cup_{n \in \mathbf{N}} \left(G_n \cap \left(n - \frac{1}{2}, n + \frac{1}{2} \right) - \{y_n\} \right)$$

contains Z but contains no member of $\{G_n : n \in \mathbf{N}\}$.

3. Collectionwise normality and paracompactness. In this section we show that if the regular closed pseudocharacter and the closed discrete character of a space X are both countable, then X is collectionwise normal (cf. 3.2) and that an isocompact space with countable closed discrete character is paracompact. We also show that the statement “every perfect space with countable closed discrete character is normal” is independent of ZFC.

Aull has proved the following theorems:

3.1. THEOREM ([2, Corollary 4]). $\psi^{fc}(X) = \omega$ if and only if X is perfectly normal.

3.2. THEOREM ([2, Theorem 8]). If $\chi_c(X) = \omega$, then X is collectionwise normal.

We will show that we can weaken the hypothesis of 3.2.

3.3. THEOREM. If $\psi^{fc}(X) = \chi_{cd}(X) = \omega$, then X is collectionwise normal.

Proof. Let $\{F_\xi : \xi < \kappa\}$ be a discrete collection of closed subsets of X . Let

$$J = \{\xi < \kappa : F_\xi - I(X) \neq \emptyset\}.$$

By 2.19, $|J| < \omega$. Since X is normal by 3.1, there exists a pairwise disjoint collection $\{U_\xi : \xi \in J\}$ of open subsets of X such that $F_\xi \subset U_\xi$ for all $\xi \in J$. For $\xi \in J$, let

$$V_\xi = U_\xi - \cup_{\beta \neq \xi} F_\beta$$

and for all $\xi \notin J$, let $V_\xi = F_\xi$. Then $\{V_\xi : \xi < \kappa\}$ is a pairwise disjoint collection of open sets with $F_\xi \subset V_\xi$ for all $\xi < \kappa$.

The hypothesis of 3.3 cannot be reduced to $\psi_c(X) = \chi_{cd}(X) = \omega$. In fact, the statement “If $\psi_c(X) = \chi_{cd}(X) = \omega$, then X is normal” is independent of ZFC. To prove this, we first show the following (which appears with a stronger hypothesis in [18, 3.1]).

3.4. THEOREM [MA + \neg CH]. *If $\psi_c(X) = \chi_{cd}(X) = \omega$, then X is paracompact.*

Proof. In [22, Corollary 3], Weiss proves that, under MA + \neg CH, a countably compact perfect space is compact. Then X is the union of a compact set and a set of isolated points and is therefore paracompact.

We next give an example of a nonnormal space X for which

$$\psi_c(X) = \chi_{cd}(X) = \omega.$$

The example, due to Wage [21], is based on Ostaszewski’s space constructed in [15] under the set-theoretic hypothesis

3.5. THEOREM []. *There exists a nonnormal space X such that*

$$\psi_c(X) = \chi_{cd}(X) = \omega.$$

Proof. The space X constructed in [21] is perfect, countably compact, first countable and nonnormal. By 2.19, $\chi_{cd}(X) = \omega$.

A space X is *isocompact* if every closed countably compact subspace of X is compact. For example, real compact spaces [11, 5H.2], spaces with various weak covering properties (e.g. weakly $\delta\theta$ -refinable spaces [3]), and spaces with a G_δ -diagonal [5] are all isocompact.

3.6. PROPOSITION. *If X is isocompact and if $\chi_{cd}(X) = \omega$, then X is paracompact.*

Proof. By 2.19, $X - I(X)$ is a closed countably compact subspace of X and is therefore compact. Then, as in the proof of 3.4, X is paracompact.

3.7. COROLLARY. *If X is a Moore space and if $\chi_{\text{cd}}(X) = \omega$, then X is metrizable.*

4. Countably compact and pseudocompact spaces. For the remainder of this paper we assume that all spaces are Tychonoff. For $f \in C^*(X)$ we denote by f^β the continuous extension of f to βX . We denote $\beta X - X$ by X^* .

Although pseudocompact spaces need not have countable zero-set character (see 2.26 (b)), zero-sets in a pseudocompact space do have a countable base for their cozero-set neighborhoods (4.2).

