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BLOWUP PROPERTIES FOR SEVERAL DIFFUSION SYSTEMS WITH LOCALISED SOURCES

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Abstract

This paper investigates the Cauchy problem for two classes of parabolic systems with localised sources. We first give the blowup criterion, and then deal with the possibilities of simultaneous blowup or non-simultaneous blowup under some suitable assumptions. Moreover, when simultaneous blowup occurs, we also establish precise blowup rate estimates. Finally, using similar ideas and methods, we shall consider several nonlocal problems with homogeneous Neumann boundary conditions.

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1. Introduction

In this paper, we study two classes of parabolic systems coupled with localised nonlinear sources of exponent type:

$$\begin{aligned} &u_t = \Delta u + \lambda e^{p_1 u(x_0(t),t)+q_1 v(x_0(t),t)}, && (x,t) \in \mathbb{R}^N \times (0,T),\\ &v_t = \Delta v + \mu e^{p_2 u(x_0(t),t)+q_2 v(x_0(t),t)}, && (x,t) \in \mathbb{R}^N \times (0,T),\\ &u(x,0) = u_0(x), \quad v(x,0) = v_0(x), && x \in \mathbb{R}^N, \end{aligned} \tag{1.1}$$

and of power type:

$$\begin{aligned} &u_t = \Delta u + u^{p_1}(x_0(t),t) v^{q_1}(x_0(t),t), && (x,t) \in \mathbb{R}^N \times (0,T),\\ &v_t = \Delta v + u^{p_2}(x_0(t),t) v^{q_2}(x_0(t),t), && (x,t) \in \mathbb{R}^N \times (0,T),\\ &u(x,0) = u_0(x), \quad v(x,0) = v_0(x), && x \in \mathbb{R}^N, \end{aligned} \tag{1.2}$$

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respectively, where $x_0 : \mathbb{R}^+ \to \mathbb{R}^N$ is Hölder continuous, and $\lambda, \mu > 0$, $p_i, q_i \geq 0$, $i = 1, 2$, are constants with $p_2 q_1 > 0$, which ensures that the equations in (1.1) and (1.2) are completely coupled, while $u_0(x)$ and $v_0(x)$ are nontrivial nonnegative continuous bounded functions in $\mathbb{R}^N$.

The equations in (1.1) and (1.2) describe chemical reaction-diffusion processes in which the nonlinear reaction in a dynamical system takes place only at a single (or sometimes several) site(s). As an example, the influence of defect structures on a catalytic surface can be modelled by a similar equation (see [1, 14]). Similar phenomena are also frequently observed in biological systems, for instance on chemically active membranes (see [5] and references therein). The additional motivation for this study comes from parabolic inverse problems and so-called nonclassical equations (see [2, 4]).

In recent years, a lot of effort has been devoted to localised/nonlocal problems, by which we mean that the problem is studied in a bounded domain with smooth boundary. For the following Cauchy problem with localised reactions:

$$\begin{aligned} &u_t = \Delta u + f(u(x_0(t), t)), && (x, t) \in \mathbb{R}^N \times (0, T), \\ &u(x, 0) = u_0(x), && x \in \mathbb{R}^N, \end{aligned} \tag{1.3}$$

Chadam *et al.* [3] proved the solution blows up in a finite time under the assumption that $x_0(t) \equiv 0$ and $u_0 \geq c > 0$. Souplet [17, 18] studied large classes of equations with localised/nonlocal reaction terms and described the blowup properties of the solution of (1.3) (see also [2, 15, 19]). In particular, Souplet in [18] proved that the solutions of (1.3) blow up globally and gave the uniform blowup rates in $\mathbb{R}^N$. For instance, if $f(u) = u^p$, then

$$((p-1)(T-t))^{-1/(p-1)} - C \leq u(x, t) \leq ((p-1)(T-t))^{-1/(p-1)} + C$$

in $\mathbb{R}^N \times [0, T)$ for some $C > 0$, where T is the maximal existence time of u.

As far as the system is concerned, Lin *et al.* in [13] considered the following problem:

$$\begin{aligned} &u_t = \Delta u + e^{v(x_0, t)}, && v_t = \Delta v + e^{u(x_0, t)}, && (x, t) \in \mathbb{R}^N \times (0, T), \\ &u(x, 0) = u_0(x), && v(x, 0) = v_0(x), && x \in \mathbb{R}^N. \end{aligned}$$

They gave the blowup conditions and showed that the blowup set is $\mathbb{R}^N$. Under the assumption that $x_0 = 0$ and initial data $u_0(x), v_0(x)$ are symmetric and radially decreasing, they also obtained the following estimates:

$$\begin{aligned} -\log(T-t) - v_0(0) \leq \sup u(x, t) = u(0, t) \leq -C \log(T-t) + C, \\ -\log(T-t) - u_0(0) \leq \sup v(x, t) = v(0, t) \leq -C \log(T-t) + C, \end{aligned} \tag{1.4}$$

for some constant $C > 0$ and all $0 \le t < T$. The corresponding Neumann boundary-value problem has also been investigated in [13].

