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DIRICHLET FINITE BIHARMONIC FUNCTIONS ON THE PLANE WITH DISTORTED METRICS

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1. The Laplace-Beltrami operator Δ on a smooth manifold M with a smooth Riemannian metric $ds^2 = \sum_{i,j} g_{ij}(x)dx^i dx^j$ applied to a smooth function φ takes the form $\Delta\varphi = g^{-1/2}\sum_{i,j}(g^{1/2}g^{ij}\varphi_{x^j})_{x^i}$. Functions in the class $H^2(M) = \{u \in C^4(M);\ \Delta^2 u = 0\}$ are called biharmonic. The class $H(M) = H^1(M) = \{u \in C^2(M);\ \Delta u = 0\}$ of harmonic functions is a subclass of $H^2(M)$. Let $D(M)$ be the class of functions φ on M having square-integrable gradients, i.e. the Dirichlet integrals $D_M(\varphi) = \int_M |\text{grad}\,\varphi|^2 * 1$ are finite. In contrast with the harmonic null class $\mathcal{O}_{HD} = \{M;\ HD(M) = \boldsymbol{R}\}$, $\boldsymbol{R}$ being the real number field (cf. Sario-Nakai [3]), we consider the biharmonic null class

$$(1) \qquad \mathcal{O}_{H^2D} = \{M;\ H^2D(M) = HD(M)\}.$$

This class was introduced and intensively studied by Nakai-Sario [1]. One of the main questions concerning the class (1) is: Does the property $M \in \mathcal{O}_{H^2D}$ have anything to do with the harmonic degeneracy of the ideal boundary of M?

Let $\boldsymbol{D}$ be the unit disk $|z| < 1$ and $\boldsymbol{D}_\alpha$ be the disk $\boldsymbol{D}$ equipped with the Riemannian metric

$$ds = (1 - |z|)^{-\alpha}|dz|.$$

Nakai-Sario [1] proved

THEOREM 1. *The manifold $\boldsymbol{D}_\alpha$ belongs to the null class $\mathcal{O}_{H^2D}$ if and only if $\alpha \geq 3/4$.*

The case $\alpha = 3/4$ was supplemented by O'Malla [2]. The significance of this assertion lies in an interesting contrast with the harmonic case:

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$D_\alpha \notin \mathcal{O}_{HD}$ for every α. Let C be the finite plane $|z| < \infty$ and C_α be the plane C equipped with the Riemannian metric

$$ds = (1 + |z|)^{-\alpha}|dz|,$$

a counter part of D_α. Nakai-Sario [1] also proved that $C_0 = C \in \mathcal{O}_{H^2D}$ and $C_\alpha \notin \mathcal{O}_{H^2D}$ if α is chosen large enough. Again its significance is revealed in an interesting contrast with the harmonic case: $C_\alpha \in \mathcal{O}_{HD}$ for every α. Although the existence of α with $C_\alpha \notin \mathcal{O}_{H^2D}$ was assured in [1], its exact determination, which may be useful for e.g. producing a more delicate examples, was left unsettled. Therefore the main object of this paper is to establish a counterpart of the above Theorem 1:

THEOREM 2. *The manifold C_α belongs to the null class $\mathcal{O}_{H^2D}$ if and onlg if $\alpha \leq 3/2$.*

2. We denote by Δ_α, dv_α, and grad_α the Laplace-Beltrami operator, the volume element, and the gradient with respect to the Riemannian manifold C_α. Let Δ, dv, grad, and C stand for the case $\alpha = 0$. By using $\lambda_\alpha(z) = (1 + |z|)^{-\alpha}$, we see that $\Delta_\alpha = \lambda_\alpha^{-2}\Delta$, $dv_\alpha = \lambda_\alpha^2 dv$, and $\text{grad}_\alpha = \lambda_\alpha^{-2}\,\text{grad}$. Therefore $H(C_\alpha) = H(C)$, $D(C_\alpha) = D(C)$, and $D_{C_\alpha}(\varphi) = D_C(\varphi)$. A fortiori the assertion $C_\alpha \notin \mathcal{O}_{H^2D}$ is equivalent to the Poisson equation