4.1. PROPOSITION. *Let X be a space and let $f \in C^*(X)$. Then $\text{cl}_{\beta X} Z(f) = Z(f^\beta)$ if and only if \mathcal{B}_f is a base for the cozero-set neighborhoods of $Z(f)$ in X .*

Proof. If $\text{cl}_{\beta X} Z(f) = Z(f^\beta)$, then by 2.7, \mathcal{B}_{f^β} is a base for the neighborhoods of $\text{cl}_{\beta X} Z(f)$ in βX . Let $Z(f) \subset P$ where P is a cozero-set of X . Since $Z(f)$ is completely separated from $X - P$, there exists $B \in \mathcal{B}_{f^\beta}$ such that

$$\text{cl}_{\beta X} Z(f) \subset B \subset \beta X - \text{cl}_{\beta X} (X - P).$$

Then $B \cap X \in \mathcal{B}_f$ and $Z(f) \subset B \cap X \subset P$.

Conversely, suppose there exists $p \in Z(f^\beta) - \text{cl}_{\beta X} Z(f)$. Let Z be a zero-set neighborhood of p in βX which misses $Z(f)$. Then $Z(f) \subset X - Z$ but for all $B \in \mathcal{B}_f$, $B \cap Z \neq \emptyset$.

4.2. THEOREM. *A space X is pseudocompact if and only if for all $f \in C(X)$, \mathcal{B}_f is a base for the cozero-set neighborhoods of $Z(f)$ in X .*

Proof. Let X be pseudocompact and let $f \in C(X)$. By 4.1, it suffices to show that

$$\text{cl}_{\beta X} Z(f) = Z(f^\beta).$$

Suppose, on the contrary, that

$$p \in Z(f^\beta) - \text{cl}_{\beta X} Z(f).$$

Let Z be a zero-set neighborhood of p in βX such that $Z \cap Z(f) = \emptyset$. We may pick, recursively, for all $n \in \mathbf{N}$, x_n and U_n such that U_n is an open set in X , $p \notin \text{cl}_{\beta X} U_n$, and

$$x_n \in U_n \subset \text{cl } U_n \subset (f^{-1}\left(-\frac{1}{n}, \frac{1}{n}\right) \cap Z \cap X) - \cup_{j < n} \text{cl } U_j.$$

Then $\{U_n:n \in \mathbf{N}\}$ is an infinite locally finite collection of open subsets of X , contradicting the assumption that X is pseudocompact.

Conversely, assume that X is not pseudocompact and let $D = \{x_n:n \in \mathbf{N}\}$ be a copy of \mathbf{N} that is C -embedded in X [11, 1.21]. Define $f:D \rightarrow \mathbf{R}$ by $f(x_{2n}) = 1/n$ and $f(x_{2n-1}) = 0$, and let $A = \{x_{2n}:n \in \mathbf{N}\}$. There exists $g \in C^*(X)$ such that $g|D = f$. Since $A \cap Z(g) = \emptyset$ and since A is C -embedded in D and hence in X , A and $Z(g)$ are completely separated in X . Therefore there exists a zero-set Z of X with $A \subset Z$ and $Z \cap Z(g) = \emptyset$. Then $Z(g) \subset X - Z$ but for all $n \in \mathbf{N}$,

$$g^{-1}\left(-\frac{1}{n}, \frac{1}{n}\right) \cap Z \neq \emptyset$$

and thus \mathcal{B}_g is not a base for the cozero-set neighborhoods of $Z(g)$ in X .

4.3. COROLLARY. *A space X is pseudocompact if and only if for all $f \in C(X)$,*

$$cl_{\beta X} Z(f) = Z(f^\beta).$$

The preceding corollary is no doubt reasonably well known. (That $cl_{\beta X} Z(f) = Z(f^\beta)$ in pseudocompact spaces is a consequence of [11, 8.8 (b)], but no proof of the converse in the literature is known to the author.)

Our next result is a characterization of countably compact spaces analogous to that of 4.2 for pseudocompact spaces.

