

Uncovering the forcing statistics in stochastic linear models for compressible wall-bounded turbulence

Xianliang Chen^{1,2}, Anjia Ying³, Jianping Gan^{1,2}, and Lin Fu^{1,3}

¹Department of Mathematics and Center for Ocean Research in Hong Kong and Macau (CORE), The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, PR China

²Department of Ocean Science, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, PR China

³Department of Mechanical and Aerospace Engineering, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, PR China

Corresponding author: Lin Fu, linfu@ust.hk

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Modelling the nonlinear forcing is critical for linear models based on resolvent or inputoutput analyses. For compressible wall-bounded turbulence, little is known on what the real forcing looks like due to limited data, so the prediction agrees more qualitatively than quantitatively with direct numerical simulations (DNSs). Here, we present detailed forcing statistics of stochastic linear models, derived from elaborate DNS datasets for channel flows with bulk Mach number reaching 3. These statistics directly explain the success and failure of current models and provide guidance for further improvements. The benchmark linearised Navier-Stokes (LNS) and eLNS models are considered; the latter is assisted by eddy-viscosity-related terms. First, we prove the self-consistency of the models by using DNS-computed forcing as the input. Second, we present the spectral distributions of the forcing and its components. Third, we quantify the acoustic components, absent in incompressible cases, within the linear models. We reveal that the LNS forcing can exhibit relatively high coherence and low rank, very different from the modelled diagonal fullrank forcing. The eddy-viscosity-related term is not partial modelling of the LNS forcing; contrarily, the former is much larger than the latter, serving to disrupt the low-rank feature, enhance diagonal dominance and increase robustness across scales. The scales narrow in either horizontal direction are most susceptible to acoustic modes, while the others are little affected (< 2% in energy). Furthermore, the extended strong Reynolds analogy is assessed in predicting the density and temperature components.

Key words: turbulent boundary layers, turbulence modelling, Navier-Stokes equations

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1. Introduction

It is known that compressible wall-bounded turbulence critically influences the surface drag and heat transfer, so deep physical understanding and accurate modelling of turbulent flows are significant for reliable vehicle design and flow control (Lele 1994; Cheng *et al.* 2024). Compared with the incompressible counterpart, current knowledge on compressible wall-bounded turbulence is still limited, likely due to a less comprehensive database and higher physical complexity resulting from extra thermodynamic processes, such as heat transfer, acoustics, dilatational work and high-enthalpy gas effects (Gatski & Bonnet 2013).

So far, numerous high-quality experiments and direct numerical simulations (DNSs) have provided us with the most comprehensive details of turbulence (e.g. Hultmark et al. 2012; Lee & Moser 2015). As an attempt to conduct operator-based modal decomposition and develop reduced-order modellings, various linear models have been developed for wall-bounded turbulence, which are promising and have been an academic hotspot in recent years; see the reviews of McKeon (2017) and Jovanović (2021). These linear models are built upon the linearised Navier-Stokes (LNS) equations relative to a timeinvariant reference state, usually the mean flow assumed known. Contrary to the laminar case where the nonlinear fluctuation terms can be neglected, the nonlinear terms are retained for turbulent cases, and are treated as feedback or forcing to this linearly stable system. The fluctuation can then be solved in the wavenumber space based on non-modal instability theory and resolvent and input-output analyses. Within such frameworks, the linear operator and nonlinear forcing represent the energy amplification and redistribution mechanisms, respectively. For incompressible flows, outcomes from these linear models include revealing the multi-scale coherent motions (del Álamo & Jiménez 2006; Hwang & Cossu 2010), deriving fluctuation scalings (McKeon & Sharma 2010; Moarref et al. 2013), constructing low-rank estimation models (Illingworth, Monty & Marusic 2018; Gupta et al. 2021; Ying et al. 2024b) and designing flow control strategies (Moarref & Jovanović 2012; Ran, Zare & Jovanović 2021). Many of these works adopt the eddy-viscosityenhanced models (termed eLNS), where the Reynolds stress fluctuation is linearised using the eddy viscosity μ_t in the same spirit as the Reynolds-averaged NS (RANS) models, to partly model the colour of the forcing. The eLNS models are shown to perform generally better than those without using μ_t , i.e. the LNS model, in terms of estimating coherent fluctuations and spectra for wall-bounded flows (Reynolds & Hussain 1972; Illingworth et al. 2018; Morra et al. 2021; Symon et al. 2023).

The linear models have also been developed for compressible wall-bounded turbulent flows, mainly in the aspects of studying multi-scale coherent fluctuations (Alizard *et al.* 2015; Bae, Dawson & McKeon 2020; Dawson & McKeon 2020; Chen *et al.* 2023*a,b*; Fan *et al.* 2024). In addition to the wide similarities to incompressible flows, some notable differences have been reported. Bae *et al.* (2020) highlighted that, for supersonic boundary layers, the fluctuation in the relatively supersonic region (phase speed faster than the free-stream speed of sound) is centred around the relative sonic line instead of the critical layer, and exhibits Mach-wave radiation and eddy shocklets (also Madhusudanan, Stroot & McKeon 2025), which are absent in incompressible flows. Chen *et al.* (2023*b*) noted that the acoustic components can be overly amplified in the eLNS models due to reduced non-normality of the linear operator and inaccurate modelling of the forcing, which can disrupt the linear coherent estimation. They designed a post-processing decomposition to remove the acoustic components, which can improve the model results regarding velocity and temperature estimations.

One of the central problems in the linear models is the modelling of the forcing. Although the linear operator can solely determine the characteristic modes of the system,

realistic fluctuations result from a combination of these modes, and the imposed forcing directly affects the energy distribution and sorting of these modes (Morra *et al.* 2021). Therefore, the accuracy of the forcing directly determines the predictability of the linear models. A feasible way to bypass the explicit evaluation of the forcing is to directly determine the weights of different modes from the DNS data (Moarref *et al.* 2013, 2014), whereas for more general cases, *a priori* modelling of the forcing is preferred if DNS is not available. Early works, for both incompressible and compressible cases, adopted the simplest form of the forcing that it is either a white noise or a harmonic signal uniform in spatial spectra and delta correlated in the wall-normal direction. The resulting agreement with the DNS data regarding the fluctuations turns out to be more qualitative than quantitative (e.g. Hwang & Cossu 2010; Alizard *et al.* 2015; Chen *et al.* 2023*a*).

Improvements on the forcing modelling have been extensively explored for incompressible flows. Successful attempts include considering the spatial and spectral dependence of the forcing (Gupta et al. 2021; Wu & He 2023), and modelling its colour through elaborate optimisation processes (Zare, Jovanović & Georgiou 2017; Hwang & Eckhardt 2020; Holford, Lee & Hwang 2023; Ying et al. 2024a). Limited instantaneous DNS data can also assist by providing additional constraints (Illingworth et al. 2018; Towne, Lozano-Durán & Yang 2020). To gain a complete understanding of the real forcing, the spatial-temporal statistics of the forcing can be derived from the DNS data, as done for incompressible cases by Amaral et al. (2021), Morra et al. (2021) and Nogueira et al. (2021). It is revealed that the forcing also features relatively high spatial-temporal coherence, which benefits the construction of more reliable forcing models. Morra et al. (2021) also clarified why the eLNS model leads to an improved response prediction than the LNS one, because the weights of principal resolvent modes from the eLNS model are closer to the DNS data. Besides, Symon, Illingworth & Marusic (2021) analysed the nonlinear forcing term from the perspective of energy transfer, and clarified how the gain or loss of the redistributed energy affects the behaviour of resolvent models for different scales.

Nevertheless, the forcing statistics from DNS and possible improved models for compressible wall-bounded turbulence have not been carefully investigated. Therefore, it is the objective of this work to reveal the real forcing statistics of the linear models from DNS for compressible channel flows, which can straightforwardly explain the success and failure of current models for different cases and provide direct guidance for model improvements.

The classic LNS and eLNS models will be focused on. Although their differences have been extensively discussed for incompressible flows, we still include both models because relevant discussions for compressible flows are inadequate due to insufficient DNS data. and some new observations are worth reporting. Meanwhile, different from the various forcing models for incompressible flows, rare attempts have been made to optimise the compressible models, so candidate models for the present work are in fact very limited. Nonetheless, the classic LNS and eLNS models, although simple, can serve as benchmarks to provide insights into the forcing physics. To accurately compute the forcing statistics, we will employ a series of DNS data with the maximum bulk Mach number reaching 3. In addition to reporting the similar features to the incompressible counterparts, we will particularly focus on the distinctions in compressible flows, regarding the forcing for density and temperature, turbulent heat fluxes, the role of acoustic components and Mach number effects. The remainder of the article is organised as follows. Section 2 introduces the mathematical framework of the employed linear models and describes the DNS datasets. Section 3 verifies the DNS data processing and demonstrates the mathematical self-consistency of the linear models. The forcing distributions of different models are summarised in § 4, and the role of acoustics modes is discussed in § 5. The work is finally concluded in § 6.

2. Problem formulations

2.1. Mean flow and fluctuation equations

We consider canonical compressible turbulent channel flow under the calorically perfect gas assumption. To obtain reliable statistics from the DNS, we need to carefully deal with each term in the mean and fluctuation equations, especially those regarding high-order fluctuation terms and the difference between Reynolds and Favre averages. The equations in compressible flows are far more complicated than the incompressible version, so it is necessary to first present the complete form of the equations to be dealt with.

The Favre-averaged quantities are chosen as basic variables for the linear models, for ease of realising a thorough linearisation (Chen *et al.* 2023*b*). The continuity, momentum and enthalpy equations of the mean flow are written as (e.g. Gatski & Bonnet 2013)

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0, \qquad (2.1a)$$

$$\bar{\rho}\left(\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j}\right) = -\frac{\partial}{\partial x_j} \left(\bar{\rho} \widetilde{u''_i u''_j}\right) - \frac{\partial \bar{p}}{\partial x_i} + \frac{\partial \bar{\tau}_{ij}}{\partial x_j}, \quad \text{for } i = 1, 2, 3,$$
(2.1b)

$$\frac{\partial}{\partial t} \left(\bar{\rho}c_{p}\widetilde{T} \right) + \frac{\partial}{\partial x_{j}} \left(\bar{\rho}c_{p}\widetilde{u}_{j}\widetilde{T} \right) - \left(\frac{\partial\bar{p}}{\partial t} + \widetilde{u}_{j}\frac{\partial\bar{p}}{\partial x_{j}} + \overline{u_{j}'}\frac{\partial\bar{p}'}{\partial x_{j}} + \overline{u_{j}''}\frac{\partial\bar{p}}{\partial x_{j}} \right) \\
= -\frac{\partial}{\partial x_{j}} \left(\bar{\rho}c_{p}\widetilde{u_{j}''T''} \right) - \frac{\partial\bar{\vartheta}_{j}}{\partial x_{j}} + \overline{\tau}_{ij}\frac{\partial\tilde{u}_{i}}{\partial x_{j}} + \overline{\tau}_{ij}\frac{\partial\overline{u_{i}''}}{\partial x_{j}}.$$
(2.1c)

Here, $\bar{\varphi}$ is the temporal (Reynolds) average and $\tilde{\varphi} = \overline{\rho \varphi}/\bar{\rho}$ is the Favre average; ρ , $u_i = [u, v, w]^T$ and T are the fluid's density, velocities and temperature, respectively; the mean pressure is $\bar{p} = \bar{\rho}R\tilde{T}$, with R the gas constant and c_p is the isobaric specific heat. Note that the temporal derivatives are retained in (2.1) for reference for the fluctuation equations. No-slip and isothermal walls are set on both sides as the boundary condition. The mean viscous stress $\bar{\tau}_{ij}$ and heat flux $\bar{\vartheta}_j$ are calculated as

$$\bar{\tau}_{ij} = \underbrace{\bar{\mu}\left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}\right) - \frac{2}{3}\bar{\mu}\frac{\partial \tilde{u}_k}{\partial x_k}\delta_{ij}}_{\equiv \bar{\tau}_{ij,lin}} + \underbrace{\overline{\mu}\frac{\partial u_i''}{\partial x_j} + \mu\frac{\partial u_j''}{\partial x_j} - \frac{2}{3}\overline{\mu}\frac{\partial u_k''}{\partial x_k}\delta_{ij}}_{\equiv \bar{\tau}_{ij,nln}}, \quad \bar{\vartheta}_j = -\bar{\kappa}\frac{\partial \widetilde{T}}{\partial x_j} - \overline{\kappa}\frac{\partial \overline{T}''}{\partial x_j}, \quad (2.2)$$

where $\mu(T)$ and $\kappa = \mu c_p/Pr$ are the viscosity and thermal conductivity; μ is calculated by Sutherland's law and Pr = 0.72 is the Prandtl number; δ_{ij} represents the Kronecker delta. Note that $\bar{\tau}_{ij}$ is divided into two parts, as underbraced, for later linearisation of the fluctuation equations. The first part $\bar{\tau}_{ij,lin}$ is a function of mean-flow variables only, while the second part $\bar{\tau}_{ij,nln}$ is contributed by the second-order moment of the fluctuations.