$$\Delta u(z) = \lambda_\alpha(z)^2 h(z) \tag{2}$$

having a nonharmonic (Euclidean) Dirichlet finite solution u on C for some harmonic function h. We denote by $H_\alpha(C)$ the class of such harmonic functions. Clearly the constant function 0 does not belong to $H_\alpha(C)$ but $H_\alpha(C) \cup \{0\}$ forms a vector space.

In order to prove Theorem 2, we only have to show that $H_\alpha(C) = \emptyset$ if and only if $\alpha \leq 3/2$. It will be convenient to provide a test for an $h \in H(C)$ to belong to $H_\alpha(C)$. We denote by $(f, g)_\alpha$ the inner product of f and g in $L^2(C_\alpha) = L^2(C, \lambda_\alpha^2 dv)$ and by (f, g) the $(f, g)_0$. Then we have (Nakai-Sario [1])

LEMMA 1. *A nonzero harmonic function h on C belongs to the class $H_\alpha(C)$ if and only if*

$$\sup_{\varphi \in C_0^1(C)} |(h, \varphi)_\alpha|^2 / D_C(\varphi) < \infty. \tag{3}$$

Here C_0^1 is the class of C^1-functions with compact supports. To prove Lemma 1 suppose $h \in H_\alpha(C)$, i.e. (2) has a solution $u \in D(C)$. For

$\varphi \in C_0^1(C)$, the Green formula yields $(h, \varphi)_\alpha = (\Delta u, \varphi) = -D_C(u, \varphi)$. By the Schwarz inequality, $|(h, \varphi)_\alpha|^2 \leq D_C(u) \cdot D_C(\varphi)$. Conversely suppose (3) is valid. Let $\mathscr{L}$ be the closure of $C_0^\infty(C)$ in $D(C)$ with respect to $D_C(\cdot)$. By the Riesz theorem, there exists $u \in \mathscr{L}$ such that $\ell(\varphi) = D_C(u, \varphi)$ for every $\varphi \in \mathscr{L}$ and in particular for every $\varphi \in C_0^\infty$, where ℓ is the bounded extension to $\mathscr{L}$ of $(h, \cdot)_\alpha$. Namely, $(h, \varphi)_\alpha = -D_C(u, \varphi)$ for every $\varphi \in C_0^\infty(C)$. By the Weyl lemma u is a genuine solution of (1) and also $u \in D(C)$.

3. Expand an $h \in H(C)$ into its Fourier series:

$$h(re^{i\theta}) = \sum_{n=0}^{\infty} r^n(a_n \cos n\theta + b_n \sin n\theta)\,, \qquad b_0 = 0 \tag{4}$$

for $r \in [0, \infty)$ and $\theta \in \boldsymbol{R}$. For the sake of simplicity we call $m(h) = \sup\{n;\ a_n^2 + b_n^2 \neq 0\} \leq \infty$ the *order* of h. We denote by E_k the class $\{h \in H(C);\ m(h) \leq k\}$ for $k = 0, 1, 2, \cdots$ and we set $E_k = \{0\}$ for $k = -1, -2, \cdots$, $E'_k = \{h \in E_k;\ h \not\equiv 0,\ a_0 = b_0 = 0\}$, for $k = 1, 2, \cdots$, and $E'_k = \emptyset$ for $k = 0, -1, -2, \cdots$. We first prove