4.4. THEOREM. *A space X is countably compact if and only if for all $f \in C(X)$, \mathcal{B}_f is a base for the neighborhoods of $Z(f)$ in X .*

Proof. Assume that for all $f \in C(X)$, \mathcal{B}_f is a base for the neighborhoods of $Z(f)$ in X . By 4.2, X is pseudocompact. Suppose now that X is not countably compact and let $A = \{x_n:n \in \mathbf{N}\}$ be an infinite closed discrete subset of X . Since X is completely regular, by 2.16 there exists a pairwise disjoint collection $\{P_n:n \in \mathbf{N}\}$ of cozero-sets of X with $x_n \in P_n$ for all $n \in \mathbf{N}$. Let

$$Z = \bigcap_{n \in \mathbf{N}} (X - P_n).$$

There exists $f \in C^*(X)$ such that $Z = Z(f)$. We show next that

$$cl_{\beta X} A \cap cl_{\beta X} Z(f) \neq \emptyset.$$

Suppose, on the contrary, that $A \subset P$ and $Z(f) \subset P'$ where P and P' are disjoint cozero-sets of X . Then $\{P \cap P_n:n \in \mathbf{N}\}$ is an infinite discrete family of open sets of X , contradicting that X is pseudocompact.

Hence there exists $p \in \text{cl}_{\beta X} A \cap \text{cl}_{\beta X} Z(f)$ and therefore for all $n \in \mathbf{N}$,

$$f^{-1}\left(-\frac{1}{n}, \frac{1}{n}\right) \cap A \neq \emptyset.$$

Then $Z(f) \subset X - A$ but $X - A$ contains no member of \mathcal{B}_f , contradicting the assumption. The converse is given by 2.7.

We note that if X is normal, then any base for the cozero-set neighborhoods of a zero-set Z is a base for the neighborhoods of Z . A consequence of 4.2 and 4.4, then, is the well-known fact that a pseudocompact normal space is countably compact.

The next result gives sufficient conditions for the converse of 2.7.

4.5. PROPOSITION. *If X is normal, $\psi(X) = \omega$, and $I(X) = \emptyset$, then X is pseudocompact if and only if $\chi_Z(X) = \omega$.*

Proof. Suppose X is not pseudocompact. Let $\{U_n : n \in \omega\}$ be an infinite discrete collection of nonempty open subsets of X . Pick $x_n \in U_n$ and let $Z = \{x_n : n \in \omega\}$. Since $\psi(X) = \omega$, Z is a closed G_δ and thus a zero-set of X , but $\chi(Z, X) > \omega$ by 2.17.

5. Zero-sets and α^+ -closed sets in βX . There are at least two questions concerning the relation between zero-sets of X and zero-sets of βX : (1) When is $\text{cl}_{\beta X} Z$ a zero-set in βX if Z is a zero-set of X ? (2) When does $\text{cl}_{\beta X} Z(f) = Z(f^\beta)$ for $f \in C^*(X)$?

Answers to these two questions have been given by several authors: We gave an answer to (2) in 4.1, Rudd in [16] has given several conditions equivalent to $\text{cl}_{\beta X} Z(f) = Z(f^\beta)$, and in [20, Lemma 5], Terada has shown that if X is realcompact, then $\text{cl}_{\beta X} Z(f)$ is a zero-set of βX if and only if

$$\chi(Z(f), X) = \omega.$$

([20] is concerned with the question of when βX is Oz for an Oz -space X (see Section 6 for the definition of an Oz -space). Neighborhood bases for regular closed subsets of X play a role in [20]. For example, if one defines

$$\chi_{\text{rc}}(X) = \sup\{\chi(A, X) : A \text{ is a regular closed subset of } X\},$$

then [20, Corollary 1] can be phrased as follows: For a normal space X , βX is Oz if and only if $\chi_{\text{rc}}(X) = \omega$.)

We are interested here in a question which is more general than (1): If a subset F of X is the intersection of $<\alpha$ zero-sets of X , is $\text{cl}_{\beta X} F$ the intersection of $<\alpha$ zero-sets of βX ? We need the following definitions: We say that a subset S of a space X is *well-embedded* in X if S is completely separated in X from every disjoint zero-set of X [14, 6.1]. For example, it is known that every zero-set, every C -embedded subset, and every pseudo-compact subset of X is well-embedded in X .