The fluctuation equations are far more complex than the incompressible counterparts. Following the derivation in Chen *et al.* (2023b), the fluctuation equations relative to (2.1) are written as

$$\underbrace{\frac{\partial \rho'}{\partial t} + \frac{\partial \bar{\rho} u''_{j}}{\partial x_{j}} + \frac{\partial \rho' \tilde{u}_{j}}{\partial x_{j}}}_{\text{linear terms}} = \underbrace{-\frac{\partial \rho' u''_{j}}{\partial x_{j}}}_{\text{nonlinear term}}, \quad (2.3a)$$

$$\bar{\rho} \left(\frac{\partial u''_{i}}{\partial t} + \tilde{u}_{j} \frac{\partial u''_{i}}{\partial x_{j}} \right) + \left(\rho' \tilde{u}_{j} + \bar{\rho} u''_{j} \right) \frac{\partial \tilde{u}_{i}}{\partial x_{j}} + \frac{\partial p'_{lin}}{\partial x_{i}} - \frac{\partial \tau'_{ij,lin}}{\partial x_{i}} \\
= \underbrace{-\frac{\partial}{\partial x_{j}} \left(\rho u''_{i} u''_{j} - \bar{\rho} \widetilde{u''_{i} u''_{j}} \right)}_{RSF} \underbrace{-\frac{\partial p'_{nln}}{\partial x_{i}}}_{PGNF} + \underbrace{\frac{\partial \tau'_{ij,nln}}{\partial x_{j}}}_{VSNF} + MMNF, \quad (2.3b)$$

$$\bar{\rho} \left(c_{v} \frac{\partial T''}{\partial t} + c_{p} \tilde{u}_{j} \frac{\partial T''}{\partial x_{j}} \right) - R \widetilde{T} \frac{\partial \rho'}{\partial t} + \left(\rho' \tilde{u}_{j} + \bar{\rho} u''_{j} \right) c_{p} \frac{\partial \widetilde{T}}{\partial x_{j}} - \left(\tilde{u}_{j} \frac{\partial p'_{lin}}{\partial x_{j}} + u''_{j} \frac{\partial \bar{p}}{\partial x_{j}} \right) \\
+ \frac{\partial \vartheta'_{j,lin}}{\partial x_{j}} - \bar{\tau}_{ij,lin} \frac{\partial u''_{i}}{\partial x_{j}} - \tau'_{ij,lin} \frac{\partial \widetilde{u}_{i}}{\partial x_{j}} = \underbrace{-c_{p} \frac{\partial}{\partial x_{j}} \left(\rho u''_{j} T'' - \bar{\rho} \widetilde{u''_{j}} T'' \right)}_{THF} \underbrace{-\frac{\partial \vartheta'_{j,nln}}{\partial x_{j}}}_{MHNF} \\
+ \bar{\tau}_{ij,nln} \frac{\partial u''_{i}}{\partial x_{j}} + \tau'_{ij,nln} \frac{\partial \widetilde{u}_{i}}{\partial x_{j}} + \left(\tau'_{ij} \frac{\partial u''_{i}}{\partial x_{j}} - \overline{\tau}_{ij,\frac{\partial u''_{i}}{\partial x_{j}}} \right) + PCNF + EMNF.$$

$$(2.3c)$$

Here, φ' and φ'' are the Reynolds and Favre fluctuations, respectively; the linear and nonlinear fluctuation terms are listed on the two sides of each equation. There are five independent variables in the system. We choose the basic variable set as $q'' = [\rho', u'', v'', w'', T'']^T$, then the derived variables p', ϑ'_j and τ'_{ij} are not strictly linear functions of q''. To achieve a thorough linearisation of the system, p' and ϑ'_j are also divided into the linear and nonlinear parts in terms of q'', as

$$p' = \underbrace{\bar{\rho}T'' + \rho'\widetilde{T}}_{\equiv p'_{lin}} + \underbrace{\rho'T''}_{\equiv p'_{nln}}, \qquad \vartheta'_{j} = \underbrace{-\bar{\kappa}\frac{\partial T''}{\partial x_{j}} - \kappa'\frac{\partial \widetilde{T}}{\partial x_{j}}}_{\equiv \vartheta'_{j,lin}} \underbrace{-\left(\kappa'\frac{\partial T''}{\partial x_{j}} - \overline{\kappa}\frac{\partial T''}{\partial x_{j}}\right)}_{\equiv \vartheta'_{j,nln}}, \quad (2.4)$$

and so is $\tau'_{ij} = \tau'_{ij,lin} + \tau'_{ij,nln}$ (not detailed for brevity). As a result, the left-hand sides in (2.3) are all linear functions of q'', ready for constructing the linear operator. The various nonlinear fluctuation terms reside on the right-hand sides of (2.3), classified as underbraced. Their names are collectively defined in table 1. The expressions of the unspecified terms MMNF, PCNF, EMNF are detailed in Appendix A.

Notably, the term RSF is the only nonlinear term in incompressible flows. For compressible cases, however, many other terms appear due to the thermodynamic processes. In previous works (e.g. Alizard *et al.* 2015), RSF and THF were of the primary concern and were modelled using algebraic RANS relations to formulate the eLNS model; other terms were usually neglected. But here, we provide the complete form of the fluctuation equations and will clarify in §4.1 the contribution of each nonlinear term.

Abbreviation	Full name	Equation
RSF	Reynolds stress fluctuation	(2.3b)
PGNF	pressure gradient nonlinear fluctuation	(2.3b)
VSNF	viscous stress nonlinear fluctuation	(2.3b)
MMNF	momentum material derivative nonlinear fluctuation	(2.3b)
THF	turbulent heat-flux fluctuation	(2.3c)
MHNF	molecular heat-flux nonlinear fluctuation	(2.3c)
VDNF	viscous dissipation nonlinear fluctuation	(2.3 <i>c</i>)
PCNF	pressure convection nonlinear fluctuation	(2.3c)
EMNF	enthalpy material derivative nonlinear fluctuation	(2.3 <i>c</i>)

Table 1. Names and abbreviations of the nonlinear terms in the fluctuation equations.

2.2. The LNS and eLNS models

Equation (2.3) constitutes a closed set of equations for q'', and can be rearranged into an operator form as

$$\frac{\partial \rho'}{\partial t} - \mathcal{L}_{\rho} \boldsymbol{q}'' = f_{\rho}'', \quad \bar{\rho} \frac{\partial u_i''}{\partial t} - \mathcal{L}_{m_i} \boldsymbol{q}'' = f_{m_i}'', \quad \bar{\rho} c_v \frac{\partial T''}{\partial t} - R \widetilde{T} \frac{\partial \rho'}{\partial t} - \mathcal{L}_h \boldsymbol{q}'' = f_h'',$$
(2.5a)

where \mathcal{L} denotes the linear operator determined only by the mean flow \tilde{q} , and f'' represents the nonlinear terms. Detailed expressions of \mathcal{L} are readily contained in (2.3), also available in Dawson & McKeon (2020) and Chen *et al.* (2023*a*). Equation (2.5*a*) is rewritten into a matrix form as

$$A\frac{\partial}{\partial t}\begin{bmatrix}\rho'\\u''_{j}\\T''\end{bmatrix} = \begin{bmatrix}\mathcal{L}_{\rho}\\\mathcal{L}_{m_{i}}\\\mathcal{L}_{h}\end{bmatrix}\boldsymbol{q}'' + \begin{bmatrix}f''_{\rho}\\f''_{m_{i}}\\f''_{h}\end{bmatrix}, \qquad A \equiv \begin{bmatrix}1\\&\bar{\rho}\delta_{ij}\\-R\widetilde{T}&&\bar{\rho}c_{v}\end{bmatrix}, \quad (2.5b)$$

or equivalently

$$\frac{\partial \boldsymbol{q}''}{\partial t} = \mathcal{L}_q \boldsymbol{q}'' + \boldsymbol{f}''_q, \qquad (2.6)$$

where A^{-1} has been absorbed into \mathcal{L}_q and $f''_q \equiv [f''_\rho, f''_u, f''_v, f''_w, f''_T]^{\mathrm{T}}$. Equation (2.6) is the linearised NS equation and serves as a classic model problem. The linear model built upon (2.6) is often referred to as the LNS model.

In addition to the LNS model, the compressible eLNS model is frequently considered for model improvement, which is built upon algebraic RANS models (Alizard *et al.* 2015; Chen *et al.* 2024, 2025). Specifically, the Boussinesq assumption and the strong Reynolds analogy (SRA) are introduced to linearise the Reynolds stress $\tau_{R,ij}$ and turbulent heat flux $\vartheta_{R,j}$

$$\tau'_{R,ij} = \mu_t \left(\frac{\partial u''_i}{\partial x_j} + \frac{\partial u''_j}{\partial x_i} \right) - \frac{2}{3} \mu_t \frac{\partial u''_k}{\partial x_k} \delta_{ij}, \qquad \vartheta'_{R,j} = -\kappa_t \frac{\partial T''}{\partial x_j}.$$
 (2.7)

Here, κ_t is the eddy diffusivity. In practice, μ_t and κ_t are either assumed known from the DNS data or from semi-empirical models (e.g. Fan *et al.* 2024). The terms τ'_R and ϑ'_R were originally designed to partially model the colour of the nonlinear terms. Taking the RSF term as an example, it is modelled as

$$-\frac{\partial}{\partial x_j} \left(\rho u_i'' u_j'' - \bar{\rho} \widetilde{u_i'' u_j''} \right) \equiv RSF_i = \frac{\partial \tau_{R,ij}'}{\partial x_j} + eRSF_i,$$
(2.8)

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where eRSF represents the residual nonlinear term of RSF, after subtracting the modelled stress flux $\partial \tau'_{R,ij}/\partial x_j$. The other two residual terms eTHF and eVDNF, after modelling the turbulent heat flux and viscous dissipation, are similarly defined (e.g. Chen *et al.* 2023*b*).

It is worth emphasising that (2.8) is the core assumption for the eLNS model. However, we will show in § 4 that (2.8) can deviate from its original intention. The supposedly small residual terms, eRSF and eTHF, turn out to essentially rebuild the forcing distributions.

Since μ_t and κ_t profiles are assumed fixed, τ'_R and ϑ'_R are linear in terms of q'', so these eddy-viscosity-related terms in the momentum and enthalpy equations are linearised as

$$\frac{\partial \tau'_{R,ij}}{\partial x_j} \equiv \mathcal{M}_{m_i} \boldsymbol{q}^{\prime\prime}, \quad -\frac{\partial \vartheta'_{R,j}}{\partial x_j} + \bar{\tau}_{R,ij} \frac{\partial u_i^{\prime\prime}}{\partial x_j} + \tau'_{R,ij} \frac{\partial \tilde{u}_i}{\partial x_j} \equiv \mathcal{M}_h \boldsymbol{q}^{\prime\prime}. \tag{2.9}$$

Consequently, parts of the LNS nonlinear terms can be linearised and moved into the linear operator, so that the model is eddy-viscosity enhanced, following the term for incompressible flows (Madhusudanan, Illingworth & Marusic 2019). The resulting operator form of the eLNS model reads

$$\begin{cases} \frac{\partial \rho'}{\partial t} = \mathcal{E}_{\rho} \boldsymbol{q}'' + \boldsymbol{e}_{\rho}'', & \text{with } \mathcal{E}_{\rho} \equiv \mathcal{L}_{\rho} \text{ and } \boldsymbol{e}_{\rho}'' \equiv \boldsymbol{f}_{\rho}'', \\ \bar{\rho} \frac{\partial u_{i}''}{\partial t} = \underbrace{\left(\mathcal{L}_{m_{i}} + \mathcal{M}_{m_{i}}\right)}_{\equiv \mathcal{E}_{m_{i}}} \boldsymbol{q}'' + \underbrace{\left(\boldsymbol{f}_{m_{i}}'' - \mathcal{M}_{m_{i}} \boldsymbol{q}''\right)}_{\equiv \boldsymbol{e}_{m_{i}}''}, \\ \bar{\rho} c_{v} \frac{\partial T''}{\partial t} - R \widetilde{T} \frac{\partial \rho'}{\partial t} = \underbrace{\left(\mathcal{L}_{h} + \mathcal{M}_{h}\right)}_{\equiv \mathcal{E}_{h}} \boldsymbol{q}'' + \underbrace{\left(\boldsymbol{f}_{h}'' - \mathcal{M}_{h} \boldsymbol{q}''\right)}_{\equiv \boldsymbol{e}_{h}''}, \end{cases}$$
(2.10)

where the linear operator \mathcal{E} and the nonlinear term e'' for eLNS are defined as underbraced. The final standard form similar to (2.6) is

$$\frac{\partial \boldsymbol{q}''}{\partial t} = \mathcal{E}_q \boldsymbol{q}'' + \boldsymbol{e}_q''. \tag{2.11}$$

Note that \mathcal{E}_q is the same as \mathcal{L}_q except for replacing $\bar{\mu}$ with $\bar{\mu} + \mu_t$ and $\bar{\kappa}$ with $\bar{\kappa} + \kappa_t$. Equations (2.6) and (2.11) constitute the model equations for LNS and eLNS, respectively.

