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ON HOLOMORPHIC DIFFERENTIALS OF SOME ALGEBRAIC FUNCTION FIELD OF ONE VARIABLE OVER C

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We give holomorphic differentials of some algebraic function field K of complex dimension one which is a generalisation of a hyperelliptic field.

1. INTRODUCTION

Let K be an algebraic function field of one variable over $\mathbf{C}$. Then K is generated over $\mathbf{C}$ by two generic points x and y of a plane curve defined by some equation $f(X,Y)=0$, if f is the irreducible polynomial vanishing on (x,y). Here x is transcendental over $\mathbf{C}$ and y is algebraic over the rational function field $F=\mathbf{C}(x)$. In the sense of the theory of Riemann surfaces one can identify K with the field of meromorphic functions on a compact Riemann surface ([**1**, **6**, **7**, **8**]). Thus all the statements parallel one another.

Let Ω be the set of differentials (= differential one forms) of K. Then Ω is a one dimensional vector space over K ([**3**, **5**, **8**]), which allows us to express it as $\Omega = K\cdot dx$. If $w = y\,dx$ is a differential of K and $\mathfrak{P}$ a prime divisor of K, we define the order of w at $\mathfrak{P}$ by $v_{\mathfrak{P}}(w) = \nu_{\mathfrak{P}}(y\,dx/dt)$ where t is a local uniformising parameter in $K_{\mathfrak{P}}$ (the completion of K with respect to the $\mathfrak{P}$-adic topology). As is well known, $K_{\mathfrak{P}}$ is isomorphic to the field $\mathbf{C}((t))$ of formal power series and hence $\nu_{\mathfrak{P}}$ is the standard valuation on $\mathbf{C}((t))$. Note that $v_{\mathfrak{P}}(w)$ is independent of the choice of t and $v_{\mathfrak{P}}(w)=0$ for almost all prime divisors $\mathfrak{P}$. We call $w\in\Omega$ a *holomorphic differential* (or a *differential of the first kind*) if $v_{\mathfrak{P}}(w)\geqslant 0$ for all $\mathfrak{P}$. If Ω_1 is the set of holomorphic differentials and g is the genus of K, then Ω_1 is a g-dimensional vector space over $\mathbf{C}$ ([**1**, **2**, **3**, **6**, **8**]). Therefore we can describe all the holomorphic differentials when we are given an algebraic function field K.

In this paper we find holomorphic differentials of $K=\mathbf{C}(x,y)$ for which x and y satisfy the following equation :

$$y^n = A(x-a_1)^{n_1}(x-a_2)^{n_2}\cdots(x-a_\ell)^{n_\ell} \tag{1}$$

where all $a_i(\in\mathbf{C})$ are distinct, $A\in\mathbf{C}^\times$, $n\geqslant 2$, $n_j\geqslant 1$ $(j=1,2,\ldots,\ell)$ and $(n,n_1,\ldots,n_\ell)=1$. Observe that $(n,n_1,\ldots,n_\ell)$ is the greatest common divisor of

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the integers $n, n_1, \ldots, n_\ell$. In particular, K is called a hyperelliptic field provided that $n = 2$, $\ell \geqslant 5$ and $n_1 = \cdots = n_\ell = 1$, in which case one can give an explicit basis $\{w_1, \ldots, w_g\}$ for Ω_1 as follows ([**2, 3, 8**]):

$$w_j = \frac{x^{j-1}\,dx}{y}, \qquad 1 \leqslant j \leqslant g = \left[\frac{\ell - 1}{2}\right]$$

where $y = \sqrt{A}\sqrt{(x-a_1)\ldots(x-a_\ell)}$ and $[r]$ denotes the largest integer $\leqslant r$.

2. Preliminaries

To start with we need to show that the polynomial

$$(*) \qquad Y^n - A(X-a_1)^{n_1}\ldots(X-a_\ell)^{n_\ell}$$

is irreducible over $\mathbf{C}$.

Lemma 1. *Let D be a unique factorisation domain of characteristic O and F be the quotient field of D. For $a \in D$, put $a = \pi_1^{e_1}\pi_2^{e_2}\cdots\pi_\ell^{e_\ell}$ with π_j prime elements in D. If n is a positive integer such that $(n, e_1, \ldots, e_\ell) = 1$, then the polynomial $\varphi(Y) = Y^n - a$ over D is irreducible over F.*

Proof: Let ξ be a primitive n-th root of unity in F and α be an element of the algebraic closure $\overline{F}$ of F for which $\varphi(\alpha) = 0$. Since $F(\alpha)$ is a (finite) Galois extension of F, let G be the Galois group of $F(\alpha)$ over F and $m = [F(\alpha) : F]$. It suffices to show that $m = n$.