Let α be an infinite cardinal. A subset A of X is α -open in X if A is the union of $<\alpha$ cozero-sets of X , and A is α -closed if $X - A$ is α -open. (Thus an ω_1 -open set is a cozero-set, and an ω_1 -closed set is a zero-set.) A subset A of X is a G_α -set in X if A is the intersection of $<\alpha$ open sets in X . (Hence a G_{ω_1} -set in X is a G_δ -set in X .)

The results below will be applied in [19].

Our first lemma generalizes the well-known fact that every closed G_δ -set in a normal space is a zero-set. The simple proof is omitted.

5.1. LEMMA. *Every closed G_α -set in a normal space X is α -closed in X .*

5.2. THEOREM. *If F is a well-embedded subset of a space X , then the following are equivalent:*

- (a) $\text{cl}_{\beta X} F$ is α^+ -closed in βX .
- (b) F has a cozero-set base of cardinality $\leq \alpha$ for its cozero-set neighborhoods.

Proof. (a) \Rightarrow (b). If $\text{cl}_{\beta X} F$ is α^+ -closed in βX , then

$$\psi(\text{cl}_{\beta X} F, \beta X) \leq \alpha$$

and hence, by 2.13,

$$\chi(\text{cl}_{\beta X} F, \beta X) \leq \alpha.$$

Since βX is normal, $\text{cl}_{\beta X} F$ has a cozero-set neighborhood base $\{Q_\xi: \xi < \alpha\}$ in βX . We will show that $\{Q_\xi \cap X: \xi < \alpha\}$ satisfies (b). Let $F \subset P$, where P is a cozero-set of X . Since F is well-embedded in X , there exists $\xi < \alpha$ such that

$$\text{cl}_{\beta X} F \subset Q_\xi \subset \beta X - \text{cl}_{\beta X} (X - P).$$

Clearly then, $F \subset Q_\xi \cap X \subset P$.

(b) \Rightarrow (a). Let $\{P_\xi: \xi < \alpha\}$ be a cozero-set base for the cozero-set neighborhoods of F in X . By 5.1, it suffices to show that

$$\text{cl}_{\beta X} F = \bigcap_{\xi < \alpha} (\beta X - \text{cl}_{\beta X} (X - P_\xi)).$$

Since F is well-embedded in X ,

$$\text{cl}_{\beta X} F \subset \bigcap_{\xi < \alpha} (\beta X - \text{cl}_{\beta X} (X - P_\xi)).$$

Now suppose $p \notin \text{cl}_{\beta X} F$. Let Z be a zero-set neighborhood of p in βX which misses F . There exists $\xi < \alpha$ such that

$$F \subset P_\xi \subset X - Z,$$

and hence $p \notin \text{cl}_{\beta X} P_\xi$. Then

$$p \notin \bigcap_{\xi < \alpha} (\beta X - \text{cl}_{\beta X} (X - P_\xi)).$$

5.3. COROLLARY. *If F is a closed subset of the normal space X , then the following are equivalent:*

- (a) $\text{cl}_{\beta X} F$ is α^+ -closed in βX .
- (b) $\chi(F, X) \cong \alpha$.

5.4. COROLLARY. *If F is a closed subset of the normal space X , then $\text{cl}_{\beta X} F$ is a zero-set of βX if and only if $\chi(F, X) = \omega$.*

6. Extremely disconnected spaces. A space X is *extremely disconnected* if every open subset of X is C^* -embedded in X [11, 1H.6], and X is an *F-space* if every cozero-set of X is C^* -embedded in X [11, 14.25]. Clearly if X is extremely disconnected, then X is an *F-space*. It is known [11, 6M.1 and 14.25] that X is extremely disconnected (resp. an *F-space*) if and only if βX is extremely disconnected (resp. an *F-space*).

In this concluding section we will show that an extremely disconnected hereditarily normal space X is close to being discrete. The additional hypotheses needed are a mild cardinality restriction and the requirement that $\chi_{\text{nz}}(X) = \omega$.