It is worth mentioning that the eLNS model was originally derived based on the triple decomposition of q (Reynolds & Hussain 1972; Alizard *et al.* 2015), where q is decomposed into the mean part, the turbulent fluctuation part and a small-amplitude organised perturbation. This approach is more physically interpretable, whereas the final linearised equation is mathematically equivalent to (2.11). Therefore, we do not present the triplydecomposed fluctuation equations, considering the already complicated (2.3).

2.3. The DNS datasets

A series of DNSs for compressible turbulent channel flows have been conducted by the present authors' group, as described at length in Cheng & Fu (2022, 2023). The simulations adopt three bulk Mach numbers $Ma_b = U_b/a_w = 0.8$, 1.5, 3.0 and a series of bulk Reynolds numbers $Re_b = \rho_b U_b h/\mu_w$. Here, ρ_b and U_b are the bulk density and streamwise velocity, $a = (\gamma RT)^{1/2}$ is the speed of sound with the specific ratio $\gamma = 1.4$, h is the channel half-width and the subscript w denotes wall quantities. Our focus is the compressibility effects on the forcing models, so we first select two DNS cases at $Ma_b = 1.5$ and 3.0 with nearly equal Re_{τ}^* . A third case with a higher Re at $Ma_b = 1.5$ is also included, to examine potential scalings. Additionally, two incompressible DNS cases

Case	Ma _b	$Re_{ au}^{*}$	Re_{τ}	Re_b	\widetilde{T}_c/T_w	L_x/h	L_z/h	$t_{total}u_{\tau}/h$	N_t
Ma15Re3k	1.5	144	219	3000	1.40	4π	2π	27.1	255
Ma30Re5k	3.0	141	455	4880	2.64	4π	2π	20.4	221
Ma15Re9k	1.5	393	598	9400	1.39	4π	2π	11.8	206
Ma00Re3k	0.0	186	186	2920	1.00	4π	2π	34.3	301
Ma00Re10k	0.0	547	547	10060	1.00	4π	2π	23.4	201

Table 2. Parameters of the DNS cases for turbulent channel flows, where $t_{total}u_{\tau}/h$ is the total eddy turnover time to accumulate statistics. Two incompressible cases are also included for later reference.

at different Re_{τ} are adopted, with data from Ying *et al.* (2023), to facilitate quantitative measure of the compressibility effects. Detailed parameters of the five cases are listed in table 2. Here, $Re_{\tau} = \rho_w u_{\tau} h/\mu_w$ and $Re_{\tau}^* = \bar{\rho} u_{\tau}^* h/\bar{\mu}$ are the friction and semi-local Reynolds numbers; u_{τ} is the friction velocity and u_{τ}^* is the semi-local counterpart; \tilde{T}_c is the temperature at the channel centre; L_x and L_z are the domain sizes. For later use, the wall viscous and semi-local length scales are defined as $\delta_v = \mu_w/(\rho_w u_{\tau})$ and $\delta_v^* = \bar{\mu}/(\bar{\rho} u_{\tau}^*)$, based on which the two sets of non-dimensional lengths are denoted by superscripts + and *, respectively.

To obtain the real forcing statistics, the nonlinear terms f''_q and e''_q are calculated by their definitions in §§ 2.1, 2.2 using the DNS data. The linear models in this work are solved within the framework of the stochastic Lyapunov equation (see § 2.4), so we do not need time-resolved DNS data of massive memory requirement as in Morra *et al.* (2021) and Ying *et al.* (2024*a*) for incompressible cases. Instead, we focus on the ensemble (temporal) average $\langle \cdot \rangle$ of variables, where time-not-resolved data are adequate. Therefore, the sampling number of the instantaneous fields N_t can be orders of magnitude lower than that required by the time-resolved data; for the latter case, N_t can reach $\sim 10^4$ at current Re_{τ} . The price is that the frequency spectrum is not resolved, but we can still reveal the forcing statistics in the spatial spectra, which provides adequate information to improve the stochastic linear models. The Fourier decomposition is applied on q'' in two homogenous directions as

$$\boldsymbol{q}''(x, y, z, t) = \iint_{-\infty}^{\infty} \hat{\boldsymbol{q}}''(y, t; k_x, k_z) \exp\left[i\left(k_x x + k_z z\right)\right] dk_x dk_z,$$
(2.12)

where k_x and k_z are the streamwise and spanwise wavenumbers, and \hat{q}'' is the shape function. The Fourier components of \hat{f}'' and \hat{e}'' are similarly defined. The value of N_t used for different cases is listed in table 2, which will be shown in § 3 to be large enough to provide converged spectral statistics.

2.4. Fluctuations in response to modelled and DNS forcing

In this subsection, we first present how the spectral correlation tensor $\widehat{\Phi}(y, y'; k_x, k_z) = \langle \hat{q}''(y, t; k_x, k_z) \hat{q}''^H(y', t; k_x, k_z) \rangle$ in response to a modelled forcing is obtained in the linear model, which requires only the mean flow (including μ_t , κ_t) as the input. Afterwards, we present how the real forcing can be computed from the DNS data, hence enabling a direct interpretation of the model errors.

In the Fourier space, the model equations (2.6) and (2.11) are expressed as

$$\frac{\partial \hat{\boldsymbol{q}}''}{\partial t} = \widehat{\mathcal{L}}_q \, \hat{\boldsymbol{q}}'' + \hat{\boldsymbol{f}}''_q, \qquad \frac{\partial \hat{\boldsymbol{q}}''}{\partial t} = \widehat{\mathcal{E}}_q \, \hat{\boldsymbol{q}}'' + \hat{\boldsymbol{e}}''_q. \tag{2.13a,b}$$

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When the real forcings \hat{f}''_q and \hat{e}''_q are unknown, the stochastic linear models provide a mean to estimate $\hat{\Phi}$. An idealised stochastic forcing is assumed as $\hat{f}_q = F\hat{f}_0$ (so is $\hat{e}_q = E\hat{e}_0$), where the matrix F(y) models the wall-normally varied forcing amplitude, and \hat{f}_0 (and \hat{e}_0) is a delta-correlated Gaussian white noise with zero mean. Consequently, the correlation tensor $\hat{\Phi}$ is obtained by solving the Lyapunov equation (Farrell & Ioannou 1993), as

$$\widehat{\mathcal{L}}\,\widehat{\boldsymbol{\Phi}} + \widehat{\boldsymbol{\Phi}}\,\widehat{\mathcal{L}}^H + (\boldsymbol{F}\boldsymbol{F}^H) = \boldsymbol{0}, \qquad \text{for the LNS model}, \qquad (2.14a)$$

$$\widehat{\mathcal{E}}\,\widehat{\boldsymbol{\Phi}} + \widehat{\boldsymbol{\Phi}}\,\widehat{\mathcal{E}}^H + (\boldsymbol{E}\boldsymbol{E}^H) = \boldsymbol{0}, \qquad \text{for the eLNS model}, \qquad (2.14b)$$

where the Hermitian transpose of $\widehat{\mathcal{L}}$ and $\widehat{\mathcal{E}}$ is defined as the discrete adjoint. In the simplest models, F and E are assumed to be diagonal, suggesting perfect zerocorrelation of the forcing between different variables and between different wall-normal heights. The simple yet well-behaved W-model (Gupta *et al.* 2021; Chen *et al.* 2023*b*) is considered here, where the forcing amplitude varies in proportion to the kinematic eddy viscosity $v_t = \mu_t / \overline{\rho}$, as

$$\boldsymbol{F}, \, \boldsymbol{E} = \operatorname{diag}\left(\left[\frac{1}{Pr_t}\frac{\partial\bar{\rho}}{\partial\tilde{u}}\boldsymbol{v}_t, \, \boldsymbol{v}_t, \, \boldsymbol{v}_t, \, \frac{1}{Pr_t}\frac{\partial\widetilde{T}}{\partial\tilde{u}}\boldsymbol{v}_t\right]^{\mathrm{I}}\right) \, \boldsymbol{M}^{-1/2}.$$
(2.15)

The weight matrix M will be defined later, and the extended SRA (Huang, Coleman & Bradshaw 1995) has been utilised in deriving (2.15) as

$$T_{rms}^{\prime\prime} \approx \frac{1}{Pr_t} \left| \frac{\partial \widetilde{T}}{\partial \widetilde{u}} \right| u_{rms}^{\prime\prime}, \qquad \rho_{rms}^{\prime} \approx \frac{1}{Pr_t} \left| \frac{\partial \bar{\rho}}{\partial \widetilde{u}} \right| u_{rms}^{\prime\prime}, \tag{2.16}$$

where *rms* denotes the root mean square. Equation (2.16) also suggests two wall viscous units for the temperature and density, $T_{\tau} = (\partial \tilde{T} / \partial \tilde{u})_w u_{\tau} / Pr$ and $\rho_{\tau} = (\partial \bar{\rho} / \partial \tilde{u})_w u_{\tau}$.

The eigenmodes of $\widehat{\Phi}$, known as the proper orthogonal decomposition (POD) or Karhunen–Loève modes, are of interest in various analyses on turbulence. They are defined as

$$\left(\widehat{\boldsymbol{\Phi}}, \, \check{\boldsymbol{q}}_{j}^{\prime\prime}\right)_{E} = \theta_{j} \, \check{\boldsymbol{q}}_{j}^{\prime\prime}, \qquad j = 1, \, 2, \, \dots,$$
(2.17)

where the eigenvalues and eigenfunctions θ_j and \check{q}''_j are sorted in the descending order of the energy of the *j*th mode (θ_j). These POD modes are orthogonal to each other under the energy norm $(\cdot, \cdot)_E$, which is defined following common usage (Chu 1965) as

$$\|\check{\boldsymbol{q}}''\|^{2} = \left(\check{\boldsymbol{q}}'',\check{\boldsymbol{q}}''\right)_{E} = \int_{0}^{2h} \left(\bar{\rho}\check{\boldsymbol{u}}''^{H}\check{\boldsymbol{u}}'' + \frac{R\widetilde{T}}{\bar{\rho}}\check{\rho}'^{\dagger}\check{\rho}' + \frac{\bar{\rho}c_{v}}{\widetilde{T}}\check{T}''^{\dagger}\widehat{T}''\right) \mathrm{d}\boldsymbol{y} = \check{\boldsymbol{\mathcal{Q}}}''^{H}\boldsymbol{M}\check{\boldsymbol{\mathcal{Q}}}''.$$
(2.18)

Here, \dagger denotes complex conjugate, the global vector \check{Q}'' contains \check{q}'' at all wall-normal grids and M is the weight matrix. Regarding the numerics, (2.14) is discretised using the Chebyshev collocation point method; 241 points are used in default, which is abundant to ensure grid independence. The solver verification can be found in Chen *et al.* (2023*a*).

Instead of the ideal modelling in (2.15), the forcings can be realistically computed by their definitions from the DNS data. Equation (2.13) indicates that the correlation tensor satisfies the following two Lyapunov equations for the LNS and eLNS models, respectively (derivations in Appendix A):

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$$\mathbf{0} \approx \frac{\partial \mathbf{\Phi}}{\partial t} = \widehat{\mathcal{L}}_{q} \widehat{\mathbf{\Phi}} + \widehat{\mathbf{\Phi}} \widehat{\mathcal{L}}_{q}^{H} + \underbrace{\left(\widehat{f}_{q}^{"} \widehat{\mathbf{q}}^{"H} + \widehat{\mathbf{q}}^{"} \widehat{f}_{q}^{"H} \right)}_{\equiv (FF^{H})_{DNS}}, \quad \text{for the LNS model,} \quad (2.19a)$$
$$\mathbf{0} \approx \frac{\partial \widehat{\mathbf{\Phi}}}{\partial t} = \widehat{\mathcal{E}}_{q} \widehat{\mathbf{\Phi}} + \widehat{\mathbf{\Phi}} \widehat{\mathcal{E}}_{q}^{H} + \underbrace{\left(\widehat{\mathbf{e}}_{q}^{"} \widehat{\mathbf{q}}^{"H} + \widehat{\mathbf{q}}^{"} \widehat{\mathbf{e}}_{q}^{"H} \right)}_{\equiv (EE^{H})_{DNS}}, \quad \text{for the eLNS model.} \quad (2.19b)$$

Here, the approximation on the left-hand side holds subject to a sufficiently large N_t . Comparing between (2.14), (2.19), we can define the real forcing matrix from DNS as $(FF^H)_{DNS}$ and $(EE^H)_{DNS}$, as underbraced in (2.19). A reasonable consequence is that when $(FF^H)_{DNS}$ and $(EE^H)_{DNS}$ are input into the stochastic linear models (2.14), the output $\hat{\Phi}$ should be identical to the DNS counterpart $\hat{\Phi}_{DNS}$, which is required by the models' mathematical self-consistency. On the other hand, $(FF^H)_{DNS}$ and $(EE^H)_{DNS}$ naturally provide the 'standard answers' to how to model FF^H and EE^H , and can directly tell why the current LNS and eLNS models succeed or fail in specific cases.