LEMMA 2. *If $2\alpha > k + 2 \geq 3$, then $E'_k \subset H_\alpha(C)$.*

We only have to show that $r^n \cos n\theta$ and $r^n \sin n\theta$ belong to $H_\alpha(C)$ for every n with $1 \leq n < 2\alpha - 2$. Since the reasoning is the same, we only show that $r^n \cos n\theta \in H_\alpha(C)$. Let $\varphi \in C_0^\infty(C)$, and expand it into its Fourier series:

$$\varphi(re^{i\theta}) = \sum_{n=0}^{\infty} (a_n(r) \cos n\theta + b_n(r) \sin n\theta) \tag{5}$$

where $a_n(r)$ and $b_n(r)$ are all in $C_0^\infty[0, \infty)$. Observe that

$$D_C(\varphi) = \sum \pi\left(\int_0^\infty (a'_n(r)^2 + b'_n(r)^2) r\,dr + n^2 \int_0^\infty (a_n(r)^2 + b_n(r)^2)\frac{dr}{r}\right). \tag{6}$$

On the other hand we have

$$\begin{aligned}(h, \varphi)_\alpha &= \int_0^\infty \left(\int_0^{2\pi} \varphi(re^{i\theta}) \cos n\theta\, d\theta\right) r^{n+1}(1 + r)^{-2\alpha} dr \\ &= \pi \int_0^\infty a_n(r) r^{n+1} (1 + r)^{-2\alpha} dr\,.\end{aligned}$$

By the Schwarz inequality

$$|(h, \varphi)_\alpha|^2 \leq \pi^2 K_\alpha \cdot \int_0^\infty a_n(r)^2 \frac{dr}{r} \tag{7}$$

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where $K_\alpha = \int_0^\infty r^{2n+3}(1+r)^{-4\alpha}dr$ is finite if and only if $2\alpha > n+2 \geq 3$. By (6) and (7), we have (3) and $r^n \cos n\theta \in H_\alpha(C)$.

LEMMA 3. *If* $k+3 \geq 2\alpha$, *then* $H_\alpha(C) \subset E'_k$.

Let the Fourier expansion of $h \in H_\alpha(C)$ be given by (4) and suppose $a_n^2 + b_n^2 \neq 0$. For $t > 1$ let

$$\rho_t(r) = \begin{cases} (r-t^{1/2})^2(t-r)^2, & r \in [t^{1/2}, t]; \\ 0, & r \in [0,\infty) - [t^{1/2}, t], \end{cases} \tag{8}$$

which belongs to $C_0^1(0,\infty)$. Then the function

$$\varphi_t(re^{i\theta}) = \rho_t(r)(a_n \cos n\theta + b_n \sin n\theta)$$

belongs to $C_0^1(C)$. By an easy computation we find the universal positive constants A, B and $t_0 > 1$ such that

$$(h, \tau\varphi_t)_\alpha \geq A\tau t^{6+n-2\alpha}, \quad D_C(\tau\varphi_t) \leq B\tau^2 t^8 \tag{9}$$

for every $t > t_0$ and $\tau > 0$. If $6+n-2\alpha > 4$, then (9) implies that $|(h,\varphi_t)_\alpha|^2/D_C(\varphi_t) \to \infty$, which contradicts (3). If $6+n-2\alpha = 4$, then (9) takes the form

$$(h, \tau\varphi_t)_\alpha \geq A\tau t^4, \quad D_C(\tau\varphi_t) \leq B\tau^2 t^8 . \tag{10}$$

Let $\{t_\nu\}_{\nu=0}^\infty$ be a sequence of real numbers such that $t_\nu + \nu < t_{\nu+1}^{1/2}$. Next consider a sequence $\{\tau_\nu\}_{\nu=1}^\infty$ given by

$$\tau_\nu t_\nu^4 = \nu^{-1} \qquad (\nu = 1, 2, \cdots) . \tag{11}$$

We then consider a sequence $\{\Phi_\mu\}_{\mu=1}^\infty$ of functions Φ_μ in $C_0^1(C)$ given by

$$\Phi_\mu(re^{i\theta}) = \sum_{\nu=1}^{\mu} \tau_\nu \varphi_{t_\nu}(re^{i\theta}) . \tag{12}$$