We need the following results:

6.1. THEOREM ([7, 9.3]). *If X is locally compact and nonpseudocompact, then X^* is not extremely disconnected.*

6.2. PROPOSITION. *If X is an F-space and if Z is a nonempty nowhere dense zero-set of βX , then Z is not extremely disconnected.*

Proof. If Z is a nonempty nowhere dense zero-set of βX , then $\beta X - Z$ is a nonpseudocompact locally compact subset of $\beta X = \beta(\beta X - Z)$. Hence by 6.1, $Z = (\beta X - Z)^*$ is not extremely disconnected.

6.3. PROPOSITION. *If X is a normal F-space, and if Z is a nonempty nowhere dense zero-set of X with $\chi(Z, X) = \omega$, then Z is not extremely disconnected.*

Proof. By 5.4, $\text{cl}_{\beta X} Z$ is a nonempty nowhere dense zero-set of βX and therefore $\text{cl}_{\beta X} Z = \beta Z$ is not extremally disconnected by 6.2. Hence Z is not extremally disconnected.

6.4. PROPOSITION. *If X is normal and hereditarily extremally disconnected and if $\chi_{\text{nz}}(X) = \omega$, then X is an almost- P -space.*

Proof. If X is not an almost- P -space, then there exists a nonempty nowhere dense zero-set Z of X . Since $\chi(Z, X) = \omega$, Z is not extremally disconnected by 6.3, contradicting that X is hereditarily extremally disconnected.

A space X is Oz if every open subset of X is z -embedded in X . Clearly every extremally disconnected space is Oz . X is *cellularly Ulam-nonmeasurable* if every cellular family in X has Ulam-nonmeasurable cardinality. In [4, 5.12], Blair has proved that every Oz almost- P -space of Ulam-nonmeasurable cardinality is discrete. The proof in [4] is actually of the following stronger result:

6.5. THEOREM ([4, 5.12]). *If X is a cellularly Ulam-nonmeasurable Oz almost- P -space, then X is discrete.*

6.6. THEOREM. *If X is a cellularly Ulam-nonmeasurable extremally disconnected hereditarily normal space with $\chi_{\text{nz}}(X) = \omega$, then X is discrete.*

Proof. By [9, 6.2 G(c)], X is hereditarily extremally disconnected and by 6.4, X is an almost- P -space. The result then follows from 6.5.

6.7. *Examples.* We give examples to show that none of the hypotheses of 6.6 can be omitted. (None of the spaces described below is discrete.)

(a) In view of 2.7, the ordinal space ω_1 satisfies all of the hypotheses of 6.6 except that ω_1 is not extremally disconnected.

(b) $\beta\omega$ satisfies all of the hypotheses of 6.6 except that it is not hereditarily normal.

(c) The space X of 2.22 (c) satisfies all of the hypotheses of 6.6 except that $\{p\}$ is a nowhere dense zero-set of X with $\chi(p, X) > \omega$. Note also that since X is perfect, the hypothesis $\chi_{\text{nz}}(X) = \omega$ cannot be omitted, even if X is perfectly normal.

(d) Let $X = D(\mu) \cup \{p\}$ where $D(\mu)$ is the discrete topology on μ , the first Ulam-measurable cardinal, and where $p \in \nu D(\mu) - D(\mu)$. X is an extremally disconnected hereditarily normal [4, 4.8] P -space and hence $\chi_Z(X) = \omega$. Thus the hypothesis that X is cellularly Ulam-nonmeasurable cannot be omitted.

6.8. *Remark.* Note that (d) is an example of a space which is extremally disconnected and hereditarily normal but not perfectly normal. The referee of this paper has asked if there exists such a space of nonmeasurable cardinality. This question is related to one asked by Blair in [4, 4.2(a)]. He defines a space X to be *weakly perfectly normal* if every subset of X is z -embedded in X and asks if there is a weakly perfectly normal space of nonmeasurable cardinality which is not perfectly normal. By [4, 4.11], if a space is extremally disconnected and hereditarily normal, then it is weakly perfectly normal, and hence an affirmative answer to the referee's question would answer Blair's question. In [4, 4.4], Blair observes that an affirmative answer to his question is implied by the existence of a weakly perfectly normal space of nonmeasurable cardinality which is not realcompact. While \diamond does imply the existence of such a space ([4, 4.10]), the question, without extra set-theoretical assumptions, remains open.

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