Suppose $m < n$. Take an F-automorphism σ from G. Then

$$\sigma(\alpha)^n = \sigma(\alpha^n) = \sigma(a) = a;$$

hence $\sigma(\alpha) = \xi^{\nu(\sigma)}\alpha$ for $\nu(\sigma) \in \mathbf{Z}/n\mathbf{Z}$. We derive from this an injective homomorphism ν from G into the additive group $\mathbf{Z}/n\mathbf{Z}$. Thus we may view G as a subgroup of $\mathbf{Z}/n\mathbf{Z}$ which is cyclic of order n. Let $G = \langle\tau\rangle$. Since $F(\alpha)$ is Galois over F, $|G| = m$ divides n. We observe by the definition of ν that $\tau(\alpha) = \xi^{\nu(\tau)}\alpha$ implies $\tau^m(\alpha) = \xi^{\nu(\tau^m)}\alpha = \xi^{m\nu(\tau)}\alpha$.

But, $\tau^m(\alpha) = \alpha$; hence

$$(2) \qquad m\nu(\tau) \equiv 0 \pmod{n}.$$

On the other hand, by (2),

$$(\tau(\alpha))^m = \xi^{m\nu(\tau)}\alpha^m = \alpha^m$$

and

$$(\tau(\alpha))^m = \tau(\alpha^m).$$

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Therefore α^m belong to F. Set $\alpha^m = b$. Then $b^{n/m} = (\alpha^m)^{n/m} = \alpha^n = a$.

Since m divides n, let $b^{n/m} = b^r$. Obviously $r > 1$. Now we have $a = b^r$ with $a = \pi_1^{e_1} \cdots \pi_\ell^{e_\ell}$, from which we conclude that r is a common divisor of $e_1, e_2, \ldots,$ and e_ℓ. Since r also divides n, $(n, e_1, \ldots, e_\ell) > 1$. This is a contradiction. ∎

Lemma 1 asserts that the polynomial in $(*)$ is irreducible over $\mathbf{C}$.

THEOREM 2 (Riemann-Hurwitz's formula). *Let K be an algebraic function field of one variable over $\mathbf{C}$ of genus g, and $F = \mathbf{C}(x)$ for $x \in K - \mathbf{C}$ such that $[K : F] = n$. Let $\mathfrak{P}$ be a prime divisor of K and $e_{\mathfrak{P}}$ be the ramification index of $\mathfrak{P}$. Then*

$$\frac{1}{2}\sum_{\mathfrak{P}}(e_{\mathfrak{P}} - 1) = n + g - 1.$$

PROOF: ([2, 3, 4, 5, 8]).

Since $\mathfrak{P}$ lies over some prime divisor $\mathfrak{F}_\alpha$ of F ($\alpha \in \widehat{\mathbf{C}}$ the extended complex plane), we have the following diagram :

$$\begin{array}{ccc} & & K_{\mathfrak{P}} \simeq \mathbf{C}((t)) \\ K & \nearrow & | \\ | & & F_{\mathfrak{F}_\alpha} \simeq \mathbf{C}((u)) \\ F & \nearrow & \end{array}$$

where $F_{\mathfrak{F}_\alpha}$ is the completion of F and u is a local uniformising parameter in $F_{\mathfrak{F}_\alpha}$. If $e = e(\mathfrak{P}|\mathfrak{F}_\alpha)$ is the ramification index of $\mathfrak{P}$ over $\mathfrak{F}_\alpha$, then $u = t^e$. Hence

$$u = \begin{cases} x - \alpha & \text{if } \alpha \neq \infty \\ 1/x & \text{if } \alpha = \infty, \end{cases}$$

implies that

(3) $$v_{\mathfrak{P}}(dx) = \nu_{\mathfrak{P}}\left(\frac{dx}{dt}\right) = \begin{cases} e - 1 & \alpha \neq \infty \\ -e - 1 & \alpha = \infty. \end{cases}$$

3. Main Theorem

Following the Riemann-Hurwitz's formula one can derive the genus g of K as follows

(4) $$g = \frac{1}{2}n(\ell - 1) - \frac{1}{2}\left\{\sum_{j=1}^{\ell}(n, n_j) + (n, N)\right\} + 1$$

where $N = \sum_{j=1}^{\ell} n_j$. We first consider the differential of the form $w = dx/y$ with $y = \sqrt[n]{A}\sqrt[n]{(x-a_1)^{n_1}\cdot\dots\cdot(x-a_\ell)^{n_\ell}}$.

