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AN INTERNAL SOLUTION TO THE PROBLEM OF LINEARIZATION OF A CONVEXITY SPACE

BY
D. A. SZAFRON AND J. H. WESTON

1. **Introduction.** Following Kay and Womble [2] an abstract *convexity structure* on a set X is a collection ζ of subsets of X which includes the empty set, X and is closed under arbitrary intersections. One of the natural problems that arises in convexity structures is to give necessary and sufficient conditions for the existance of a linear structure on X such that the collection of all convex sets in the resulting linear space is precisely ζ. An associated problem is to consider a set with a convexity structure and a topology and find necessary and sufficient conditions for the existance of a linear structure on X such that X becomes a linear topological space with again ζ the collection of convex sets. These problems were solved in [3] and [1] respectively in terms of the existance of families of functions from X to the real line. In this paper we give internal solutions to both problems.

We will follow the notation and terminology of [3] and [1]. If X is a set and ζ is a convexity structure on X then (X, ζ) will be called a *convexity space* and the members of ζ *convex sets*. If $S \subseteq X$ then the *convex hull* of S, written $\zeta(S)$, is the set $\zeta(S) = \bigcap \{C \in \zeta \mid S \subseteq C\}$. If $S = \{s_1, \ldots, s_n\}$ is finite write $\zeta(s_1, \ldots, s_n)$ for $\zeta(\{s_1, \ldots, s_n\})$.

If $x \in X$, $S \subseteq X$ the *ζ-join* of x and S is the set $x_\zeta S = \bigcup \{\zeta(x, s) \mid s \in S\}$. ζ is said to be *join-hull commutative* if $\zeta(x_\zeta S) = \zeta(\{x\} \cup S) = x_\zeta \zeta(S)$. ζ is said to be *domain finite* if for each $S \subseteq X$, $\zeta(S) = \bigcup \{\zeta(t) \mid T \subseteq S,\ T \text{ finite}\}$. In [2] (theorem 2) it is shown that for a domain finite, join-hull commutative convexity structure ζ, S is convex if and only if $\zeta(x, y) \subseteq S$ for each $x, y \in S$.

For any two distinct points $x, y \in X$ the *line* determined by x and y is the set $\langle x, y\rangle = \{z \in X \mid z \in \zeta(x, y), \text{ or } x \in \zeta(z, y), \text{ or } y \in \zeta(x, z)\}$. Notice that if $s, t \in \langle x, y\rangle$ and $s \neq t$ then $\langle s, t\rangle = \langle x, y\rangle$.

2. **Linearization.**

Definition 2.1. A convexity space (X, ζ) is said to be *gridable over a field F* if there is a set A satisfying the following.

(i) For each $\alpha \in A$, $a \in F$ there is a $P_a^\alpha \in \zeta$ such that $\bigcup \{P_a^\alpha \mid a \in F\} = X$, $P_a^\alpha \cap P_b^\alpha = \phi$ if and only if $a \neq b$, and if $x, y \in P_a^\alpha$ then $\langle x, y\rangle \subseteq P_a^\alpha$. Write $P^\alpha = P_0^\alpha$.

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(ii) If $x, y \in X$, $x \neq y$, $\langle x, y\rangle \cap P_a^\alpha$ is a singleton and $P_a^\alpha \cap P_b^\beta = \phi$ then $\langle x, y\rangle \cap P_b^\beta \neq \phi$.

(iii) There is a distinguished point $x_0 \in X$ such that $\bigcap \{P^\alpha \mid \alpha \in A\} = \{x_0\}$.

(iv) If $x, y \in X$, $x \neq y$ then there is an $\alpha \in A$ such that $\langle x, y\rangle \cap P^\alpha$ is a singleton.

(v) For each $\alpha, \beta \in A$ and $\langle x, y\rangle \subseteq X$ for which $\langle x, y\rangle \cap P^\alpha$ and $\langle x, y\rangle \cap P^\beta$ are singletons, there are $\ell, m \in F$ such that for each $a \in F$ $\langle x, y\rangle \cap P_a^\alpha = \langle x, y\rangle \cap P_{\ell a+m}^\beta$.