It is worth noting that, when deriving (2.19), we do not and need not assume the DNS signal to be a white noise in the temporal domain; it is indeed not a white noise. The consequence is that $(FF^H)_{DNS}$ (and also $(EE^H)_{DNS}$) may be negative definite for some scales, so the matrix $(F)_{DNS}$ may not really exist to satisfy this product. That is also the case when the white noise assumption and (2.15) fail. Nonetheless, this consequence does not affect the proof of the models' self-consistency because we simply treat $(FF^H)_{DNS}$ as a whole and do not pursue further matrix decompositions.

3. Data verification and models' self-consistency

Two central objectives in this section are to confirm that (i) N_t in table 2 is adequate to obtain converged forcing statistics, and (ii) the stochastic linear models in § 2.4 are mathematically self-consistent. Note that the different variable components in $\hat{\Phi}$, (FF^H) and (EE^H) below will be normalised by ρ_b , U_b , T_w and h, unless otherwise stated.

First, we inspect the mean-flow budgets in (2.1) for all the cases (shown in Appendix B). A term balance is realised throughout the field for each case, demonstrating the reliability of the averaged DNS data and the post-processing schemes. Second, we calculate the real forcing matrices (FF^H)_{DNS} and (EE^H)_{DNS} in (2.19) at different k_x , k_z , and input them into the LNS and eLNS models in (2.14). Both models are anticipated to output $\hat{\Phi}$ values that are identical to $\hat{\Phi}_{DNS}$. This is realised in figure 1 at nearly all k_x and k_z that are resolved. Specifically, close resemblance among $\hat{\Phi}_{LNS}$, $\hat{\Phi}_{eLNS}$ is not displayed for conciseness), at (k_x , k_z)h = (0.5, 3) (wavelengths $\lambda_x^+ = 2750$, $\lambda_z^+ = 460$). A more quantitative comparison is shown in figure 1(c) by plotting the leading fortieth θ_j of the POD modes. They cover nearly four orders of magnitude, and the three sets of θ_j are still in close agreement with each other. Such agreement in θ_j is also achieved at other k_x and k_z . Two representative results are displayed in figures 1(d) and 1(e), where $\lambda_x^+ < \lambda_z^+$ for the former scale and λ_x^+ , λ_z^+ are both small (<70) for the latter. The θ_j from the eLNS model matches slightly better the DNS than the LNS model for different scales, and the reason will be clear in § 4.1.

Consequently, we arrive at the following two points. First, the value of N_t in table 2 is adequate to obtain converged ensemble averages of the variables and nonlinear forcing at different scales. The computation of the complicated fluctuation terms in (2.3) is



Figure 1. (*a,b*) Ensemble-averaged correlation tensor $\langle \hat{q}'' \hat{q}''^H \rangle$ (logarithmic scale to display all structures) from (*a*) the DNS data and (*b*) the eLNS model using the DNS-computed forcing, at $(k_x, k_z)h = (0.5, 3)$ for case Ma15Re3k; only 15 components of the correlations out of 25 are shown because the tensor is Hermitian. (*c*-*e*) Leading eigenvalues of the correlation from DNS and the LNS/eLNS models using the DNS-computed forcing with $(k_x, k_z)h$ equal to (*c*) (0.5, 3), (*d*) (3.5, 1) and (*e*) (20, 40).

also accurate enough. Second, the framework of the stochastic linear models in §2 is mathematically self-consistent. Both the LNS and eLNS models can provide accurate $\hat{\Phi}$ at different k_x and k_z , if the forcings FF^H and EE^H are accurate enough. In that case, the derived variables of $\hat{\Phi}$, like the fluctuation variance, spatial spectra and the linear stochastic estimations (e.g. Madhusudanan *et al.* 2019), can also closely agree with DNS.

4. Real forcing distributions from DNS

The characteristics of the real matrices $(FF^H)_{DNS}$ and $(EE^H)_{DNS}$ are detailed here. Prior to that, the nonlinear terms f''_q and e''_q from the DNS, required by $(FF^H)_{DNS}$ and $(EE^H)_{DNS}$ (see (2.19)), are discussed in § 4.1. Note that we concern with three central problems. The first is to clarify the relative contributions of different nonlinear components in (2.3) and table 1. The second is to examine whether the diagonally modelled forcing



Figure 2. Root mean squares of different nonlinear fluctuation terms from DNS in the (*a*) streamwise, (*b*) wallnormal and (*c*) spanwise momentum equations, and (*d*) the enthalpy equation, for case Ma15Re3k. Panels (*a*–*c*) are normalised by u_{τ} and panel (*d*) is by T_{τ} . See table 1 for term abbreviations. Note that all the components have been scaled by the mean-flow coefficients in (2.5*b*), as in (2.6).

(2.15) aligns with the DNS statistics, which directly explains the success or failure of current models. The third is to reveal the differences between the LNS and eLNS models, and clarify the role of the critical eddy-viscosity-related terms.

4.1. Distributions of the nonlinear terms

Given the significance of accurately computing the forcing (§ 3), we scrutinise each nonlinear fluctuation term defined in (2.3). Their root mean squares in the LNS and eLNS models are computed from the DNS. The wall-normal distributions of these components are plotted in figures 2 and 3 for cases Ma15Re3k and Ma30Re5k, respectively. The LNS model is discussed first. For $f''_{u,rms}$ and $f''_{T,rms}$, RSF and THF are respectively the dominant terms in nearly all regions. The terms related to molecular viscosity and conductivity (VSNF, VDNF and MHNF) are negligible except in the viscous sublayer, and the pressure-related terms PGNF and PCNF remain small throughout. For $f''_{v,rms}$ and $f''_{w,rms}$, RSF still dominates away from the wall, but within and below the buffer layer, PGNF is significant and leads to an additional peak of f''_{rms} , especially for case Ma30Re5k (figures 3b and 3c). Therefore, the compressibility effect is prominent near the wall for the nonlinear terms in the wall-normal and spanwise momentum equations. The physics behind this additional peak will be clarified later using its spectra. Moreover, the convection-related terms MMNF and EMNF remain moderate throughout the field. They become increasingly



Figure 3. Same as figure 2 except for case Ma30Re5k. See table 1 for the term abbreviations.

important as Ma_b rises, suggesting notable convection of momentum and energy caused by ρ' . The results of case Ma15Re9k are qualitatively the same as case Ma15Re3k, and are hence not displayed.

Next, we turn to the eLNS model. The original intention of introducing τ'_R and ϑ'_R is to partially model RSF and THF, so the residual terms eRSF and eTHF should be smaller than RSF and THF (see (2.8)). In figures 2 and 3, however, the results are the opposite. For all the equations shown, eRSF and eTHF are close to RSF and THF at $y^* \leq 15$, where $\mu_t/\bar{\mu}$ (and $\kappa_t/\bar{\kappa}$) is small. In the outer region with $\mu_t \gg \bar{\mu}$, however, eRSF and eTHF can be over two times larger than RSF and THF, so e''_{rms} are much higher than f''_{rms} in all the momentum and enthalpy equations. The Re_{τ} in the present cases are relatively low, so the ratio e''_{rms}/f''_{rms} will become even larger under higher Re_{τ} , due to the increased $\mu_t/\bar{\mu}$.

Considering the unusual trend of eRSF and eTHF, the modelling assumption (2.8) deserves more discussion. The variances of the three components in (2.8), i.e. RSF_i , $\partial \tau'_{R,ij}/\partial x_j$ and $eRSF_i$, are plotted in figure 4 for case Ma30Re5k, along with the correlation between RSF_i and $\partial \tau'_{R,ij}/\partial x_j$. For both the streamwise and wall-normal momentum equations (and also the spanwise one not shown), the modelled stress flux $\partial \tau'_{R,ij}/\partial x_j$ does not follow RSF_i throughout, regarding both shapes and amplitudes. The modelled flux highly overestimates, and has moderately negative correction with RSF in the outer layer, which leads to the much higher amplitudes of eRSF and RSF. The same conclusion applies to THF and eTHF. Also, we have examined that the trends in figure 4 are analogous for other cases, insensitive to Re_{τ} and Ma_b .



Figure 4. Variance of the three terms in (2.8) and the correlation between RSF and the modelled stress flux (all normalised by u_{τ}^2) for case Ma30Re5k: (*a*) streamwise and (*b*) wall-normal momentum equations.

Consequently, we arrive at a conclusion that, the τ'_R and ϑ'_R fluxes are not partial modellings of RSF and THF, as previously believed (Hwang & Cossu 2010; Pickering *et al.* 2021; Chen *et al.* 2023*a*; Ying *et al.* 2023). On the contrary, the 'residual' terms eRSF and eTHF can be several times larger than RSF and THF, so e''_{rms} is more dominant by eRSF and eTHF, than the dominance of RSF and THF over f''_{rms} . As a result, the effects of other unmodelled nonlinear fluctuations (VSNF, MMNF, etc.) become minor, which increases the robustness of the shapes and amplitudes of the forcing. More discussions on this point will be provided in § 4.2. This robustness of the eLNS forcing also explains the trend in figure 1 that its ensemble average can reach convergence using a lower N_t than the LNS counterpart.

The spectra of these nonlinear fluctuations are calculated, to identify the active wavenumbers of different components. Case Ma30Re5k is studied first, as it features the strongest density and temperature fluctuations. The pre-multiplied one-dimensional (1-D) spectra of \hat{f}''_{q} and \hat{e}''_{q} in the streamwise and spanwise directions are displayed in figure 5. The semi-local coordinates y^* , λ_x^* and λ_z^* are adopted, and the spectra of the *u* component are used as references in each row to examine the spectral similarity among different variables. A notable observation is that, compared with the LNS model, the spectra of \hat{e}''_{u} , \hat{e}''_{w} , \hat{e}''_{w} and \hat{e}''_{T} in eLNS have greater resemblance to each other in shapes and peak locations for both the streamwise and spanwise ones; see figures 5(b) and 5(d). This similarity among variables can be explained by the dominance of eRSF and eTHF over other terms, as observed in figure 3. A further decomposition of $\partial \tau'_{R,ij}/\partial x_j$ and $\partial \vartheta'_{R,j}/\partial x_j$ (difference between f''_{q} and e''_{q}) shows that the dominant terms for the four terms are $\mu_t \Delta \hat{u}''$, $\mu_t \Delta \hat{v}''$, $\mu_t \Delta \hat{w}''$ and \hat{r}''_{q} and \hat{r}''_{q} , and the similarity among \hat{e}''_{u} , \hat{e}''_{w} and \hat{e}''_{T} is more easily established than \hat{f}'' .

An additional peak appears in the \hat{f}_v'' and \hat{e}_v'' spectra in the viscous sublayer, which is contributed by PGNF (equal to $-\partial(\rho'T'')/\partial y$) from figure 3. Compared with the outer peak, this inner peak resides in a larger scale of $\lambda_z^* \approx 180$. The value of λ_x^* of this inner peak exceeds 1500, but due to the relatively low Re_τ in this case, its streamwise spectrum is not fully resolved. The $\lambda_x^* \ll \lambda_z^*$ feature of this inner peak indicates that it is not caused by the near-wall acoustic motions (where $\lambda_x < \lambda_z$; see § 5), but is attributed to the near-wall density and temperature streaks which are most active at $\lambda_x^+ \sim 1000$ and



Figure 5. Pre-multiplied one-dimensional spectra of the nonlinear fluctuation terms in the (a,b) streamwise and (c,d) spanwise directions in semi-local units, for the (a,c) LNS model (\hat{f}'_q) and (b,d) eLNS model (\hat{e}''_q) , respectively, for case Ma30Re5k. The contours in each panel are normalised by their extreme values labelled on the top (in wall units ρ_{τ} , u_{τ} and T_{τ}). The blue dotted lines denote the peak location of the *u*-spectrum, and the blue dashed lines denote the *u*-contour of 0.4.

 $\lambda_z^+ \sim 100$; these two streaks are also in nearly perfect negative correlation (Coleman, Kim & Moser 1995; Patel, Boersma & Pecnik 2016; Cheng & Fu 2024).

To examine the robustness of the spectral features of the eLNS nonlinear terms at different Ma_b and Re_τ , the 1-D spectra of \hat{e}''_q for cases Ma15Re3k and Ma15Re9k are depicted in figure 6. As in case Ma30Re5k, similarity between the spectra of \hat{e}''_u , \hat{e}''_v , \hat{e}''_w and \hat{e}''_T is observed in both the streamwise and spanwise directions. Meanwhile, the inner peak of \hat{e}''_v due to the PGNF is less evident than in figure 5 owing to the reduced Ma_b . Comparing between figures 6 and 5, we note that the (outer) peak location of these 1-D spectra is not sensitive to Ma_b and Re_τ after using the semi-local units, which is measured to be $\lambda_x^* \approx 90$, $\lambda_x^* \approx 30$ and $y^* \approx 40$.