By (10) and (11) we deduce that

$$(h, \Phi_\mu)_\alpha \geq A \sum_{\nu=1}^{\mu} \nu^{-1} . \tag{13}$$

By definition (12) we see that $(\partial\Phi_\mu/\partial x^i)^2 = \sum_{\nu=1}^{\mu}(\tau_\nu \partial\varphi_\nu/\partial x^i)^2$ $(i = 1, 2)$, and a fortiori $D_C(\Phi_\mu) = \sum_{\nu=1}^{\mu} D_C(\tau_\nu\varphi_\nu)$. Again by (10) and (11) we obtain

$$D_C(\Phi_\mu) \leq B \sum_{\nu=1}^{\mu} \nu^{-2} . \tag{14}$$

From (13) and (14) it follows that

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$$|(h, \Phi_\mu)_\alpha|^2 / D_C(\Phi_\mu) \geq (A^2/B)\left(\sum_{\nu=1}^{\mu} \nu^{-1}\right)^2 \Big/ \left(\sum_{\nu=1}^{\mu} \nu^{-2}\right) \to \infty$$

as $\mu \to \infty$, in violation of (3). Hence n must satisfy $6 + n - 2\alpha < 4$, i.e. $n + 2 < 2\alpha \leq k + 3$. Then $n \leq k$, and $H_\alpha(C) \subset E_k$. Because of Lemma 2

$$a_0 = h(re^{i\theta}) - \sum_{n=1}^{m(h)} r^n(a_n \cos n\theta + b_n \sin n\theta)$$

must belong to $H_\alpha(C)$ unless $a_0 = 0$. It is easy to find a bounded sequence $\{\varphi_\mu\}_1^\infty \subset C_0^1(C)$ such that φ_μ converges to 1 and $D_C(\varphi_\mu) \to 0$. If $a_0 \in H_\alpha(C)$, then $|(a_0, \varphi_\mu)|^2 \to \left(2\pi a_0 \int_0^\infty (1 + r)^{-2\alpha} r dr\right)^2 > 0$; but $D_C(\varphi_\mu) \to 0$ as $\mu \to \infty$, in violation of (3). Therefore $a_0 = 0$ and $h \in E'_k$, i.e. $H_\alpha(C) \subset E'_k$.

4. Suppose that $H_\alpha(C) = \emptyset$. If $2\alpha > 1 + 2 = 3$, then by Lemma 2, $E'_1 \subset H_\alpha(C)$, a contradiction. Therefore $2\alpha \leq 3$. Conversely suppose that $2\alpha \leq 3$, i.e. $0 + 3 \geq 2\alpha$. By Lemma 3 we see that $H_\alpha(C) \subset E'_0 = \emptyset$. Thus $H_\alpha(C) = \emptyset$ if and only if $\alpha \leq 3/2$. This completes the proof of Theorem 2.

5. Let u_1 and u_2 be Dirichlet finite solutions of (2). Then $u_1 - u_2$ is a Dirichlet finite harmonic function on C, i.e. $u_1 - u_2 \in HD(C) = \mathbf{R}$. Therefore the vector space $H^2D(C_\alpha)/\mathbf{R}$ is isomorphic to $H_\alpha(C) \cup \{0\}$. By Lemmas 2 and 3, $H_\alpha(C) \cup \{0\} = E'_k(2\alpha - 2 > k \geq 2\alpha - 3)$. Since $\dim E'_k = 2k$ for $k > 0$ and $= 0$ for $k \leq 0$, as a more precise form of Theorem 2, we obtain

THEOREM 3. *Let d_α be the dimension of the vector space $H^2D(C_\alpha)/HD(C_\alpha) = H^2D(C_\alpha)/\mathbf{R}$. If $\alpha \leq 3/2$, then $d_\alpha = 0$. If $\alpha > 3/2$, then $d_\alpha = 2k_\alpha$ with $2\alpha - 2 > k_\alpha \geq 2\alpha - 3$.*

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