The set A will be called the *grid* and the sets P_a^α *hyper-planes.*

LEMMA 2.2. *If (X, ζ) is a gridable convexity space, P is a hyper-plane in X and $\langle x, y\rangle$ is a line in X then exactly one of the following is true.*

(i) $\langle x, y\rangle \cap P = \phi$

(ii) $\langle x, y\rangle \cap P$ *is a singleton*

(iii) $\langle x, y\rangle \subseteq P$.

Proof. If $s, t \in \langle x, y\rangle \cap P$ and $s \neq t$ then $\langle s, t\rangle \subseteq P$ and $\langle s, t\rangle = \langle x, y\rangle$. The following result is an immediate consequence of definition 2.1(ii) and the above lemma.

LEMMA 2.3. *If $\alpha \in A$, $x, y \in X$, $x \neq y$, and $\langle x, y\rangle \cap P_a^\alpha$ is a singleton then $\langle x, y\rangle \cap P_b^\alpha$ is a singleton.*

DEFINITION 2.4. Given a convexity structure (X, ζ) which is gridable over a field F with grid A we define *scalar multiplication* as follows.

Let $a \in F$, $x \in X$.

(i) If $x = x_0$ or $a = 0$ define $ax = x_0$.

(ii) If $x \neq x_0$ and $a \neq 0$ then by 2.1(iii) there is an $\alpha \in A$ with $x \notin P^\alpha$. Hence, by 2.1(i), there is a $b \in F$, $b \neq 0$, and $x \in P_b^\alpha$. Since $ab \neq 0$. $P^\alpha \cap P_{ab}^\alpha = \phi$. But, by 2.2, $\langle x_0, x\rangle \cap P_b^\alpha$ is a singleton and then by lemma 2.3 there is a $z \in X$ with $\langle x_0, x\rangle \cap P_{ab}^\alpha = \{z\}$. Define $ax = z$.

LEMMA 2.5. *Scalar multiplication is well defined.*

Proof. Assume $x \neq x_0$, $a, b, d \in Fa, b, d \neq 0$, $\alpha, \beta \in A$, and $x \in P_b^\alpha$, $x \in P_d^\beta$. As in definition 2.4 $\langle x_0, x\rangle \cap P_{ab}^\alpha = \{z\}$ and $\langle x_0, x\rangle \cap P_{ad}^\beta = \{w\}$. We must show $w = z$. By 2.1(v) there are $\ell, m \in F$ so that for each $c \in F$, $\langle x_0, x\rangle \cap P_c^\alpha = \langle x_0, x\rangle \cap P_{\ell c+m}^\beta$. But $\{x_0\} = \langle x_0, x\rangle \cap P_0^\alpha = \langle x_0, x\rangle \cap P_m^\beta$ thus $m = 0$. Thus $\{x\} = \langle x_0, x\rangle \cap P_b^\alpha = \langle x_0, x\rangle \cap P_{\ell b}^\beta$. But $x \in P_d^\beta$ thus $\ell b = d$. Finally $\{z\} = \langle x, x_0\rangle \cap P_{ab}^\alpha = \langle x_0, x\rangle \cap P_{\ell ab}^\beta = \langle x_0, x\rangle \cap P_{ad}^\beta = \{w\}$. Thus $w = z$.

DEFINITION 2.6. Given a convexity structure (X, ζ) which is gridable over a field F with grid A, with the characteristic of F not 2, we define *addition* as follows.

Let $x, y \in X$.

(i) If $x = y$ define $x + y = 2x$.