In short, we emphasise that the introduction of τ'_R and ϑ'_R is not a partial modelling of RSF and THF. On the contrary, it highly lifts the amplitudes of eRSF and eTHF, and hence the dominance of \hat{e}''_q by eRSF and eTHF. Therefore, \hat{e}''_q is more robust than \hat{f}''_q for different variables in terms of the peaks and shapes of their physical and spectral distributions. Consequently, \hat{e}''_q can be more easily modelled than \hat{f}''_q by featuring more robust composition and structures, although this robustness may not always improve the model results; see § 4.2.





Figure 6. Same as figure 5, but for (a,b) case Ma15Re3k and (c,d) case Ma15Re9k and only the results of the eLNS model are shown. Panels (a,c) are the streamwise spectra and panels (b,d) are the spanwise ones.

4.2. Distributions of the forcing matrix

As mentioned in § 3, $(FF^H)_{DNS}$ and $(EE^H)_{DNS}$ provide standard answers to how to model (FF^H) and (EE^H) in the stochastic linear models, so the distributions of these forcing are discussed at different k_x and k_z . The spectral fluctuation is usually classified into two branches according to its aspect ratio λ_x/λ_z . The structure with $\lambda_x > \lambda_z$ is considered as streamwise elongated, which is energetic and has strong linear coherence in the wall-normal direction (e.g. del Álamo & Jiménez 2006). In comparison, the structure with $\lambda_x < \lambda_z$ is regarded as spanwise elongated, which receives energy from the nonlinear terms and can be worse predicted by the linear models than the streamwise-elongated one (Gupta *et al.* 2021; Symon *et al.* 2021). These two types of fluctuations are discussed separately below.

The streamwise-elongated fluctuations are considered first. Figure 7 vividly demonstrates what the real forcing matrices $(FF^H)_{DNS}$ and $(EE^H)_{DNS}$ look like for case Ma30Re5k at $(k_x, k_z)h = (0.5, 15) (\lambda_x = 12.6h, \lambda_z = 0.4h)$. A direct observation is that these two matrices are far more complicated than the simple diagonal modelling in (2.15). Recall that two levels of simplification are made in (2.15). First, only the selfcorrelation matrices of the forcing variables $(\hat{f}''_{\rho} \hat{f}''_{\rho}^{H}, \hat{f}''_{u} \hat{f}''_{u}^{H}$, etc.) are retained, while all

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Figure 7. Contours of the forcing matrix (a) $|(FF^H)_{DNS}|$ and (b) $|(EE^H)_{DNS}|$, at $(k_x, k_z)h = (0.5, 15)$ for case Ma30Re5k. Their diagonal terms, as labelled in blue dotted lines, are plotted in panels (c,d) in normalised values; the purple reference lines are from (2.15).

the inter-variable correlations $(\hat{f}_{\rho}^{"}\hat{f}_{u}^{"H}, \text{ etc.})$ are neglected. Second, each self-correlation matrix is set to be diagonal, signifying perfect zerocorrelation between different y. In figure 7, however, there are significant off-diagonal terms for both $(FF^{H})_{DNS}$ and $(EE^H)_{DNS}$, suggesting non-negligible inter-variable and inter-height correlations. Regarding the difference between the LNS and eLNS forcing, a visual observation seems to suggest that $(EE^{H})_{DNS}$ is more diagonal than $(FF^{H})_{DNS}$, in terms of wall-normal correlations. This is reasonable because the dominant terms τ'_R and ϑ'_R in eLNS are linear functions of the fluctuations and are related to local mean-flow gradients, so they are more localised than the other nonlinear terms which contain high-order fluctuation moments. More quantitative evidence will be provided later. Consequently, the diagonal modelling in (2.15) benefits more the forcing in the eLNS model, than the LNS one. The diagonal elements of $(FF^H)_{DNS}$ and $(EE^H)_{DNS}$ are extracted in figures 7(c) and 7(d). The modelled distributions in (2.15) are also displayed for reference. As discussed in § 4.1, the diagonal terms for eLNS reflect the large contributions of eRSF and eTHF, so the diagonal terms of the u, v, w and T components exhibit stronger resemblance to each other, hence they are more robust for modelling compared with the LNS model. Nevertheless, the prediction of (2.15) is not very consistent with the DNS results; the real forcing is more concentrated near the wall under this small λ_7 .

From the modelling perspective, it is interesting to show whether the DNS forcing is of low rank. The diagonal modelling in (2.15) means that the forcing is assumed to be of full rank, while Morra *et al.* (2021) demonstrate using the incompressible DNS data that the forcing has clear spatial-temporal coherence. The rank characteristics of the forcing matrix can be measured using the singular value decomposition (SVD), as

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Figure 8. (*a*,*c*) Energy ratios occupied by the leading 10 singular values of the forcing matrix in figure 7, and (*b*,*d*) shape functions of the principal forcing mode (case Ma30Re5k, (k_x , k_z)h = (0.5, 15)). Panels (*a*,*b*) are for the LNS model, and (*c*,*d*) for the eLNS one.

$$(FF^{H})_{DNS} = \sum_{j} \sigma_{j} \check{f}_{j}^{"} \check{f}_{j}^{"H} = \sum_{j} \sigma_{j} \check{\Psi}_{j}, \qquad (4.1)$$

where σ_j is the singular value sorted in the descending order; \check{f}''_j is the singular vector satisfying the orthogonal condition $(\check{f}''_i, \check{f}''_j)_E = \delta_{ij}$, so each component $\check{\Psi}_j$ is of rank one. The left and right singular vectors are the same because the forcing matrix is Hermitian. Besides, the sum of the singular values, $\sum_j \sigma_j \equiv V_\sigma$, represents the total energy input by the forcing. The decomposition for $(EE^H)_{DNS}$ is defined likewise. Based on (4.1), we would regard a forcing matrix as of low rank if the first or first few SVD modes occupy a large portion of the total forcing energy.

The leading ten $\sigma_{F,j}$ and $\sigma_{E,j}$ of the matrices in figure 7 are plotted in figures 8(*a*) and 8(c), expressed in relative ratios $r_{\sigma,j} = \sigma_j / V_{\sigma}$. Note that σ_j appears in pairs due to the flow symmetry or antisymmetry on the two sides of the channel regarding the centreline, so every two σ_i are combined to reflect their joint contributions. The forcing $(FF^{H})_{DNS}$ for the LNS model is of relatively low rank, with $r_{\sigma F,1}$ over 57 %. This low-rank feature is very different from the full-rank FF^{H} (fully diagonal) assumed in the linear model. The shape of the principal mode \check{f}_1'' is plotted in figure 8(b). The $\check{f}_{u,1}''$ and $\check{f}_{T,1}''$ components turn out to be the two largest, which is different from a previous suggestion of the linear models that, for the principal forcing mode, the $\check{f}_{v,1}''$ and $\check{f}_{w,1}''$ components are dominant for streamwise-elongated modes in the form of streamwise vortices to induce streaks (Hwang & Cossu 2010; Chen *et al.* 2023*a*). In comparison, $r_{\sigma E,1} = 26 \%$ for the eLNS model is lower than $r_{\sigma F,1}$. In other words, the modelled terms eRSF and eTHF lessen the spatial coherence of the forcing, and add to its diagonal dominance. Meanwhile, eRSF and eTHF markedly enhance the outer-layer forcing (figure 3), so the shapes of the principal mode in figure 8(d) is also lifted away from the wall, compared with its LNS counterpart. This amplified outer-layer forcing energy can be rapidly dissipated within the linear operator, where the stronger damping in eLNS is due to the replacement of $\bar{\mu}$ with $\bar{\mu} + \mu_t$ in the linear operator (see § 2.2). Consequently, the response mode can still take the form of near-wall streaky motions (c.f. figure 1b also with $\lambda_x > \lambda_z$).

The forcing pattern for spanwise-elongated modes is demonstrated in figure 9, where the wavenumbers are selected to be $(k_x, k_z)h = (20, 3)$ $(\lambda_x = 0.6h, \lambda_z = 4.2h)$.



Figure 9. Same as figure 7 except for scale $(k_x, k_z)h = (20, 3)$.

Compared with figure 7, the matrices $(FF^H)_{DNS}$ and $(EE^H)_{DNS}$ become visually more diagonal in the wall-normal direction. Equivalently, the two forcings are both of higher rank, compared with those in figures 7, 8, which is inferred from their more uniform σ_i (not shown; $r_{\sigma F,1}, r_{\sigma E,1} < 12\%$). This stronger diagonality for spanwise-elongated structures is reasonable, because they tend to be more localised and exhibit weak interlayer linear coherence (e.g. Marusic, Baars & Hutchins 2017). The diagonal elements of the two forcing matrices are plotted in figures 9(c) and 9(d). It is interesting to show that the prediction of (2.15) is in good agreement with the DNS results, especially for the eLNS model. Nevertheless, both incompressible and compressible works suggest that the linear models perform worse for spanwise-elongated fluctuations (Gupta *et al.* 2021; Symon et al. 2021; Chen et al. 2023b). From the modelling aspect, a prominent disrupting factor is the unmodelled inter-variable correlations, particularly $\hat{f}_{\rho}'' \hat{f}_{u}''H$, $\hat{f}_{\rho}'' \hat{f}_{T}''H$, $\hat{f}_{u}'' \hat{f}_{v}''H$ and $\hat{f}''_{\mu}\hat{f}''_{T}^{H}$ (and also the \hat{e}'' counterparts). They are all assumed zero in (2.15). From the physical perspective, previous studies showed that the linear dynamics (lift-up mechanism, transient growth, etc.) is important for generating streamwise-elongated structures, but contrarily, the travelling-wave-like spanwise-elongated fluctuations are mainly caused by nonlinear effects related to turbulent bursting, vortex stretching and streak breakdown (Schoppa & Hussain 2002; Yu et al. 2024).

The diagonality of the forcing matrices in figures 7 and 9 relies on visual measurement. For more quantitative estimation, we project the forcing into the space spanned by the POD modes of $\langle \hat{q}'' \hat{q}''^H \rangle$. The projection coefficients for models LNS and eLNS are then computed as

$$\alpha_{ij} = \check{\boldsymbol{q}}_i^{\prime\prime H} \boldsymbol{M} \left(\boldsymbol{F} \boldsymbol{F}^H \right)_{DNS} \boldsymbol{M} \check{\boldsymbol{q}}_j^{\prime\prime}, \qquad \beta_{ij} = \check{\boldsymbol{q}}_i^{\prime\prime H} \boldsymbol{M} \left(\boldsymbol{E} \boldsymbol{E}^H \right)_{DNS} \boldsymbol{M} \check{\boldsymbol{q}}_j^{\prime\prime}. \tag{4.2}$$

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Figure 10. Projection coefficients of the DNS forcing to the space spanned by the POD modes for case Ma30Re5k: (a,b,c,d) LNS part α_{ij} , (e,f,g,h) eddy-viscosity part ξ_{ij} and (i,j,k,l) eLNS part β_{ij} ; see (4.2). Panels show $(a,e,i) \lambda_x = 6.3h$, $\lambda_z = 3.1h$; $(b,f,j) \lambda_x = 6.3h$, $\lambda_z^+ = 179$; $(c,g,k) \lambda_x^+ = 143$, $\lambda_z = 3.1h$; $(d,h,l) \lambda_x^+ = 143$, $\lambda_z^+ = 179$. The input energy V_{σ} shown is amplified by 10⁵ for all panels for convenience.

This projection reduces the forcing in the physical space into discrete coefficients, and can jointly reflect inter-variable and inter-height correlations. If the forcing is ideally a white noise modelled by (2.15), then α_{ij} (and β_{ij}) is also a diagonally dominant matrix for nearly all spatial scales. The difference between α_{ij} and β_{ij} is also of interest, denoted as $\xi_{ij} = \beta_{ij} - \alpha_{ij}$, which reflects the contribution of the eddy-viscosity-related terms.

The Ma30Re5k case is discussed first using four representative scales. The first two are streamwise- and spanwise-elongated fluctuations discussed above; the third type is a large-scale fluctuation in the outer region with $(\lambda_x, \lambda_z)/h = (6.3, 3.1)$, and the fourth is a small-scale fluctuation with λ_x^+ , $\lambda_z^+ \sim 100$. Their α_{ij} , ξ_{ij} and β_{ij} values are depicted in figure 10 for the leading indices i, j < 10. These coefficients are normalised by their input energy V_{σ} , as labelled, such that their trace is unity: $\sum_i \alpha_{ii}/V_{\sigma,\alpha} = 1$ (so are ξ_{ij} and β_{ij}). For the large-scale fluctuation in figure 10(a), α_{ij} for the LNS model has pronounced off-diagonal elements, indicating strong inter-variable and/or inter-height correlations. In comparison, the eddy-viscosity term ξ_{ij} has much higher energy (V_{σ}) than α_{ij} , as discussed for figure 3, and is highly diagonally dominant. As a result, their summation $\beta_{ij} = \alpha_{ij} + \xi_{ij}$ for the eLNS model has a similar distribution to ξ_{ij} and features high diagonal dominance. The above trend is analogous to streamwise-elongated structures in the second column of figure 10, which has been visually shown in figure 9. In comparison, for the structures of small λ_x (irrespective of λ_z), α_{ij} , ξ_{ij} and β_{ij} are all diagonally dominant, as shown in the last two columns of figure 10, which is in line with the observations in figure 9.