In the case of $x_0(t) \equiv x_0$, systems (1.1) and (1.2) with initial-boundary data have also been investigated by a number of authors. For instance, Zhao and Zheng [21] and Li and Wang [11, 12] investigated (1.1) and (1.2) with homogeneous Dirichlet boundary conditions. They obtained the critical exponents and established the uniform blowup profiles in the interior. For other systems with space-integral sources, we refer to Li *et al.* [10]. Recently, Xiang *et al.* [20] also studied (1.1) and (1.2) with $p_2 = q_1 = 0$ subject to homogeneous Neumann boundary values.

We remark that in connection with the local semilinear parabolic systems

$$u_t = \Delta u + \lambda e^{p_1 u + q_1 v}, \qquad v_t = \Delta v + \mu e^{p_2 u + q_2 v},$$

and

$$u_t = \Delta u + u^{p_1} v^{q_1}, \qquad v_t = \Delta v + u^{p_2} v^{q_2}, \tag{1.5}$$

with initial conditions or initial-boundary values, a lot of work has been done in the past few years on the blowup of their solutions (see survey papers [6, 8, 9] and references therein).

Motivated by the above cited papers, in this paper, we investigate the blowup properties of solutions of the Cauchy problems (1.1) and (1.2) and extend the results of [13, 20] to more generalised cases. Using similar ideas, we shall also consider systems (1.1), (1.2) and some other nonlocal problems with homogeneous Neumann boundary conditions (see Section 4 for details). Note that the assumption $x_0(t) \equiv x_0$ in [11–13, 20, 21] is very important, which ensures that one may directly construct a super-solution or sub-solution to system (1.1) or (1.2). Moreover, for the Dirichlet problem, the estimates of $-\Delta u$, $-\Delta v$ are crucial to obtain the blowup rates. In the present case, however, the comparison principle is invalid since $x_0(t)$ can move with t and it seems to be hard to obtain the bounds of $-\Delta u$ and $-\Delta v$. To overcome these difficulties, we use some ideas of Souplet [18] and define a pair of functions $(\underline{u}, \underline{v})$, which are the integral of the reaction terms in time and depend only on the time variable t. More importantly, we find (u, v) is equivalent to $(\underline{u}, \underline{v})$ and then mainly investigate the properties of $(\underline{u}, \underline{v})$, which makes the arguments very concise (see Sections 2 and 3 for details).

The local existence of a nonnegative solution to problem (1.1) or (1.2) can be shown by standard methods [7, 15, 17], so we omit it here. First, we give the blowup properties of the nonnegative solution of problem (1.1).

THEOREM 1.1. *Assume* (u, v) *is a nonnegative solution of* (1.1), *then* (u, v) *blows up in a finite time. Moreover, the blowup set is the whole space* $\mathbb{R}^N$.

Let T be the maximal existence time of the solution (u, v) of system (1.1). Theorem 1.1 suggests $T < \infty$ and $\lim_{t\to T}(\|u\|_\infty + \|v\|_\infty) = +\infty$. However, *a priori* there is no reason for both components of the system to blow up simultaneously. In fact, it could happen that one of the components blows up as $t \to T$, while the other remains bounded on $[0, T)$. This phenomenon is called *non-simultaneous blowup*. To summarise, we have the following results.

THEOREM 1.2. *Assume (u, v) is a nonnegative solution of* (1.1). *The following conclusions hold*:

(i) *if $p_2 \geq p_1$ and $q_1 \geq q_2$, then u and v must blow up simultaneously*;

(ii) *if $p_2 < p_1$ and $q_1 \geq q_2$, or $p_2 \geq p_1$ and $q_1 < q_2$, then only non-simultaneous blowup occurs.*

THEOREM 1.3. *Assume (u, v) is a classical solution of problem* (1.1), *which blows up at a finite time T.*

(i) *If $p_2 > p_1 \geq 0$ and $q_1 > q_2 \geq 0$, then*

$$\lim_{t\to T}\frac{u(x,t)}{|\log(T-t)|} = \frac{q_1-q_2}{p_2q_1-p_1q_2}, \quad \lim_{t\to T}\frac{v(x,t)}{|\log(T-t)|} = \frac{p_2-p_1}{p_2q_1-p_1q_2}$$

uniformly in $\mathbb{R}^N$.