(ii) If $x \neq y$ then by 2.1(iv) there is an $\alpha \in A$ with $\langle x, y\rangle \cap P^\alpha$ a singleton. Also there are $a, b \in F$ with $x \in P^\alpha_a$, $y \in P^\alpha_b$.
If $(a+b)/2 \neq 0$ then $P^\alpha_{(a+b)/2} \cap P^\alpha = \phi$ so by lemma 2.3 $\langle x, y\rangle \cap P^\alpha_{(a+b)/2}$ is a singleton. If $(a+b)/2 = 0$ then $\langle x, y\rangle \cap P^\alpha_{(a+b)/2} = \langle x, y\rangle \cap P^\alpha$ is a singleton. Hence, in either case, there is a $z \in X$ with $\langle x, y\rangle \cap P^\alpha_{(a+b)/2} = \{z\}$. Define $x + y = 2z$.

LEMMA 2.7. *Addition is well defined.*

Proof. Let $\alpha, \beta \in A$ be such that $\langle x, y\rangle \cap P^\alpha$ and $\langle x, y\rangle \cap P^\beta$ are singletons. Then there are $a, b, c, d \in F$ with $x \in P^\alpha_a \cap P^\beta_c$ and $y \in P^\alpha_b \cap P^\beta_d$. As in definition 2.6 there are $u, v \in X$ with $\{u\} = \langle x, y\rangle \cap P^\alpha_{(a+b)/2}$ and $\{v\} = \langle x, y\rangle \cap P^\beta_{(c+d)/2}$. We must show $u = v$.

By 2.1(v) there are $\ell, m \in F$ such that for each $e \in F$ $\langle x, y\rangle \cap P^\alpha_e = \langle x, y\rangle \cap P^\beta_{\ell e+m}$. Now $\{x\} = \langle x, y\rangle \cap P^\alpha_a = \langle x, y\rangle \cap P^\beta_{\ell a+m}$ and $\{y\} = \langle x, y\rangle \cap P^\alpha_b = \langle x, y\rangle \cap P^\beta_{\ell b+m}$. Thus $x \in P^\beta_c \cap P^\beta_{\ell a+m}$ and hence $c = \ell a + m$. Similarly $d = \ell b + m$. Thus $\ell[(a+b)/2] + m = \frac{1}{2}(\ell a + m + \ell b + m) = (c+d)/2$ and hence $\{u\} = \langle x, y\rangle \cap P^\alpha_{(a+b)/2} = \langle x, y\rangle \cap P^\beta_{(c+d)/2} = \{v\}$. Thus $u = v$.

For the remainder of this section we will assume that (X, ζ) is a gridable convexity structure over a field F with grid A, the characteristic of F is not two, and scalar multiplication and addition are defined as above.

The following two results will prove useful in showing that X is a vector space over F.

LEMMA 2.8. *Let $\alpha \in A$, $a, b \in F$, and $x \in P^\alpha_a$. Then $bx \in P_{ab}$.*

Proof. If $x = x_0$, then $a = 0 = ab$ and $bx = x_0 \in P^\alpha = P^\alpha_{ab}$.

If $x \neq x_0$ and $a \neq 0$ then $x \notin P^\alpha$ and hence, by the definition of scalar multiplication, $\{bx\} = \langle x_0, x\rangle \cap P^\alpha_{ab}$.

If $x \neq x_0$ and $a = 0$ then $bx \in \langle x_0, x\rangle$ by definition. But $x, x_0 \in P^\alpha$ thus $\langle x_0, x\rangle \subseteq P^\alpha$ and hence $bx \in P^\alpha = P^\alpha_{ab}$.

LEMMA 2.9. *Let $\alpha \in A$, $a, b \in F$, $x \in P^\alpha_a$ and $y \in P^\alpha_b$ then $x + y \in P^\alpha_{a+b}$.*

Proof. If $x = y$ the result follows from the previous lemma. If $x \neq y$ and $\langle x, y\rangle \cap P^\alpha$ is a singleton then $x + y = 2u$ where $\{u\} = \langle x, y\rangle \cap P^\alpha_{(a+b)/2}$. By the previous lemma $x + y = 2u \in P^\alpha_{a+b}$.

