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A REMARK ON A PAPER OF WALTER AND ZAYED

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ABSTRACT. One result concerning the series representation for the continuous Jacobi transform in Walter and Zayed [1] is improved, the same thought also can be applied to the related results in [1].

1. For any real numbers a, b and c with $c \neq 0, -1, -2, \ldots$, the hypergeometric function $F(a, b, c, z)$ is given by

$$F(a,b,c,z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} z^k, \ |z| < 1,$$

where the series converges at $z = -1$ and $z = 1$ provided that $c - a - b + 1 > 0$ and $c - a - b > 0$ respectively.

The Jacobi function $P_\lambda^{(\alpha,\beta)}(x)$ of the first kind is defined by

$$P_\lambda^{(\alpha,\beta)}(x) = \frac{\Gamma(\lambda + \alpha + 1)}{\Gamma(\alpha+1)\Gamma(\lambda+1)} F(-\lambda,\ \lambda + \alpha + \beta + 1,\ \alpha + 1,\ (1-x)/2),\ x \in (-1, 1],$$

where $\alpha, \beta > -1, \lambda + \alpha + 1 \neq 0, -1, -2, \ldots$, and without loss of generality, $\lambda \geqq -(\alpha + \beta + 1)/2$ (cf. [1]). For integer values of $\lambda, P_\lambda^{(\alpha,\beta)}$ reduces to the usual Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ as defined in [2],

$$2^{-\alpha-\beta-1} \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x)\, dx = \delta_{nm} h_n^{(\alpha,\beta)},$$

where

$$h_n^{(\alpha,\beta)} = \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)} \frac{1}{2n+\alpha+\beta+1}.$$

Define further

$$\hat{P}_\lambda^{(\alpha,\beta)}(n) = 2^{-\alpha-\beta-1} \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta P_\lambda^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x) dx,$$

hence for $\lambda \neq n$,

$$\hat{P}_\lambda^{(\alpha,\beta)}(n) = \frac{(-1)^n \Gamma(\lambda+\alpha+1)\Gamma(\lambda+\beta+1) \sin \pi\lambda}{\pi(\lambda - n)(\lambda + n + \alpha + \beta + 1) n! \Gamma(\lambda + \alpha + \beta + 1)}.$$

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G. G. Walter and A. I. Zayed [1] introduced the continuous Jacobi transform as follows. Let $f(x) \in L^1\{(-1,1), W^{\alpha\beta}(x)\}$, $W^{\alpha\beta}(x) = (1-x)^\alpha(1+x)^\beta$, then the continuous Jacobi transform $\hat{f}^{(\alpha,\beta)}(\lambda)$ of $f(x)$ will be defined by

$$\hat{f}^{(\alpha,\beta)}(\lambda) = 2^{-\alpha-\beta-1}\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta P_\lambda^{(\alpha,\beta)}(x)f(x)dx, \lambda > -(\alpha+\beta+1)/2.$$

They gave a series representation for the continuous Jacobi transform $\hat{f}^{(\alpha,\beta)}(\lambda)$.

THEOREM A. *Let $f(x)$ be 2p times continuous differentiable with support in* $(-1,1)$, $2p > \max(\alpha,\beta)+3/2$, *then*

$$\hat{f}^{(\alpha,\beta)}(\lambda) = \sum_{n=1}^{\infty}\frac{1}{h_n^{(\alpha,\beta)}}\hat{f}^{(\alpha,\beta)}(n)\hat{P}_\lambda^{(\alpha,\beta)}(n),$$

where the series converges uniformly on any compact subset of $[0,\infty)$.

In the present paper, we indicate that the condition on Theorem A can be weakened by methods in approximation theory. One can similarly improve other results in [1] (e.g. Proposition 4.1). We will, however, omit the details.

2. **Main Result And Proof.**

THEOREM. *Let f(x) be 2p times continuous differentiable with support in* $(-1,1)$, $2p > \max\{\beta-1, 0\}$, *then*

$$\hat{f}^{(\alpha,\beta)}(\lambda) = \sum_{n=1}^{\infty}\frac{1}{h_n^{(\alpha,\beta)}}\hat{f}^{(\alpha,\beta)}(n)\hat{P}_\lambda^{(\alpha,\beta)}(n),$$

where the series converges uniformly on any compact subset of $[0,\infty)$. *Furthermore,*

$$\sum_{n=[2\lambda]+1}^{\infty}\frac{1}{h_n^{(\alpha,\beta)}}\hat{f}^{(\alpha,\beta)}(n)\hat{P}_\lambda^{(\alpha,\beta)}(n) = 0(\lambda^{-2p-1}\omega(f^{(2p)},\lambda^{-1})),$$

where $\omega(f,\delta)$ is the modulus of continuity of $f \in C_{(-1,1)}$.