It follows from (3) that

$$\text{(5)}\qquad v_{\mathfrak{P}}(dx) = \begin{cases} 1-1=0 & \text{if } \alpha \neq \infty \text{ and } a_j,\\ \frac{n}{(n,n_j)} - 1 & \text{if } \alpha = a_j,\\ -\frac{n}{(n,N)} - 1 & \text{if } \alpha = \infty. \end{cases}$$

Next, we obtain from (1) that

$$n\nu_{\mathfrak{P}}(y) = \nu_{\mathfrak{P}}((x-a_1)^{n_1}) + \dots + \nu_{\mathfrak{P}}((x-a_\ell)^{n_\ell})$$

because $v_{\mathfrak{P}}(y) = \nu_{\mathfrak{P}}(y)$ and $\nu_{\mathfrak{P}}(A) = 0$. Since all $(x-a_j)$ belong to the field F,

$$\text{(6)}\qquad n\nu_{\mathfrak{P}}(y) = n_1 e_{\mathfrak{P}}\nu_{\mathfrak{F}_\alpha}((x-a_1)) + \dots + n_\ell e_{\mathfrak{P}}\nu_{\mathfrak{F}_\alpha}((x-a_\ell))$$

where $\nu_{\mathfrak{F}_\alpha}$ is the discrete (exponential) valuation of F. Thus, by (6),

(7)

$$\nu_{\mathfrak{P}}(y) = \begin{cases} \frac{1}{n}\{n_1\cdot 1\cdot 0 + \dots + n_\ell\cdot 1\cdot 0\} = 0, & \alpha \neq \infty \text{ and } a_j;\\ \frac{1}{n}\{0+\dots+0+n_j\cdot\frac{n}{(n,n_j)}\cdot 1 + 0 + \dots + 0\} = \frac{n_j}{(n,n_j)}, & \alpha = a_j;\\ \frac{1}{n}\{n_1\cdot\frac{n}{(n,N)}\cdot -1 + \dots + n_\ell\cdot\frac{n}{(n,N)}\cdot -1\} = -\frac{N}{(n,N)}, & \alpha = \infty. \end{cases}$$

Since $v_{\mathfrak{P}}(w) = v_{\mathfrak{P}}(dx) - \nu_{\mathfrak{P}}(y)$, it follows from (5) and (7) that for a prime divisor $\mathfrak{P}$ over $\mathfrak{F}_\alpha$

$$\nu_{\mathfrak{P}}(w) = \begin{cases} 0 & \text{if } \alpha \neq \infty \text{ and } a_j,\\ \frac{n-(n_j+(n,n_j))}{(n,n_j)} & \text{if } \alpha = a_j,\\ \frac{N-(n+(n,N))}{(n,N)} & \text{if } \alpha = \infty. \end{cases}$$

Therefore, dx/y is holomorphic if and only if $n \geqslant n_j + (n,n_j)$ $(1 \leqslant j \leqslant \ell)$ and $N \geqslant n + (n,N)$. We readily see from this that the necessary conditions are

$$\text{(8)}\qquad n > n_j \qquad (j = 1,2,\dots,\ell) \quad \text{and} \quad N > n.$$

For instance, look at the case $n = 3$ and $\ell = 3$. Then by (4)

$$g = 3 - \frac{1}{2}\{(3,n_1) + (3,n_2) + (3,n_3) + (3,n_1+n_2+n_3)\} + 1,$$

and hence, by the restriction (8), we come up with the following table :

n_1	n_2	n_3	validity	g
1	1	1	×	1
1	1	2	O	2
1	2	2	O	2
2	2	2	O	1

In the first case dx/y is not a holomorphic differential; then what is the basis element of Ω_1? In the other cases $dx/y \in \Omega_1$, moreover in the last case every holomorphic differential is a constant multiple of dx/y. But, there still remains a question in the second and the third case; what is another basis element w independent of dx/y?

To have complete answers we consider the differentials of the form

$$w = \frac{\prod_{j=1}^{\ell} (x-a_j)^{k_j}\, dx}{y^m} \tag{9}$$

where $m \geqslant 1$, $k_j \geqslant 0$ and $y = \sqrt[n]{A}\sqrt[n]{(x-a_1)^{n_1}\cdot\dots\cdot(x-a_\ell)^{n_\ell}}$.