If $x \neq y$ and $\langle x, y\rangle \cap P^\alpha$ is not a singleton then there is a $d \in F$ with $\langle x, y\rangle \subseteq P^\alpha_d$ then $a = b = d$. In this case $x + y = 2v$ where $v \in \langle x, y\rangle \subseteq P^\alpha_d$. Hence, by 2.8, $x + y \in P^\alpha_{2d} = P^\alpha_{a+b}$.

THEOREM 2.10. *X is a vector space over F.*

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Proof. I. $1x=x$: $1x_0=x_0$. If $x\neq x_0$ then $x\in P^\alpha_a$ for some $a\neq 0$. Thus $\{1x\}=\langle x_0,x\rangle\cap P^\alpha_{1a}=\langle x_0,x\rangle\cap P^\alpha_a=\{x\}$.

II. $a(bx)=(ab)x$: Clear if $x=x_0$, $a=0$, or $b=0$. If $x\neq x_0$, $a\neq 0$, and $b\neq 0$ then $x\in P^\alpha_c$ for some $\alpha\in A$, $c\in F$, $c\neq 0$. Hence, by definition 2.4 $\{(ab)x\}=\langle x_0,x\rangle\cap P^\alpha_{(ab)c}$ and $\{bx\}=\langle x_0,x\rangle\cap P^\alpha_{bc}$. Therefore $bx\in\langle x_0,x\rangle$ and, since $b\neq 0$, $c\neq 0$, $bx\neq x_0$. Thus $\langle x_0,bx\rangle=\langle x_0,x\rangle$ and hence $\{a(bx)\}=\langle x_0,bx\rangle\cap P^\alpha_{a(bc)}=\langle x_0,x\rangle\cap P^\alpha_{(ab)c}=\{(ab)x\}$.

III. $x+y=y+x$: clear.

IV. $x+x_0=x$: $x_0+x_0=2x_0=x_0$. If $x\neq x_0$ then $x+x_0=2u$ where $\{u\}=\langle x_0,x\rangle\cap P^\alpha_{(0+a)/2}$, $x\in P^\alpha_a$, and $a\neq 0$. But $u\in P^\alpha_{a/2}$ implies, by lemma 2.8, $\{x_0+x\}=\{2u\}=\langle x_0,x\rangle\cap P^\alpha_{2a/2}=\langle x_0,x\rangle\cap P^\alpha_a=\{x\}$.

V. *For each* $x\in X$ *there is an* $x'\in X$ *with* $x+x'=x_0$: Let $x'=(-1)x$. $x_0+(-1)x_0=x_0+x_0=x_0$. If $x\neq x_0$ then $x\in P^\alpha_a$ for some $\alpha\in A$, $a\in F$, $a\neq 0$, and $(-1)x=\langle x_0,x\rangle\cap P^\alpha_{-a}$. Hence $x+(-1)x=2u$ where $\{u\}=\langle x,(-1)x\rangle\cap P^\alpha_{(a-a)/2}=\{x_0\}$ since $x_0\in\langle x,(-1)x\rangle$. Thus $x+x'=2x_0=x_0$.

VI. $(a+b)x=ax+bx$: For each $\alpha\in A$ if $x\in P^\alpha_c$ then, by lemma 2.8 and 2.9, $(a+b)x\in P^\alpha_{(a+b)c}$, $ax\in P^\alpha_{ac}$, $bx\in P^\alpha_{bc}$, and thus $ax+bx\in P^\alpha_{ac+bc}$. If $(a+b)x\neq ax+bx$ by 2.1(iv) there is an $\alpha\in A$ with $\langle(a+b)x,ax+bx\rangle\cap P^\alpha$ a singleton. Hence $(a+b)x\in P^\alpha_e$, $ax+bx\in P^\alpha_f$ with $e\neq f$ which is impossible.

VII. $a(x+y)=ax+ay$: Similar to VI.

VIII. $(x+y)+z=x+(y+z)$: Similar to VI.

In order to show that the convexity structure ζ on X is the convexity structure induced on X by the linear structure just defined, we first show the following lemma.

LEMMA 2.11. *If* $x,y\in X$, $x\neq y$ *and* $k\in F$ *then* $kx+(1-k)y\in\langle x,y\rangle$.