LEMMA 1. *Let $f \in L^2\{(-1,1), W^{\alpha\beta}\}$, $E_n(f)$ be the nth best approximation to f(x) by polynomials in L^2 $W^{\alpha\beta}$-weight norm, then*

$$\left(\sum_{k=n+1}^{2n}\left(\frac{1}{\sqrt{h_n^{(\alpha,\beta)}}}\hat{f}^{(\alpha,\beta)}(n)\right)^2\right)^{1/2} \leqq C(\alpha,\beta)E_n(f).$$

PROOF. It is well-known that $\{(2^{\alpha+\beta+1}h_n^{(\alpha,\beta)})^{-1/2}P_n^{(\alpha,\beta)}\}$ forms an orthonomal system in $[-1,1]$ under the weight $W^{\alpha\beta}(x)$, $\alpha,\beta > -1$. Therefore any $f \in L^1\{(-1,1), W^{\alpha\beta}\}$ has the expansion in Fourier-Jacobi series

$$f(x) \sim \sum_{n=1}^{\infty}\frac{1}{h_n^{(\alpha,\beta)}}\hat{f}^{(\alpha,\beta)}(n)P_n^{(\alpha,\beta)}(x).$$

If $f(x) \in L^2\{(-1,1), W^{\alpha\beta}(x)\}$, then it holds true that

$$\int_{-1}^{1}(f(x)-S_n(f,x))^2 W^{\alpha\beta}(x)dx \leqq \int_{-1}^{1}(f(x)-q_n(x))^2 W^{\alpha\beta}(x)dx$$

for all nth degree polynomials q_n, where $S_n(f,x)$ is the nth partial sum of the Fourier-Jacobi series of $f(x)$. Hence

$$\left(\int_{-1}^{1}(S_{2n}(f,\ x)-S_n(f,\ x))^2 W^{\alpha\beta}(x)dx\right)^{1/2} \leqq C(\alpha,\beta)E_n(f),$$

that is the required result. ∎

LEMMA 2. *Let* $f \in L^2\{(-1,1), W^{\alpha\beta}\}$. *Then in any closed subinterval* $[s,t] \subset (-1,1)$,

$$\sum_{n=1}^{\infty} \frac{1}{h_n^{(\alpha,\beta)}} \hat{f}^{(\alpha,\beta)}(n) P_n^{(\alpha,\beta)}(x)$$

coverges uniformly and absolutely if $E_n(f) = 0(n^{-\delta})$ *for some* $\delta > 1/2$.

PROOF. From Lemma 1,

$$\sum_{i=2^k+1}^{2^{k+1}} \left|\frac{1}{h_i^{(\alpha,\beta)}} \hat{f}^{(\alpha,\beta)}(i)\right| = 0(2^k E_{2^k}(f)),$$

so we can get immediately

$$\sum_{k=1}^{\infty}\left|\frac{\hat{f}^{(\alpha,\beta)}(k)}{h_k^{(\alpha,\beta)}}\right| = \left|\frac{\hat{f}^{(\alpha,\beta)}(1)}{h_l^{(\alpha,\beta)}}\right| + \sum_{k=1}^{\infty}\sum_{s=2^{k-1}+1}^{2^k}\left|\frac{\hat{f}^{(\alpha,\beta)}(s)}{h_s^{(\alpha,\beta)}}\right| = 0\left(\sum_{k=0}^{\infty} 2^k E_{2^k}(f)\right)$$

$$= 0(1)\sum_{n=1}^{\infty} E_n(f).$$

At same time noting that (cf. [2])

$$P_n^{(\alpha,\beta)}(x) = 0(1)\begin{cases}\left(\dfrac{(1-x^2)^{1/2}}{n}+\dfrac{1}{n^2}\right)^{-\alpha-1/2} n^{-\alpha-1}, \ 0 \leqq x \leqq 1, \\ \left(\dfrac{(1-x^2)^{1/2}}{n}+\dfrac{1}{n^2}\right)^{-\beta-1/2} n^{-\beta-1}, \ -1 \leqq x \leqq 0,\end{cases}$$

with the condition $E_n(f) = 0(n^{-\delta})$ for $\delta > 1/2$, we have completed the proof of Lemma 2. ∎

LEMMA 3. *Let* $f \in L^2\{[-1,1], W^{\alpha\beta}\}$. *If* $E_n(f) = 0(n^{-s})$, $s > \beta - 1$, *then the series*

$$\sum_{n=1}^{\infty} \frac{1}{h_n^{(\alpha,\beta)}} \hat{f}^{(\alpha,\beta)}(n) \hat{P}_{\lambda}^{(\alpha,\beta)}(n)$$