Theorem 3. *A differential w in (9) is holomorphic if and only if $n(k_j+1) \geqslant mn_j + (n,n_j)$ $(1 \leqslant j \leqslant \ell)$ and $mN \geqslant \left(\sum_{j=1}^{\ell} k_j + 1\right) n + (n,N)$.*

Proof: For a prime divisor $\mathfrak{P}$ over $\mathfrak{Z}_\alpha$,

$$\begin{aligned} v_{\mathfrak{P}}(w) &= v_{\mathfrak{P}}(dx) + \sum_{j=1}^{\ell} k_j \nu_{\mathfrak{P}}((x-a_j)) - m\nu_{\mathfrak{P}}(y) \\ &= v_{\mathfrak{P}}(dx) + \sum_{j=1}^{\ell} k_j e_{\mathfrak{P}} \nu_{\mathfrak{Z}_\alpha}((x-a_j)) - m\nu_{\mathfrak{P}}(y). \end{aligned}$$

Here

$$\sum_{j=1}^{\ell} k_j e_{\mathfrak{P}} \nu_{\mathfrak{Z}_\alpha}((x-a_j)) = \begin{cases} 0 & \text{if } \alpha \neq \infty \text{ and } a_j, \\ k_j \cdot \frac{n}{(n,n_j)} & \text{if } \alpha = a_j, \\ \sum_{j=1}^{\ell} k_j \cdot \frac{-n}{(n,N)} & \text{if } \alpha = \infty. \end{cases} \tag{10}$$

Therefore, by (5), (7) and (10)

$$v_{\mathfrak{P}}(w) = \begin{cases} 0, & \alpha \neq \infty \text{ and } a_j, \\ \frac{n(k_j+1) - (mn_j + (n,n_j))}{(n,n_j)}, & \alpha = a_j, \\ \frac{mN - \left(\left(\sum_{j=1}^{\ell} k_j + 1\right)n + (n,N)\right)}{(n,N)}, & \alpha = \infty, \end{cases}$$

from which we have the theorem. ∎

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4. Applications

In Section 3 we see that dx/y is not a holomorphic differential when $n = 3$ and $n_1 = n_2 = n_3 = 1$. Applying Theorem 3, however, we can get appropriate basis element to this case. Since $N = n_1 + n_2 + n_3 = 3$, $(n, n_j) = 1$ and $(n, N) = 3$, w in (9) is holomorphic if and only if $3(k_1 + 1) \geqslant m+1$, $3(k_2 + 1) \geqslant m+1$, $3(k_3 + 1) \geqslant m+1$ and $3m \geqslant 3(k_1 + k_2 + k_3 + 1) + 3$, or equivalently $3k_1 \geqslant m - 2$, $3k_2 \geqslant m - 2$, $3k_3 \geqslant m - 2$ and $m \geqslant k_1 + k_2 + k_3 + 2$ $(m > 1)$.

Taking smallest possible integers, $m = 2$ and $k_1 = k_2 = k_3 = 0$. Therefore every holomorphic differential in this case is a constant multiple of dx/y^2 with $y = \sqrt[3]{A}\sqrt[3]{(x - a_1)(x - a_2)(x - a_3)}$.

Likewise, in the second and third cases holomorphic bases are

$$\left\{\frac{dx}{y}, \frac{(x - a_3)\,dx}{y^2}\right\} \quad \left(y = \sqrt[3]{A}\sqrt[3]{(x - a_1)(x - a_2)(x - a_3)^2}\right)$$

and

$$\left\{\frac{dx}{y}, \frac{(x - a_2)(x - a_3)\,dx}{y^2}\right\} \quad \left(y = \sqrt[3]{A}\sqrt[3]{(x - a_1)(x - a_2)^2(x - a_3)^2}\right),$$

respectively.

In the case of a hyperelliptic field K, we can derive a different basis for Ω_1 from the one described in Section 1 such as $\{w_1', \dots, w_g'\}$ with

$$w_k' = \frac{\prod_{j=1}^{\ell} (x - a_j)^{k-1}\,dx}{y^{2k-1}} \qquad (1 \leqslant k \leqslant g)$$

and

$$y = \sqrt{A}\sqrt{(x - a_1) \cdot \dots \cdot (x - a_\ell)}.$$

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