Proof. If $k=0$ the result is clear. Assume $k\neq 0$ then $x\in P^\alpha_a$, $y\in P^\alpha_b$ for some $\alpha\in A$, $a,b\in F$, $a\neq b$. Hence, by lemmas 2.8 and 2.9, $z=kx+(1-k)y\in P^\alpha_{ka+(1-k)b}$. Let $\{w\}=\langle x,y\rangle\cap P^\alpha_{ka+(1-k)b}$. We need only show $w=z$.

Assume $w\neq z$ then, by 2.1(iv), there is a $\beta\in A$ with $\langle z,w\rangle\cap P^\beta$ a singleton.

Assume $\langle x,y\rangle\cap P^\beta$ is not a singleton then, for some $e\in F$, $\langle x,y\rangle\subseteq P^\beta_e$. Hence $w\in\langle x,y\rangle\subseteq P^\beta_e$ and $z=kx+(1-k)y\in P^\beta_{ke+(1-k)e}=P^\beta_e$ by lemmas 2.8 and 2.9. Thus $\langle z,w\rangle\subseteq P^\beta_e$ which is impossible. Hence $\langle x,y\rangle\cap P^\beta$ is a singleton.

Since $\langle x,y\rangle\cap P^\alpha$ and $\langle x,y\rangle\cap P^\beta$ are singletons, by definition 2.1(v) there exists $\ell,m\in F$ such that for each $e\in F$ $\langle x,y\rangle\cap P^\alpha_e=\langle x,y\rangle\cap P^\beta_{\ell e+m}$. Thus $x\in P^\beta_{\ell a+m}$, $y\in P^\beta_{\ell b+m}$ and hence, by lemmas 2.8 and 2.9, $z\in P^\beta_f$ where $f=k(\ell a+m)+(1-k)(\ell b+m)=\ell(ka+(1-k)b)+m$. Since $w\in P^\alpha_{ka+(1-k)b}$, $w\in P^\beta_{\ell(ka+(1-k)b)+m}$. Hence $z,w\in\langle z,w\rangle\cap P^\beta_{\ell(ka+(1-k)b)+m}$ which is a singleton. Thus $w=z$.

THEOREM 2.12. *Let* (X,ζ) *be a convexity space and* F *an ordered field. Necessary and sufficient conditions that there is a linear structure on* X *over* F *in*

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which ζ is the usual convexity structure are:

(i) (X, ζ) *is join-hull commutative and domain finite.*

(ii) (X, ζ) *is gridable over F with grid A.*

(iii) *If $\zeta(x, y) = \zeta(x, z)$ then $y = z$.*

(iv) *Let $\alpha \in A$. If $\langle x, y\rangle \cap P^\alpha$ is a singleton say $x \in P^\alpha_a$, $y \in P^\alpha_b$ and $\{w\} = \langle x, y\rangle \cap P^\alpha_c$ then $a \le c \le b$ implies $w \in \zeta(x, y)$.*

Proof. Necessity is clear taking $\{P^\alpha \mid \alpha \in A\}$ to be the maximal linear subspaces and $P^\alpha_a = a + P^\alpha$.

To show sufficiency it remains to show that $\zeta = \zeta'$ where ζ' is the family of convex sets generated by the linear structure on X.

Let $C \in \zeta$, $x, y \in C$, $x \ne y$. Suppose $h, k \in F$, $h, k \ge 0$, and $h + k = 1$. By 2.1(iv) there is an $\alpha \in A$ such that $\langle x, y\rangle \cap P^\alpha$ is a singleton and there are $a, b \in F$ with $x \in P^\alpha_a$, $y \in P^\alpha_b$. Thus, by lemmas 2.8 and 2.9, $w = kx + hy \in P^\alpha_{ha+kb}$.