The projection results for cases Ma15Re3k and Ma15Re9k are shown in figure 11. The distributions of α_{ij} and β_{ij} for different scales resemble a lot those in figure 10, suggesting



Figure 11. Same as figure 10 but for (a,b,e,f) case Ma15Re3k and (c,d,g,h) case Ma15Re9k. Panels (a,c,e,g) are fluctuations of large λ_x , λ_z and panels (b,d,f,h) are of small λ_x , λ_z .

weak dependence on Ma_b and Re_{τ} . Therefore, the role of the eddy-viscosity terms is clarified again (as in § 4.1) that it adds a large-amplitude diagonallydominant term to the LNS forcing matrix (specifically, the self-correlation part of the forcing variables), so that the eLNS forcing is more diagonally dominant and has similar shapes at different scales. From the physical point of view, this enhanced diagonal dominance was shown to build a better balance between local turbulent energy dissipation and transport (Symon *et al.* 2023; Ying *et al.* 2024*a*). Consequently, the eLNS model is more robust among different scales in predicting variable correlations than LNS.

Based on the above conclusion, a natural thought to improve the model arises that the robustness of the eLNS model can be further strengthened by artificially adding very large diagonal elements, for example, 10 times eRSF, although unphysical, to the forcing matrix (EE^{H}) . Nevertheless, we note that the model behaviours will not be necessarily improved for two reasons. First, although the inter-height correlation is more robustly modelled, the off-diagonal inter-variable correlation for the forcing also matters (see figure 9), so the relative amplitudes of different variables still need to be carefully determined. Second, artificial enhancement of the diagonal terms can destroy the non-normality of the NS operator, as already reflected in the eLNS model to some extent (Symon *et al.* 2021; Chen *et al.* 2023*b*). As a result, not only can the linear amplification mechanism be disrupted, but also the energies of different branches of modes can be disorderly sorted, possibly leading to excessive amplification of the acoustic and other components for compressible flows.

Finally, we provide a global view of the coherence of the LNS and eLNS forcing at all scales. The contours of $r_{\sigma,1}$, which measures the energy ratio of the most energetic forcing mode to all modes, are plotted in figure 12 for all three cases. For the LNS model, the forcing for streamwise-elongated fluctuations $(\lambda_x > \lambda_z)$ has relatively high coherence and low rank, compared with the spanwise-elongated parts. The maximum $r_{\sigma,1}$ for case Ma15Re3k reaches over 70 % at $(\lambda_x, \lambda_z)/h \approx (12.6, 0.7)$. The maxima in other two cases (figures 12b and 12c) also appear at approximately the same scale, suggesting a robust mechanism of scale selection for the forcing at $\lambda_x > \lambda_z$ explains its poor behaviours

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Figure 12. Energy ratio occupied by the leading POD mode $(r_{\sigma,1} = \sigma_1 / \sum_j \sigma_j)$ for the forcing in the (a,b,c) LNS and (d,e,f) eLNS models at different λ_x , λ_z for cases (a,d) Ma15Re3k, (b,e) Ma30Re5k and (c,f) Ma15Re9k. The black dashed line denotes $\lambda_x = \lambda_z$.

in predicting these energetic fluctuations (e.g. Madhusudanan *et al.* 2019). Meanwhile, we notice that the distribution of $r_{\sigma,1}$ resembles a lot the linear coherence spectra of u and T between different heights (Marusic *et al.* 2017; Gupta *et al.* 2021; Chen *et al.* 2023*b*) within the inner–outer interaction model. Therefore, the LNS forcing for the scales with strong inter-height linear coherence of velocity and temperature is of particularly low rank.

For a more quantitative comparison of $r_{\sigma,1}$ between different cases, its variation with λ_z/h is plotted in figure 13 for all three compressible cases, at the largest streamwise scale $\lambda_x = 4\pi h$. The two incompressible cases are also included for reference. Cases Ma00Re3k, Ma15Re3k and Ma30Re5k have comparable $Re_{\tau}^* = 141 \sim 186$, and their $r_{\sigma,1}$ values are also close to each other, except at the very small scale. For the two higher Re_{τ}^* cases Ma15Re9k and Ma00Re10k, their $r_{\sigma,1}$ also have analogous distributions. Therefore, it is concluded that the low-rank feature of the LNS forcing is not sensitive to Ma_b , although a higher Ma_b can slightly decrease $r_{\sigma,1}$ due to the increasing acoustic components (see § 5). Meanwhile, the low-rank feature is weakened by a rising Re_{τ} , suggesting reduced spatial-temporal coherence as turbulence intensifies.

Regarding the eLNS results, figure 12 shows that the eLNS model largely disrupts the low-rank feature of the LNS forcing for scales $\lambda_x > \lambda_z$, by including the eddy-viscosity-enhanced terms. The large-scale fluctuations of high linear coherence are particularly affected, with $r_{\sigma,1}$ down to $\leq 30 \%$, so these scales are the region where the prediction of the linear coherence is mostly improved by the eLNS model over LNS. For other scales ($\lambda_x < \lambda_z$, or small λ_z), the improvement by eLNS is not obvious. Further improvements are anticipated by optimising the μ_t profile for different scales, as realised by Symon *et al.* (2023) and Ying *et al.* (2024*a*) for incompressible flows, and by considering inter-variable correlations.



Figure 13. Energy ratio of the leading POD mode $(r_{\sigma,1})$ for the LNS forcing with different λ_z for different cases (at largest $\lambda_x = 4\pi h$). Cases Ma00Re3k, Ma15Re3k and Ma30Re5k have comparable $Re_{\tau}^* = 141 \sim 186$; cases Ma15Re9k and Ma00Re10k have higher $Re_{\tau}^* = 393$ and 547 (see table 2).

5. Role of acoustic modes

It is known that the fluctuations in compressible flows can be decomposed into the vortical, acoustic and entropy components (Kovasznay 1953). The latter two are absent in incompressible flows, and are primarily responsible for intrinsic compressibility effects. The acoustic modes were shown to result in distinct features of the linear models from the incompressible counterpart (Jeun, Nichols & Jovanović 2016; Bae *et al.* 2020; Madhusudanan *et al.* 2025), such as centred fluctuations around the relative sonic line and notable Mach-wave radiation into the freestream. Particularly, Chen *et al.* (2023*b*) found that overlypredicted acoustic modes can severely degrade the performance of stochastic linear models in estimating wall-normally coherent density and temperature fluctuations (i.e. the components of $\langle \hat{q}'' \hat{q}''^H \rangle$). When using (2.15) as the forcing model, for example, they showed that the predicted acoustic and entropy modes can contribute up to 55 % of the response growth. Although, they pointed out that such a high energy portion contradicted the DNS data (their figures 5 and 14), the real energy portion was not revealed.

Here, we assess quantitatively the role of the acoustic components within the current linear-model framework using the DNS data. We first seek to extract the acoustic components out of $\langle \hat{q}'' \hat{q}''^H \rangle$ and the forcing matrix, and then discuss the key characteristics of these acoustic components.

5.1. Pressure decomposition for acoustic fluctuations

One viable path to extract the acoustic fluctuations is to first decompose the pressure fluctuation p' based on the pressure Poisson equation, and then formulate other acoustic variables based on the compressible part of p'. The first step is conducted in this subsection.

Following Sarkar (1992) and Zhang *et al.* (2024), p' is decomposed into the quasiincompressible part p'_{ic} and the compressible part p'_{c} . The governing equations for the two are obtained by classifying the right-hand-side terms of the pressure Poisson equation, as

$$\frac{\partial^2 p'_{ic}}{\partial x_i \partial x_i} = -2 \frac{\partial \tilde{u}_i}{\partial x_j} \frac{\partial \rho u''_j}{\partial x_i} - \frac{\partial^2}{\partial x_i \partial x_j} \left(\rho u''_i u''_j - \bar{\rho} \widetilde{u''_i u''_j} \right) + \frac{\partial^2 \tau'_{ij}}{\partial x_i \partial x_j},$$
(5.1*a*)

$$\frac{\partial p_c'}{\partial x_i \partial x_i} = \frac{\partial^2 \rho'}{\partial t^2} - \frac{\partial^2 (\rho' \tilde{u}_i \tilde{u}_j)}{\partial x_i \partial x_j} - \tilde{u}_i \frac{\partial^2 (2\rho u_j'')}{\partial x_i \partial x_j} - \frac{\partial}{\partial x_i} \left(2\rho u_i'' \frac{\partial \tilde{u}_j}{\partial x_j} \right).$$
(5.1b)

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Figure 14. (a) Variance and correlation of pressure fluctuation components for cases Ma15Re3k and Ma30Re5k. (b-e) Pre-multiplied 2-D spectra for the (b,d) incompressible and (c,e) compressible parts of the wall pressure fluctuations for cases (b,c) Ma15Re3k and (d,e) Ma30Re5k. The black dashed line in the right four panels denotes $\lambda_x = \lambda_z$.

The boundary conditions are $\partial p'_{ic}/\partial y = \partial \tau'_{i2}/\partial x_i$ and $\partial p'_c/\partial y = 0$ at both wall sides (Yu, Xu & Pirozzoli 2020). For incompressible flows, p'_c naturally reduces to zero since $\rho' = \partial \tilde{u}_j/\partial x_j = \partial u''_j/\partial x_j = 0$. The part p'_{ic} can be further decomposed (into a rapid term, etc.), but here, we simply treat it as a whole, as we focus on the compressibility effects. In this sense, p'_{ic} reflects the collective contribution of vortical modes, while p'_c is mainly contributed by acoustic modes. Equation (5.1) is solved using the Fourier–Galerkin scheme. Since the temporal derivative in (5.1*b*) is not available in our time-not-resolved data, we obtain p'_c simply from $p'_c = p' - p'_{ic}$ after solving p'_{ic} from (5.1*a*). The accuracy of p'_c is reliable because the realistic p' and that solved from the Poisson equation were shown to differ by less than 1 % for similar channel cases (Yu *et al.* 2020).

The variances of the pressure and its components for cases Ma15Re3k and Ma30Re5k are plotted in figure 14(*a*), where the normalisation is $p'^+ = p'/\tau_w$. After the decomposition, the $p'^+_{ic,rms}$ in the two cases are close to each other under the same Re^*_{τ} , suggesting robust contributions from the quasi-incompressible part. If p'^+_{ic} is regarded as the result at $Ma_b = 0$, then the wall pressure variance roughly follows a linear relation

$$\overline{p_w'^{+2}}(Ma_b) \approx \overline{p_w'^{+2}}(0) + 0.216Ma_b^2, \quad \text{for } Re_\tau^* \approx 140.$$
 (5.2)

This scaling is consistent with Yu *et al.* (2020) although the slope differs due to a different *Re*. With *Mab* increased, $p'^+_{c,rms}$ quickly rises and surpasses $p'^+_{ic,rms}$ at $Ma_b = 3$ at the wall. Moreover, $p'^+_{c,rms}$ is mostly amplified near the wall, suggesting locally intensified fluctuations of mass flux. The correlation $p'^+_{ic} p'^+_{c}$ is also plotted, which remains small (<0.2) throughout the field for both cases.

Furthermore, the pre-multiplied 2-D spectra of \hat{p}'_{ic} and \hat{p}'_{c} at the wall are displayed in figure 14. The pressure spectra in compressible flows have been comprehensively studied before (Bernardini & Pirozzoli 2011; Duan, Choudhari & Zhang 2016). Our intention here is to provide further support to the decomposition (5.1), by demonstrating the resemblance of the \hat{p}'_{ic} spectra under different Ma_b and the clear scale separation between \hat{p}'_{ic} and \hat{p}'_{c} . The part \hat{p}'_{c} resides at much smaller streamwise scales than \hat{p}'_{ic} , taking the form of

travelling waves, so \hat{p}'_c turns out to be spanwiseelongated with the peak locating at $(\lambda_x, \lambda_z)/h \approx (0.5, 1)$ for case Ma30Re5k.