(ii) *If $p_2 > p_1 \geq 0$ and $q_1 = q_2 > 0$, then*

$$\lim_{t\to T}\frac{u(x,t)}{\log|\log(T-t)|} = \frac{1}{p_2-p_1}, \qquad \lim_{t\to T}\frac{v(x,t)}{|\log(T-t)|} = \frac{1}{q_2}$$

uniformly in $\mathbb{R}^N$.

(iii) *If $p_2 = p_1 > 0$ and $q_1 > q_2 \geq 0$, then*

$$\lim_{t\to T}\frac{u(x,t)}{|\log(T-t)|} = \frac{1}{p_1}, \quad \lim_{t\to T}\frac{v(x,t)}{\log|\log(T-t)|} = \frac{1}{q_1-q_2}$$

uniformly in $\mathbb{R}^N$.

(iv) *If $p_2 = p_1 > 0$ and $q_1 = q_2 > 0$, then there exist constants $C, c > 0$ such that*

$$-c\log(T-t) - C \leq u(x,t) \leq -C\log(T-t) + C, \quad x \in \mathbb{R}^N,\ t \to T,$$
$$-c\log(T-t) - C \leq v(x,t) \leq -C\log(T-t) + C, \quad x \in \mathbb{R}^N,\ t \to T.$$

REMARK. For $p_1 = q_2 = 0$, Theorem 1.3 (i) is sharper than (1.4). So we improve the blowup estimates in [13, 20]. Moreover, we remove the restriction that u_0, v_0 are symmetric or radially decreasing. We also permit $x_0(t) \in \mathbb{R}^N$ to move with t.

Next, we investigate problem (1.2). Throughout this paper, we take $D = p_2q_1 - p_1q_2 + p_1 + q_2 - 1$. If $D \neq 0$, we also define $\alpha = (q_1 - q_2 + 1)/D$, $\beta = (p_2 - p_1 + 1)/D$.

THEOREM 1.4. (i) *If* $p_1 < 1$, $q_2 < 1$ *and* $D \le 0$, *then all solutions of* (1.2) *exist globally.*

(ii) *If* $p_1 > 1$ *or* $q_2 > 1$, *or* $D > 0$, *then all solutions of* (1.2) *blow up in a finite time. Moreover, the blowup set is the whole space* $\mathbb{R}^N$.

REMARK. It is well known that for the local semilinear system (1.5) with Cauchy data there exists a finite Fujita critical exponent. However, Theorem 1.4 implies that for the localised problem (1.2) the Fujita critical exponent is $+\infty$.

THEOREM 1.5. (i) *If* $p_2 \ge p_1 - 1 > 0$ *and* $q_1 \ge q_2 - 1 > 0$, *or* $p_2 > p_1 - 1$ *and* $q_1 > q_2 - 1$ *and* $D > 0$, *then for the solution* (u, v) *of* (1.2), u *and* v *must blow up simultaneously*.

(ii) *If* $p_2 < p_1 - 1$ *and* $q_1 \ge q_2 - 1 > 0$, *or* $p_2 \ge p_1 - 1 > 0$ *and* $q_1 < q_2 - 1$, *then only non-simultaneous blowup occurs.*

THEOREM 1.6. *Assume* (u, v) *is a nonnegative solution of* (1.2), *which blows up at a finite time* T.

(i) *If* $p_2 > p_1 - 1$, $q_1 > q_2 - 1$ *and* $D > 0$, *then*

$$\lim_{t\to T}(T-t)^{\alpha}u(x,t) = \left(\frac{1}{\alpha}\left(\frac{\alpha}{\beta}\right)^{q_1/(q_1-q_2+1)}\right)^{-\alpha},$$

$$\lim_{t\to T}(T-t)^{\beta}v(x,t) = \left(\frac{1}{\beta}\left(\frac{\beta}{\alpha}\right)^{p_2/(p_2-p_1+1)}\right)^{-\beta},$$

uniformly in $\mathbb{R}^N$;

(ii) *If* $p_2 > p_1 - 1 > 0$ *and* $q_1 = q_2 - 1 > 0$, *then*

$$\lim_{t\to T}|\log(T-t)|^{-1/(p_2-p_1+1)}u(x,t) = \left(\frac{1}{q_1}(p_2-p_1+1)\right)^{1/(p_2-p_1+1)},$$

$$\lim_{t\to T}(T-t)v^{q_1}(x,t)(\log v(x,t))^{p_2/(p_2-p_1+1)} = \frac{1}{q_1}(p_2-p_1+1)^{-p_2/(p_2-p_1+1)},$$

uniformly in $\mathbb{R}^N$;