We may assume $a < b$ then $a \le ha + kb \le b$ and, by lemma 2.11, $w \in \langle x, y\rangle$ and thus by (iv), $w \in \zeta(x, y)$. Since $\zeta(x, y) = \bigcap\{E \in \zeta \mid x, y \in E\}$, $\zeta(x, y) \subseteq C$. Hence for each $x, y \in C$, $h, k \in F$ with $h, k \ge 0$ and $h + k = 1$, $hx + ky \in C$. Thus $C \in \zeta'$.

Let $D \in \zeta'$, $x, y \in D$, $x \ne y$. By 2.1(iv) there is an $\alpha \in A$ such that $\langle x, y\rangle \cap P^\alpha$ is a singleton, say $x \in P^\alpha_a$, $y \in P^\alpha_b$, $a, b \in F$. We may assume $a < b$.

Let $z \in \zeta(x, y)$ and $c \in F$ with $z \in P^\alpha_c$. If $c < a < b$ then by (iv) $x \in \zeta(z, y)$ and thus $\zeta(x, y) = \zeta(z, y)$. Hence $x = z$ and thus $a = c$ which is impossible. Similarly if $a < b < c$, and hence we have $a \le c \le b$. Thus there are $h, k \in F$ with $c = ha + kb$, $h, k \ge 0$, and $h + k = 1$.

Let $w = hx + ky$ then, by lemmas 2.8, 2.9, and 2.11 $\{w\} = \langle x, y\rangle \cap P^\alpha_{ha+kb} = \langle x, y\rangle \cap P^\alpha_c = \{z\}$. Hence, since $D \in \zeta'$, $z \in D$. Therefore $\zeta(x, y) \subseteq D$. Since (X, ζ) is domain finite and join-hull commutative this is sufficient to show $D \in \zeta$.

3. **Linear topological spaces.** If (X, ζ) is a convexity space and τ is a T_1 topology on X then the triple (X, τ, ζ) is called a *topological convexity space.*

The following definitions are taken from [1]. The *convex topology* τ_c of the triple (X, τ, ζ) is the topology with sub-base S, the collection of complements of all τ-closed members of ζ. A net $(x_d \mid d \in D)$ in X is said to *converge convexly* to $x \in X$ if for each subnet $(x_e \mid e \in E)$ of $(x_d \mid d \in D)$, $x \in \zeta(S)^-$ where S is the range of $(x_e \mid e \in E)$ and $-$ is τ-closure. The triple (X, τ, ζ) is *convexly regular* if for each $A \in \zeta$, $x \in X$, $x \notin A^-$ there are disjoint sets S, T containing x and A respectively such that $X\backslash S$ and $X\backslash T$ are closed members of ζ. Moreman [4] has shown that in a convexly regular space closures of convex sets are convex. Also it is an easy exercise to show that in such spaces a net is convexly convergent to $x \in X$, if and only if it τ_c-converges to x.

In order to consider a linear topological space over an ordered field F we need a topology on F which makes F a linear topological space when considered as a vector space over itself.

DEFINITION 3.1. If F is an ordered field then the *interval topology* on F is the topology with base $\{(a, b) \mid a, b \in F, a < b\}$ where $(a, b) = \{c \in F \mid a < c < b\}$.

LEMMA 3.2. *An ordered field F with the interval topology is a linear topological space when considered as a vector space over itself.*

Proof. To show that addition is continuous suppose $(x_d \mid d \in D)$ and $(y_d \mid d \in D)$ are nets in F converging to x and y respectively. Let $x+y \in (a, b)$, $a, b \in F$, then $x \in (x+(a-x-y)/2, \quad x+(b-x-y)/2) = U$ and $y \in (y+(a-x-y)/2, y+(b-x-y)/2) = V$. If $d \in D$ and $x_d \in U$, $y_d \in V$ then $x_d + y_d \in (a, b)$ thus $(x_d + y_d \mid d \in D)$ converges to $x + y$.