5.2. Variance and structure of acoustic components

The \hat{p}'_c obtained in § 5.1 can be utilised to recover the other variables of acoustic modes. From 1-D eigenmode analysis, the \hat{q}'' of acoustic modes, denoted as \hat{q}''_{ac} , is related to \hat{p}'_c as

$$\hat{\boldsymbol{q}}_{ac}^{\prime\prime} = \left[\rho_{ac}^{\prime}, \ \boldsymbol{u}_{ac}^{\prime\prime}, \ \boldsymbol{v}_{ac}^{\prime\prime}, \ \boldsymbol{w}_{ac}^{\prime\prime}, \ \boldsymbol{T}_{ac}^{\prime\prime}\right]^{T} = \frac{\boldsymbol{C}_{vis}}{\gamma \, \bar{p}} \left[\bar{\rho}, \ \frac{k_{x}a}{k}, -\frac{\mathrm{i}a}{k} \frac{d}{\mathrm{d}y}, \ \frac{k_{z}a}{k}, \ (\gamma - 1)\widetilde{T}\right]^{T} \hat{p}_{c}^{\prime},$$
(5.3)

where $k^2 = k_x^2 + k_z^2$. Derivation of (5.3) is detailed in Appendix C. Consequently, the acoustic part $\langle \hat{q}''_{ac} \hat{q}''_{ac} \rangle \equiv \langle \hat{q}'' \hat{q}''^H \rangle_{ac}$ can be extracted out of $\langle \hat{q}'' \hat{q}''^H \rangle$, based on $\langle \hat{p}'_c \hat{p}'_c^H \rangle$ and (5.3). The residual $\langle \hat{q}'' \hat{q}''^H \rangle_{nac} = \langle \hat{q}'' \hat{q}''^H \rangle - \langle \hat{q}'' \hat{q}''^H \rangle_{ac}$ is regarded as the non-acoustic part, which mostly comes from vortical modes although it also includes weak vortical–acoustic coupling. In the following, the relative contributions of $\langle \hat{q}'' \hat{q}''^H \rangle_{ac}$ and $\langle \hat{q}'' \hat{q}''^H \rangle_{nac}$ are assessed, and the two are further utilised to decompose the forcing, which will uncover the quantitative contribution of the acoustic components in the stochastic linear model.

Equation (5.3) explicitly indicates the scale dependence of \hat{q}''_{ac} . The ratios $\hat{\rho}'_{ac}/\hat{p}'_{c}$ and $\hat{T}''_{ac}/\hat{p}'_{c}$ turn out to be nearly independent of k_x and k_z , so the spectral distributions of $\hat{\rho}'_{ac}$ and \hat{T}''_{ac} are primarily determined by \hat{p}'_{c} and the mean flow. Regarding the velocity components, the relation $\hat{u}''_{ac} \ll \hat{w}''_{ac}$ holds if $k_x \ll k_z$, so the acoustic motion for streamwise-elongated structures is mainly active in the y-z plane with velocities \hat{v}''_{ac} and \hat{w}''_{ac} . Conversely, the acoustic motion is constrained in the x-y plane for spanwise-elongated structures when $k_x \gg k_z$.

The \hat{p}'_c value in the two $Ma_b = 1.5$ cases is relatively low, so we focus on case Ma30Re5k to see the potential effects of acoustic modes. Before inspecting \hat{q}''_{ac} , we first examine the spectral distribution of the relative strength of \hat{p}'_c and \hat{p}'_{ic} . To reflect their amplitudes at different scales, the energy norm for pressure is defined as (Chu 1965)

$$V_{p} = (\hat{p}', \, \hat{p}')_{E} = \int_{0}^{2h} \left[\bar{\rho} a^{2} \frac{\hat{p}'^{\dagger} \hat{p}'}{(\gamma \, \bar{p})^{2}} \right] \mathrm{d}y.$$
(5.4)

Figure 15 provides the distributions of three norm ratios of the pressure components among all scales: the compressible part $r_{p_c} = (\hat{p}'_c, \hat{p}'_c)_E/V_p$, the incompressible part $r_{p_i} = (\hat{p}'_{ic}, \hat{p}'_{ic})_E/V_p$ and their coupling $r_{p_cp_i} = (\hat{p}'_c, \hat{p}'_{ic})_E/V_p$. Based on these ratios, three distinct regions in terms of the horizontal scales (λ_x, λ_z) can be divided, as denoted by the dashed lines (see figure 15c). The first region (I) is of either small λ_x^+ or λ_z^+ (≤ 30), where r_{p_c} exceeds 40 % and r_{p_i} falls below 50 % and can down to 10 %. These fluctuations reside near the wall where p'_c is the strongest, so they are most affected by acoustic modes, resulting in deviations from their incompressible counterpart. The second region (II) comprises the streamwise-elongated fluctuations of high linear coherence $(\lambda_z^+ > 30)$. They are nearly unaffected by p'_c with r_{p_i} over 90 %, so they are dominated by vortical modes. The slightly higher r_{p_c} for the largest scale in the domain $(\lambda_x, \lambda_z)/h = (4\pi, 2\pi)$ may be attributed to the artificial scale truncation by the domain size. In comparison, the spanwise-elongated ones in region III feature higher $r_{p_c} \gtrsim 20$ % and stronger $\hat{p}'_c - \hat{p}'_i$ coupling with $|r_{p_cp_i}|$ over 10 %, which is in line with the active zone of $p'_{c,w}$ in figure 14(e).

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Figure 15. Energy norm ratios of (a) the compressible part $\langle \hat{p}'_c \hat{p}'_c^H \rangle$, (b) the incompressible part $\langle \hat{p}'_{ic} \hat{p}'_{ic}^H \rangle$ and (c) their coupling $\langle \hat{p}'_c \hat{p}'_{ic}^H \rangle$ relative to $\langle \hat{p}' \hat{p}'^H \rangle$ for case Ma30Re5k. Three featured regions (I, II, III) are divided by the three black dashed lines $\lambda_x = \lambda_z$, $\lambda_x^+ = 30$ and $\lambda_z^+ = 30$. Extra contours outside the shaded levels are shown in grey dashed lines with labelled levels.



Figure 16. (a) Pre-multiplied 2-D spectrum of the acoustic energy $V_{q_{ac}}$ (normalised by $\rho_b U_b^2/h$) and the energy-norm ratios of (b) $\langle \hat{q}'' \hat{q}''^H \rangle_{ac}$ and (c) $\langle \hat{\rho}' \hat{\rho}'^H \rangle_{ac}$ relative to $\langle \hat{q}'' \hat{q}''^H \rangle$ and $\langle \hat{\rho}' \hat{\rho}'^H \rangle$, respectively, for case Ma30Re5k. Three regions (I, II, III) are divided as in figure 15. The dotted lines in panel (a) are the contours (0.2, 0.4, 0.6, 0.8) of the normalised pre-multiplied spectrum of the total energy V_q .

Next, the variance of \hat{q}_{ac}'' , i.e. $V_{qac} = \|\hat{q}_{ac}''\|^2$, is computed, which reflects how much energy is contributed by acoustic modes to the total fluctuation energy in the linear models. The pre-multiplied 2-D spectrum of V_{qac} is first plotted in figure 16(*a*), along with the V_q spectrum to show the region of energetic total fluctuations. The energy V_{qac} is more pronounced at scales $\lambda_x < \lambda_z$, consistent with figure 14(*e*). The ratio $r_{qac} = V_{qac}/V_q$ for all scales is displayed in figure 16(*b*), whose pattern is qualitatively the same as in figure 15(*a*). The three distinct regions (I, II, III) analogous to figure 15 can be identified. The near-wall fluctuations of either small λ_x^+ or λ_z^+ (<30) are most susceptible to acoustic modes, with $r_{qac} > 10$ %. This is consistent with Chen *et al.* (2023*a*) that the near-wall streak in the buffer layer, becomes non-existent with the rise of Ma_b . In other words, the travelling-wave-like acoustic components disrupt the near-wall scaling of streamwise-elongated streaks and tend to dominate the near-wall turbulent motions.

When λ_x^+ , $\lambda_z^+ > 30$, $r_{q_{ac}}$ is generally less than 2% for this $Ma_b = 3$ case, indicative of minor contributions from acoustic modes and dominance of vortical modes. This fact justifies the post-processing procedure of the cospectrum decomposition designed by Chen *et al.* (2023*b*), where the acoustic components are artificially removed from $\langle \hat{q}'' \hat{q}''^H \rangle$ to improve the prediction of the linear coherence spectra of velocity and temperature in the eLNS model. The overly amplified acoustic components before the processing ($r_{q_{ac}}$ can



Figure 17. Profiles of the total density fluctuation variance and those contributed by the acoustic and nonacoustic parts for scales of (a) $(\lambda_x, \lambda_z)/h = (0.6, 6.3)$, (b) $(\lambda_x, \lambda_z)/h = (12.6, 6.3)$ and (c) $(\lambda_x, \lambda_z)/h = (12.6, 2.1)$ for case Ma30Re5k. Those predicted by the SRA relation (2.16) are also shown.

reach 20%-55% in their eLNS model under (2.15)) is due to misordered energies of the POD modes subject to an inaccurate forcing model.

Regarding different components of \hat{q}''_{ac} , the density fluctuation is most affected by the acoustic components. As shown in figure 16(*c*), $r_{\rho_{ac}} = \|\hat{\rho}'_{ac}\|^2 / \|\hat{\rho}'\|^2$ can surpass 10% for spanwise-elongated fluctuations (region III). In comparison, the corresponding maximum ratios of the velocity components are all less than 3%, so the velocities are dominated by vortical modes.

Considering the sensitivity of $r_{\rho_{ac}}$ to acoustic modes, the wall-normal structure of $\hat{\rho}'_{ac}$ is presented in figure 17 using three selective scales. The first scale is $(\lambda_x, \lambda_z)/h = (0.6, 6.3)$, corresponding to the peak of $r_{q_{ac}}$ (and $r_{\rho_{ac}}$) in figure 16(b) in region III. The other two scales are both streamwise-elongated ones with $\lambda_x = 12.6h$ and in different λ_z . In figures 17(b) and 17(c), $\hat{\rho}'_{ac}$ is much smaller than $\hat{\rho}'_{nac}$, and the latter can be well approximated by the SRA relation (2.16). Therefore, $\hat{\rho}'$ is transported like a passive scalar controlled by the mean-flow gradients for these streamwise-elongated motions (Coleman *et al.* 1995). Compared with the temperature part, the SRA for density has received less attention. Its effectiveness for these energetic motions also supports the usage of (2.15) for the density and temperature components. In figure 17(a) where $r_{q_{ac}}$ peaks, however, $\hat{\rho}'_{ac}$ is more prominent and is higher than $\hat{\rho}'_{nac}$ near the wall. The $\hat{\rho}'$ at the wall is almost solely contributed by $\hat{\rho}'_{ac}$, leaving the residual $\hat{\rho}'_{nac,w}$ nearly zero. This is reasonable because $\hat{\rho}'_{nac}$ is induced by vortical modes, then $\hat{\rho}'_{nac}$ at this scale because the localised spanwise motion (\hat{w}'') is active for these spanwise-elongated fluctuations.

The above decomposition of $\langle \hat{q}'' \hat{q}''^H \rangle$ into the acoustic and non-acoustic elements enables a decomposition of $(FF^H)_{DNS}$ and $(EE^H)_{DNS}$, which aids the respective modelling of the acoustic and non-acoustic parts of the forcing. From (2.19), the LNS forcing can be decomposed into $(FF^H)_{DNS} = (FF^H)_{DNS,ac} + (FF^H)_{DNS,nac}$, with the two parts satisfying

$$\left(\boldsymbol{F}\boldsymbol{F}^{H}\right)_{DNS,ac} = -\left(\widehat{\mathcal{L}}_{q} \left\langle \hat{\boldsymbol{q}}^{\prime\prime} \hat{\boldsymbol{q}}^{\prime\prime H} \right\rangle_{ac} + \left\langle \hat{\boldsymbol{q}}^{\prime\prime} \hat{\boldsymbol{q}}^{\prime\prime H} \right\rangle_{ac} \widehat{\mathcal{L}}_{q}^{H}\right), \tag{5.5a}$$

$$(\boldsymbol{F}\boldsymbol{F}^{H})_{DNS,nac} = -(\widehat{\mathcal{L}}_{q}\langle \hat{\boldsymbol{q}}^{''} \hat{\boldsymbol{q}}^{''H} \rangle_{nac} + \langle \hat{\boldsymbol{q}}^{''} \hat{\boldsymbol{q}}^{''H} \rangle_{nac} \widehat{\mathcal{L}}_{q}^{H}), \qquad (5.5b)$$

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Figure 18. Contours of the decomposed eLNS forcing $(EE^H)_{DNS}$ into the (*a*) non-acoustic part and (*b*) acoustic part for the scale $(\lambda_x, \lambda_z)/h = (0.6, 6.3)$ for case Ma30Re5k.

respectively. The decomposition for the eLNS forcing is defined likewise. Different from vortical modes featuring strong non-normality, acoustic modes are nearly perpendicular to each other. Taking the leading acoustic modes (lowest frequency $|\omega|$) as a representative, $(FF^H)_{DNS,ac}$ can be approximated as $-2\omega_{ac}\langle \hat{q}''\hat{q}''^H \rangle_{ac}$ since $\hat{\mathcal{L}}_q\check{q}''=\omega_{ac}\check{q}''$ (details in Appendix C). Thereby, it can be conjectured that $(FF^H)_{DNS,ac}$ resembles the distribution of $\langle \hat{q}''\hat{q}''^H \rangle_{ac}$ except in different amplitudes.