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converges on any compact subset of $[0, \infty)$. *Furthermore*

$$\sum_{n=[2\lambda]+1}^{\infty} \frac{1}{h_n^{(\alpha,\beta)}} \hat{f}^{(\alpha,\beta)}(n)\hat{P}_{\lambda}^{(\alpha,\beta)}(n) = 0(\lambda^{-1/2}E_{[2\lambda]+1}(f)).$$

PROOF. Using a well-known result

$$\frac{\Gamma(x)}{\Gamma(x+\alpha)} \sim x^{-\alpha}, \ x \longrightarrow \infty,$$

we give an estimate to $\hat{P}_{\lambda}^{(\alpha,\beta)}(n)$:

$$\hat{P}_{\lambda}^{(\alpha,\beta)}(n) = 0\left(\frac{\lambda^{-\beta}n^{\beta}}{(|\lambda - n| + 1)(\lambda + n + \alpha + \beta + 1)}\right), \ \lambda \geqq -\frac{\alpha + \beta + 1}{2},$$

together with Lemma 1,

$$\sum_{n=1}^{\infty} \left|\frac{1}{h_n^{(\alpha,\beta)}} \hat{f}^{(\alpha,\beta)}(n)\hat{P}_{\lambda}^{(\alpha,\beta)}(n)\right| = \left|\frac{1}{h_1^{(\alpha,\beta)}} \hat{f}^{(\alpha,\beta)}(1)\hat{P}_{\lambda}^{(\alpha,\beta)}(1)\right| +$$

$$\sum_{k=1}^{\infty} \sum_{i=2^{k-1}+1}^{2^k} \left|\frac{1}{h_i^{(\alpha,\beta)}} \hat{f}^{(\alpha,\beta)}(i)\hat{P}_{\lambda}^{(\alpha,\beta)}(i)\right| \leq \left|\frac{1}{h_1^{(\alpha,\beta)}} \hat{f}^{(\alpha,\beta)}(1)\hat{P}_{\lambda}^{(\alpha,\beta)}(1)\right| +$$

$$\sum_{k=1}^{\infty} \left(\sum_{i=2^{k-1}+1}^{2^k} \left(\frac{1}{h_i^{(\alpha,\beta)}} \hat{f}^{(\alpha,\beta)}(i)\right)^2\right)^{1/2} \left(\sum_{i=2^{k-1}+1}^{2^k} |\hat{P}_{\lambda}^{(\alpha,\beta)}(i)|^2\right)^{1/2}$$

$$= 0\left(\sum_{k=0}^{\infty} 2^{k(\beta-1)}E_{2^k}(f)\right) = 0\left(\sum_{k=0}^{\infty} 2^{k(\beta-s-1)}\right),$$

under the condition $s > \beta - 1$, it is clear that

$$\sum_{n=1}^{\infty} \left|\frac{1}{h_n^{(\alpha,\beta)}} \hat{f}^{(\alpha,\beta)}(n)\hat{P}_{\lambda}^{(\alpha,\beta)}(n)\right| < +\infty.$$

On the other hand due to the estimate for $\hat{P}_{\lambda}^{(\alpha,\beta)}(n)$,

$$\sum_{n=[2\lambda]+1}^{\infty} \frac{1}{h_n^{(\alpha,\beta)}} \hat{f}^{(\alpha,\beta)}(n)\hat{P}_{\lambda}^{(\alpha,\beta)}(n) = 0\left(\lambda^{-\beta}E_{[2\lambda]+1}(f) \sum_{n=[2\lambda]+1}^{\infty} n^{\beta-3/2}\right)$$

$$= 0\left(\lambda^{-1/2}E_{[2\lambda]+1}(f)\right),$$

thus Lemma 3 is proved. ∎

PROOF OF THE THEOREM. We only need to prove

$$\hat{f}^{(\alpha,\beta)}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{h_n^{(\alpha,\beta)}} \hat{f}^{(\alpha,\beta)}(n)\hat{P}_{\lambda}^{(\alpha,\beta)}(n),$$

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it follows that by the definition of $\hat{f}^{(\alpha,\beta)}(\lambda)$ we exchange the order of the integration and the sum, as it is made in [1]. Theorem is proved. ∎

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REFERENCES

1. G. G. Walter and A. I. Zayed, *The continuous* (α, β)*-Jacobi transform and its inverse when* $\alpha + \beta + 1$ *is a positive integer*, Trans. Amer. Math. Soc. **305** (1988), 653–664.
2. G. Szegö, Orthogonal Polynomials, Amer. Math. Soc. Colloq. Publ. vol. 23, Amer. Math. Soc., Providence, R.I., 1974.

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