(iii) *If* $p_2 = p_1 - 1 > 0$ *and* $q_1 > q_2 - 1 > 0$, *then*

$$\lim_{t\to T}(T-t)u^{p_2}(x,t)\,(\log u(x,t))^{q_1/(q_1-q_2+1)} = \frac{1}{p_2}(q_1-q_2+1)^{-q_1/(q_1-q_2+1)},$$

$$\lim_{t\to T}|\log(T-t)|^{-1/(q_1-q_2+1)}v(x,t) = \left(\frac{1}{p_2}(q_1-q_2+1)\right)^{1/(q_1-q_2+1)},$$

uniformly in $\mathbb{R}^N$;

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(iv) *If* $p_2 = p_1 - 1 > 0$ *and* $q_1 = q_2 - 1 > 0$, *then*

$$\lim_{t\to T} |\log(T-t)|^{-1} \log u(x,t) = \lim_{t\to T} |\log(T-t)|^{-1} \log v(x,t) = \frac{1}{p_2+q_1},$$

uniformly in $\mathbb{R}^N$.

The results for the Neumann problem will be stated in Section 4. We shall also use C, c to denote various generic positive constants whenever there is no chance of confusion.

This paper is organised as follows. In Section 2, we consider problem (1.1) and prove Theorems 1.1–1.3. The proof of Theorems 1.4–1.6 is the subject of Section 3. Finally, in Section 4, we consider several nonlocal problems with homogeneous Neumann boundary values.

2. Blowup properties for problem (1.1)

In this section, we investigate the Cauchy problem (1.1). We denote

$$\underline{u}(x,t) = \underline{u}(t) = \lambda \int_0^t e^{p_1 u(x_0(s),s)+q_1 v(x_0(s),s)}\,ds, \quad \overline{u} = \underline{u} + C_0, \tag{2.1}$$

$$\underline{v}(x,t) = \underline{v}(t) = \mu \int_0^t e^{p_2 u(x_0(s),s)+q_2 v(x_0(s),s)}\,ds, \quad \overline{v} = \underline{v} + C_0, \tag{2.2}$$

where $C_0 = \|u_0\|_\infty + \|v_0\|_\infty$. It is clear that

$$\underline{u}_t - \Delta\underline{u} = \overline{u}_t - \Delta\overline{u} = \lambda e^{p_1 u(x_0(t),t)+q_1 v(x_0(t),t)} = u_t - \Delta u,$$
$$\underline{v}_t - \Delta\underline{v} = \overline{v}_t - \Delta\overline{v} = \mu e^{p_2 u(x_0(t),t)+q_2 v(x_0(t),t)} = v_t - \Delta v.$$

Since $\underline{u}(x,0) = 0 \le u_0(x) \le \overline{u}(x,0)$ and $\underline{v}(x,0) = 0 \le v_0(x) \le \overline{v}(x,0)$, we have $\underline{u} \le u \le \overline{u}$ and $\underline{v} \le v \le \overline{v}$ by the maximal principle as long as (u,v) exists. In particular, we obtain $\underline{u}(t) \le u(x_0(t),t) \le \overline{u}(t)$ and $\underline{v}(t) \le v(x_0(t),t) \le \overline{v}(t)$, which play important roles in the following proof.

PROOF OF THEOREM 1.1. It follows from $(u,v) \ge (\underline{u},\underline{v})$ that

$$\underline{u}_t = \lambda e^{p_1 u(x_0(t),t)+q_1 v(x_0(t),t)} \ge \lambda e^{p_1\underline{u}(t)+q_1\underline{v}(t)} \ge \lambda e^{q_1\underline{v}(t)}, \quad t>0,$$
$$\underline{v}_t = \mu e^{p_2 u(x_0(t),t)+q_2 v(x_0(t),t)} \ge \mu e^{p_2\underline{u}(t)+q_2\underline{v}(t)} \ge \mu e^{p_2\underline{u}(t)}, \quad t>0.$$

Therefore $(\underline{u}+\underline{v})_t \ge \gamma\left(e^{k\underline{u}(t)} + e^{k\underline{v}(t)}\right) \ge 2\gamma e^{k(\underline{u}+\underline{v})/2}$, $t>0$, where $\gamma = \min(\lambda,\mu)$, $k = \min(p_2,q_1)$. Then we have

$$-\left(e^{-k(\underline{u}+\underline{v})/2}\right)_t = \frac{k}{2} e^{-k(\underline{u}+\underline{v})/2}\left(\underline{u}+\underline{v}\right)_t \ge c, \quad t>0.$$