To show that scalar multiplication is continuous suppose $(x_d \mid d \in D)$ and $(y_d \mid d \in D)$ are nets in F converging to x and y respectively. Assume $x > 0$, $y > 0$ then we can also assume $x_d > 0$, $y_d > 0$ for each $d \in D$. Suppose $xy \in (a, b)$ where $0 < a < b$, $a, b \in F$. Let $x' = (xy+a)/2y$, $y' = (x'y+a)/2x'$, $x'' = (b+xy)/2y$, and $y'' = (b+x''y)/2x''$. Hence $a < xy$ implies $x'y = (xy+a)/2 > a$ and thus $a < x'y$ implies $x'y' = (x'y+a)/2 > a$. Also $x' = x/2 + a/2y < x/2 + xy/2y = x$. Similarly $x < x''$ so $x \in (x', x'')$. Similarly $y \in (y', y'')$. If $d \in D$, $x_d \in (x', x'')$ and $y_d \in (y', y'')$ then $x_d y_d \in (a, b)$ and thus $(x_d y_d \mid d \in D)$ converges to xy.

It is clear that if $(x_d \mid d \in D)$ converges to $x \in F$ then $(-x_d \mid d \in D)$ converges to $-x$ and if $(y_d \mid d \in D)$ converges to $0 \in F$ then $(x_d y_d \mid d \in D)$ converges to 0. Hence scalar multiplication is continuous.

For the remainder of the paper F will designate an ordered field with the interval topology and $(X, \mathcal{T}, \zeta)$ a topological convexity space such that (X, ζ) is gridable over F and satisfies the conditions of theorem 2.12.

DEFINITION 3.3. For each $\alpha \in A$, $a \in F$ define the *left hand hyperplane* G_a^α by $G_a^\alpha = \bigcup \{P_b^\alpha \mid b < a\}$ and the *right half hyperplane* H_a^α by $H_a^\alpha = \bigcup \{P_b^\alpha \mid a < b\}$.

DEFINITION 3.4. For each $\alpha \in A$ define the function $f_\alpha : X \to F$ by $f_\alpha(x) = c$ where $x \in P_c^\alpha$. This function is well defined by 2.1(i).

LEMMA 3.5. *For each $\alpha \in A$, f_α is linear.*

Proof. Let $x, y \in X$ and $\ell, m \in F$ then $x \in P_a^\alpha$, $y \in P_b^\alpha$ for some $a, b \in F$. By lemma 2.8 and 2.9 $\ell x + my \in P_{\ell a + mb}^\alpha$ so $f_\alpha(\ell x + my) = \ell a + mb$. But $\ell f_\alpha(x) + m f_\alpha(y) = \ell a + mb$ and hence f_α is linear.

LEMMA 3.6. *For each $\alpha \in A$ and $a \in F$, H_a^α, G_a^α, $X \setminus H_a^\alpha$,* and *$X \setminus G_a^\alpha \in \zeta$.*

Proof. Let $x, y \in H_a^\alpha$ then $x \in P_b^\alpha$, $y \in P_c^\alpha$ where $a < b, c$. Let $h, k \in F$, $h, k \geq 0$

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and $h+k=1$ then $hx+ky\in P^{\alpha}_{hb+kc}$ by lemmas 2.8 and 2.9. But $hb+kc>ha+ka=a$ and thus $hx+ky\in H^{\alpha}_a$. Thus, using theorem 2.12, $H^{\alpha}_a\in\zeta$. Similarly G^{α}_a, $X\backslash H^{\alpha}_a$, and $X\backslash G^{\alpha}_a\in\zeta$.

LEMMA 3.7. *If, for each $\alpha\in A$, $a\in F$, $H^{\alpha}_a=X\backslash G^{\alpha-}_a$ and $G^{\alpha}_a=X\backslash H^{\alpha-}_a$ then f_α is τ_c-continuous.*

Proof. Let $x\in X$, $\alpha\in A$, $x\in P^{\alpha}_c$, and $c\in(a,b)$ $a,b,c\in F$, $a<b$. Let $M=H^{\alpha}_b\cap G^{\alpha}_a$. Since $G^{\alpha-}_a=X\backslash H^{\alpha}_b$ and $H^{\alpha-}_a=X\backslash G^{\alpha}_a$, Lemma 3.6 implies that M is τ_c-open. Also $x\in M$ since $a<c<b$.