This resemblance also applies to $(\boldsymbol{EE}^{H})_{DNS,ac}$ because the eddy-viscosity-enhanced terms reside in the outer layer, while the acoustic components are mostly amplified near the wall (see figure 14). As a demonstration, $(\boldsymbol{EE}^{H})_{DNS,nac}$ and $(\boldsymbol{EE}^{H})_{DNS,ac}$ are depicted in figure 18 for the scale in figure 17(*a*), at which acoustic modes are prominent. It is observed that $(\boldsymbol{EE}^{H})_{DNS,ac}$ is indeed confined in the very near-wall region like $\langle \hat{\boldsymbol{q}}'' \hat{\boldsymbol{q}}''^{H} \rangle_{ac}$. The relative amplitudes of different components ($\langle \hat{e}'_{\rho} \hat{e}'^{H}_{\rho} \rangle_{ac}$, $\langle \hat{e}''_{\rho} \hat{e}''^{H}_{u} \rangle_{ac}$, etc.) are thus readily obtained in (5.3). Most of the eLNS forcing is produced by $(\boldsymbol{EE}^{H})_{DNS,nac}$ resulting from vortical modes, and so is the LNS forcing. The modelling of these non-acoustic parts can learn a lot from the various forcing models for incompressible flows, as introduced in § 1. Systematic improvements on the modelling of $(\boldsymbol{FF}^{H})_{DNS}$ and $(\boldsymbol{EE}^{H})_{DNS}$ for compressible flows will be explored in future works.

6. Summary

Modelling of the nonlinear forcing is critical to the predictability of the linear models based on the resolvent or input–output analyses. In this work, we employ elaborate DNS datasets for compressible turbulent channel flows with Ma_b reaching 3, and uncover the forcing statistics of stochastic linear models directly computed from DNS for the first time for such flows. The results straightforwardly explain the success and failure of current models and directly guide model improvements.

Both the compressible LNS and eLNS models are considered. The framework of stochastic Lyapunov equation is adopted, which does not need time-resolved data and thus largely relaxes the requirement on the number of DNS snapshots. First, we prove the self-consistency of the linear models, by showing that they produce the same correlation tensor as DNS if the DNS-computed forcing is the input. Second, we present the distributions of each nonlinear forcing term (listed in table 1) in the momentum and energy equations. In most wall-normal regions, the nonlinear terms are dominated by the fluctuations of the Reynolds stress and turbulent heat flux. Nonetheless, the PGNF term leads to increasingly strong $f''_{v,rms}$ below the buffer layer with the rise of Ma_b , which results from the near-wall density and temperature streaks. In the eLNS model, eRSF and eTHF were designed to partially model the colour of the forcing. However, we show that the resulting forcing e''_q is actually several times larger than f''_q in the outer layer. Consequently, the other unmodelled nonlinear fluctuations become minor, and e''_q can be more easily modelled than f''_q by featuring more robust composition and structures among scales.

Furthermore, we show that the LNS forcing matrix for streamwise-elongated fluctuations has relatively high coherence and low rank, which is very different from the diagonal full-rank forcing model in previous works. This low-rank feature is insensitive to Ma_b but is weakened by a rising Re_τ . The eLNS model largely disrupts the low rank of the forcing and adds to its diagonal dominance for the $\lambda_x > \lambda_z$ fluctuations. Therefore, these scales are the region where the prediction of the velocity and temperature fluctuations is mostly improved by the eLNS model over the LNS one. The forcing matrix for small- λ_x scales is indeed diagonally dominant, for both the LNS and eLNS models. However, off-diagonal inter-variable correlations (particularly the $\hat{\rho}' - \hat{T}'', \hat{u}' - \hat{v}''$ components) become prominent, which degrades the behaviour of the linear models.

Lastly, we quantitatively assess the acoustic components in the correlation tensor and the forcing matrix, which are absent in incompressible flows. Based on the pressure Poisson equation, the correlation tensor and forcing matrix are decomposed into the quasiincompressible part coming from vortical modes, and the compressible part contributed by acoustic modes. Three distinct regions in the spectral space are identified. The scales of either small λ_x^+ or λ_z^+ (<30) are most susceptible to acoustic modes. Streamwise-elongated fluctuations ($\lambda_x^+ > \lambda_z^+ > 30$) are dominated by vortical modes, and are nearly unaffected by acoustic modes. Their density and temperature fluctuations are well predicted by the SRA (2.16). In comparison, spanwise-elongated fluctuations ($\lambda_z^+ > \lambda_x^+ > 30$) receive higher acoustic energy, but V_{qac}/V_q is still less than 2 % at $Ma_b = 3$. Therefore, acoustic modes make minor contributions to the total fluctuation energy except at very small scales. The acoustic components of the forcing matrices resemble the distribution of $\langle \hat{q}'' \hat{q}''' h \rangle_{ac}$ except at different amplitudes. The remaining large part is contributed by vortical modes, whose modelling can be learned from the incompressible cases.

Future works will be paid to systematic improvements of the forcing models, combining the lessons learned from various incompressible models and the outcomes of this work for compressible flows.

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Data availability statement. The data that support the findings of this study are available from the corresponding author upon reasonable request.

Appendix A. Unspecified terms in (2.3) and the derivation of (2.19)

The expressions of the unspecified terms in (2.3) are

$$\begin{bmatrix} MMNF = -\left[\frac{\partial \rho' u_i''}{\partial t} + \tilde{u}_j \frac{\partial \rho' u_i''}{\partial x_j} + \rho' \left(u_i'' \frac{\partial \tilde{u}_j}{\partial x_j} + u_j'' \frac{\partial \tilde{u}_i}{\partial x_j}\right)\right],\\ PCNF = u_j'' \frac{\partial p'}{\partial x_j} - \overline{u_j'' \frac{\partial p'}{\partial x_j}} - \overline{u_j'' \frac{\partial \bar{p}}{\partial x_j}} + \tilde{u}_j \frac{\partial p'_{nln}}{\partial x_j},\\ EMNF = -\left[c_v \frac{\partial \rho' T''}{\partial t} + c_p \tilde{u}_j \frac{\partial \rho' T''}{\partial x_j} + \rho' c_p \left(T'' \frac{\partial \tilde{u}_j}{\partial x_j} + u_j'' \frac{\partial \tilde{T}}{\partial x_j}\right)\right].$$
(A1)

Regarding the derivation of (2.19), the following two equations are obtained directly from (2.13a):

$$\frac{\partial \hat{\boldsymbol{q}}''}{\partial t} \hat{\boldsymbol{q}}''^{H} = \widehat{\mathcal{L}}_{q} \hat{\boldsymbol{q}}'' \hat{\boldsymbol{q}}''^{H} + \hat{\boldsymbol{f}}_{q}'' \hat{\boldsymbol{q}}''^{H}, \qquad \hat{\boldsymbol{q}}'' \frac{\partial \hat{\boldsymbol{q}}''^{H}}{\partial t} = \hat{\boldsymbol{q}}'' \hat{\boldsymbol{q}}''^{H} \widehat{\mathcal{L}}_{q}^{H} + \hat{\boldsymbol{q}}'' \hat{\boldsymbol{f}}_{q}''^{H}.$$
(A2)

. . . .

Their summation leads directly to (2.19a) after taking the ensemble average. The derivation of (2.19b) is the same.

Appendix B. Data verification

Before processing the fluctuation data, we need to confirm that the terms and derivatives in (2.1), (2.3) are correctly computed, and N_t is large enough to ensure $\langle \partial \phi / \partial t \rangle \approx 0$. Therefore, we first calculate the budget of (2.1) for all the datasets, as shown in figure 19 for cases Ma30Re5k and Ma15Re9k. Note that, in the DNS, a spatially uniform body force $f = \tau_w$ is added in the streamwise momentum equation to ensure fixed mass flux (Huang *et al.* 1995). The sums of all terms are all close to zero for different datasets except for minor discretisation errors, which demonstrates the reliability of the present calculations. The verification for the computation of the forcing and linear operators has been presented in figure 1.

Appendix C. Derivation for the shapes of acoustic modes

For a uniform inviscid freestream $[\bar{\rho}, \tilde{u}, 0, 0, \tilde{T}]^T$, the linear operator has four eigenmodes $\mathcal{L}_q \check{q}'' = -i\omega \check{q}''$, related to vortical modes, slow and fast acoustic modes and entropy modes, respectively. The eigenvalues of slow and fast acoustic modes are $\omega = k_x \tilde{u} \pm |\mathbf{k}|a$, and their eigenfunctions $(\mathbf{q}'' = \check{\mathbf{q}}'' \exp[i(k_x x + k_y y + k_z z)])$ take the form

$$\check{\boldsymbol{q}}_{ac,uni}^{\prime\prime} = \frac{\check{p}^{\prime}}{\gamma \bar{p}} \left[\pm \bar{\rho}, \ \frac{k_x}{|\boldsymbol{k}|} a, \ \frac{k_y}{|\boldsymbol{k}|} a, \ \frac{k_z}{|\boldsymbol{k}|} a, \ (\gamma - 1) \widetilde{T} \right]^T, \tag{C1}$$

where k_y is a virtual wall-normal wavenumber and $|\mathbf{k}|^2 = k_x^2 + k_y^2 + k_z^2$. In the presence of channel walls, the boundary condition requires $\check{v} = 0$ at y = 0, 2h, so $\check{v} \sim i \sin(n\pi y/2h)$



Figure 19. Mean-flow budgets of the (a,c) streamwise momentum equation (normalised by τ_w) and (b,d) enthalpy equation (normalised by ϑ_w) for cases (a,b) Ma30Re5k and (c,d) Ma15Re9k.

and $\check{u}, \check{w}, \check{\rho}, \check{T}, \check{p} \sim \cos(n\pi y/2h)$, where n = 0, 1, 2, ... As a result, the eigenfunction $(\boldsymbol{q}'' = \check{\boldsymbol{q}}''(y) \exp[i(k_x x + k_z z)])$, as in (5.3), is expressed as

$$\check{\boldsymbol{q}}_{ac,inv} = \frac{\check{p}}{\gamma \bar{p}} \left[\pm \bar{\rho}, \ \frac{k_x}{k} a, -\frac{\mathrm{i}a}{k} \frac{\mathrm{d}}{\mathrm{d}y}, \ \frac{k_z}{k} a, \ (\gamma - 1)\widetilde{T} \right]^T.$$
(C2)

Note that n = 0 is further adopted in (C2). This lowest-order mode was shown to contain the highest energy compared with other higher-order modes because its eigenvalue is the closest to the vortical and entropy modes in the eigen-spectrum (e.g. Chen *et al.* 2023*b*). Although the wall-normal variation of the mean flow is not considered in deriving the analytical (C2), this simple relation will be shown to be highly accurate in most wallnormal regions.

Equation (C2) is not compatible with the viscous wall boundary which is no slip and isothermal. This inconsistency leads to non-zero $\check{u}_{i,w}^{"}$, $\check{w}_{i,w}^{"}$ and $\check{T}_{w}^{"}$ for $\langle \hat{q}^{"} \hat{q}^{"H} \rangle_{nac}$, which can disrupt the decomposition of the forcing based on the full NS operator. To solve this problem, an acoustic boundary layer is required. Learning from Pierce (2019, chapter 10), a simple exponential profile is assumed within the boundary layer to damp the finite $\check{u}_{i}^{"}$

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Figure 20. Eigenfunctions (normalised by the energy norm) of the leading fast acoustic mode for fluctuations of (a,b) (λ_x , λ_z)/h = (12.6, 1.3) with $\omega h/U_b \approx 3.1$ and (c,d) (λ_x , λ_z)/h = (0.6, 6.3) with $\omega h/U_b \approx 17.5$ for case Ma30Re5k: (a,c) pressure from both the inviscid and viscous linear operators, and (b,d) the components from the viscous linear operator and from (5.3).

and \check{T}'' to zero at the wall. The damping function, applied to \check{u}'' , \check{w}'' and \check{T}'' , is

$$c_{vis} = 1 - \exp\left[-(1 - i)\frac{|y - y_w|}{\delta_{vort}}\right], \qquad \delta_{vort} = \sqrt{\frac{2\bar{\mu}}{\omega_{vort}\bar{\rho}}}, \tag{C3}$$

where δ_{vort} is a characteristic boundary layer thickness, and the frequency of vortical modes is approximated as $\omega_{vort} \approx k_x U_b$. The introduction of (C3) finally leads to the viscous relation in (5.3).

Equation (5.3) is examined in figure 20 for two sets of wavenumbers as in figure 17. Here, the eigenfunctions of the leading fast acoustic mode are displayed, which is selected from the eigen-spectrum of \mathcal{L}_q with the minimum $|\omega| \approx k_x \tilde{u}_c + ka_c$. The results for slow acoustic modes are quantitatively similar. Figures 20(*a*) and 20(*c*) demonstrate that including viscosity or not in \mathcal{L}_q negligibly affects the shape function of \check{p}' , validating the additional formulation of the acoustic boundary layer. The shape functions of velocity, density and temperature from \mathcal{L}_q , and from (5.3) and \check{p}' , are compared in figures 20(*b*) and 20(*d*). Equation (5.3) is shown to estimate well all the five components of \check{q}'' in the outer layer for different (λ_x , λ_z). Using (C3) leads to slightly larger deviations near the wall, but it does not deteriorate the evaluation of $\langle \hat{q}'' \hat{q}'' H' \rangle_{ac}$ in § 5.2.

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