If $m\in M$ then $m\in P^{\alpha}_d$ where $a<d<b$ and hence $f_\alpha(m)=d\in(a,b)$. Thus f_α is τ_c-continuous.

DEFINITION 3.8. (X,τ,ζ) is said to have the *Hahn–Banach property* if for each $D\in\zeta$ and $p\in X$ with $p\notin D^-$ there exists an $\alpha\in A$ and $a\in F$ so that either

(i) $P^{\alpha}_c\cap D\neq\phi$ and $p\in P^{\alpha}_b$ implies $b<a<c$. or

(ii) $P^{\alpha}_c\cap D\neq\phi$ and $p\in P^{\alpha}_b$ implies $c<a<b$.

LEMMA 3.9. *If (X,τ,ζ) has the Hahn–Banach property, $(x_e\mid e\in E)$ is a net in X, $p\in X$ and for each $\alpha\in A$ $(f_\alpha(e)\mid e\in E)$ converges to $f(p)$ in the interval topology on F, then $(x_e\mid e\in E)$ converges convexly to p.*

Proof. Suppose not. Then there is a subnet $(x_d\mid d\in D)$ of $(x_e\mid e\in E)$ such that $p\notin\zeta(S)^-$ where S is the range of $(x_d\mid d\in D)$. By the Hahn–Banach property there is an $\alpha\in A$ and $a\in F$ with say $f_\alpha(p)<a<f_\alpha(x)$ for all $x\in\zeta(S)^-$. But then $(f_\alpha(x_d)\mid d\in D)$ does not converge to $f_\alpha(p)$ which is impossible.

THEOREM 3.10. *Let (X,τ,ζ) be a topological convexity space, τ a T_1 topology, and F an ordered field with the interval topology. The following are equivalent:*

(1) *There is a linear structure on X over F and a topology on X such that X is a linear topological space over F, ζ is the usual convexity structure and the weak topology on X is τ_c.*

(2) *(X,τ,ζ) satisfies conditions* (i), (ii), (iii), *and* (iv) *of theorem* 2.12 *and in addition* (v) *$H_a=X\backslash G^{\alpha-}_a$ and $G_a=X\backslash H^{\alpha-}_a$ for each $\alpha\in A$, $a\in F$* (vi) *(X,τ,ζ) has the Hahn–Banach property.*

Proof. Necessity is clear. Using theorem 2.12 sufficiency will follow by showing that addition and scalar multiplication are continuous in the convex topology τ_c. Note that conditions (v) and (vi) imply that (X,τ,ζ) is convexly regular and hence a net in X τ_c-converges to $x\in X$ if and only if it converges convexly to x.

Suppose $(a_d\mid d\in D)$ converges to $a\in F$ and $(x_d\mid d\in D)$ τ_c-converges to $x\in X$. Let $\alpha\in A$ then $f_\alpha(a_dx_d)=a_df_\alpha(x_d)$ by lemma 3.5 and $(f_\alpha(x_d)\mid d\in D)$ converges to $f_\alpha(x)$ by lemma 3.7. Hence by lemma 3.2 $(a_df_\alpha(x_d)\mid d\in D)$

converges to $af_\alpha(x)=f_\alpha(ax)$. Thus by lemma 3.8 $(a_dx_d \mid d \in D)$ τ_c-converges to ax and scalar multiplication is τ_c-continuous.

Suppose $(x_d \mid d \in D)$ and $(y_d \mid d \in D)$ τ_c-converge to x and y respectively. By lemmas 3.5 and 3.7 $(f_\alpha(x_d+y_d) \mid d \in D)$ converges to $f_\alpha(x)+f_\alpha(y)=f_\alpha(x+y)$. Hence, by lemma 3.9 $(x_d+y_d \mid d \in D)$ τ_c-converges to $x+y$ and thus addition is τ_c-continuous.

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Department of Mathematics
University of Regina
Regina, Saskatchewan
Canada S4S OA2