Integrating this inequality from 0 to t, we obtain $t \le C - Ce^{-(k/2)(\underline{u}+\underline{v})(t)} \le C$, which implies that $(\underline{u}+\underline{v})$ blows up in a finite time. Using $\underline{u} \le u$, $\underline{v} \le v$ again, we see that (u, v) blows up in a finite time and that the blowup set is the whole space $\mathbb{R}^N$. □

PROOF OF THEOREM 1.2. We denote by T the maximal existence time of (u, v). Theorem 1.1 implies $T < \infty$. As in the proof of Theorem 1.1, we have

$$\underline{u}_t = \lambda e^{p_1 u(x_0(t),t)+q_1 v(x_0(t),t)} \ge \lambda e^{p_1\underline{u}(t)+q_1\underline{v}(t)}, \quad t \in (0, T),$$
$$\underline{v}_t = \mu e^{p_2 u(x_0(t),t)+q_2 v(x_0(t),t)} \ge \mu e^{p_2\underline{u}(t)+q_2\underline{v}(t)}, \quad t \in (0, T).$$

On the other hand, by $\underline{u} = \overline{u} - C_0$, $\underline{v} = \overline{v} - C_0$, we also get

$$\underline{u}_t = \overline{u}_t \le \lambda e^{p_1(\underline{u}+C_0)+q_1(\underline{v}+C_0)} = \lambda e^{(p_1+q_1)C_0} e^{p_1\underline{u}+q_1\underline{v}}, \quad t \in (0, T),$$
$$\underline{v}_t = \overline{v}_t \le \mu e^{p_2(\underline{u}+C_0)+q_2(\underline{v}+C_0)} = \mu e^{(p_2+q_2)C_0} e^{p_2\underline{u}+q_2\underline{v}}, \quad t \in (0, T).$$

By the above inequalities, we see

$$ce^{(p_1-p_2)\underline{u}+(q_1-q_2)\underline{v}} \le \frac{d\underline{u}}{d\underline{v}} \le Ce^{(p_1-p_2)\underline{u}+(q_1-q_2)\underline{v}}, \quad t \in (0, T).$$

That is,

$$ce^{(q_1-q_2)\underline{v}}d\underline{v} \le e^{(p_2-p_1)\underline{u}}d\underline{u} \le Ce^{(q_1-q_2)\underline{v}}d\underline{v}, \quad t \in (0, T). \tag{2.3}$$

The proof of simultaneous blowup is divided into four cases.

Case (i): $p_2 > p_1 \ge 0$, $q_1 > q_2 \ge 0$. Integrating (2.3) from 0 to t, we have

$$c\left(e^{(q_1-q_2)\underline{v}} - 1\right) \le e^{(p_2-p_1)\underline{u}} - 1 \le C\left(e^{(q_1-q_2)\underline{v}} - 1\right), \quad t \in (0, T), \tag{2.4}$$

which implies that $\underline{u}$ and $\underline{v}$ blow up at the same time.

Case (ii): $p_2 > p_1 \ge 0$, $q_1 = q_2 > 0$. As for Case (i), integrating (2.3) on $[0, t]$ yields

$$c\underline{v} \le e^{(p_2-p_1)\underline{u}} - 1 \le C\underline{v}, \quad t \in (0, T). \tag{2.5}$$

Then we see $\underline{u}$ and $\underline{v}$ simultaneously blow up.

Case (iii): $p_2 = p_1 > 0$, $q_1 > q_2 \ge 0$. Similar to (ii), we have

$$c\underline{u} \le e^{(q_1-q_2)\underline{v}} - 1 \le C\underline{u}, \quad t \in (0, T).$$

Therefore $\underline{u}$ and $\underline{v}$ have the same blowup time.

Case (iv): $p_2 = p_1 > 0$, $q_1 = q_2 > 0$. We can deduce the following inequalities from (2.3):

$$c\underline{v}(t) \le \underline{u}(t) \le C\underline{v}(t), \quad t \in (0, T). \tag{2.6}$$

Hence, by Cases (i)–(iv), we have completed the proof of simultaneous blowup.

We now investigate non-simultaneous blowup. Firstly, we consider the case $p_2 < p_1$ and $q_1 \geq q_2$. Divide it into two subcases.

If $p_2 < p_1$ and $q_1 > q_2$, we integrate (2.3) on $[0, t]$ to obtain

$$\frac{c}{q_1 - q_2}\left(e^{(q_1-q_2)\underline{v}} - 1\right) \leq \frac{1}{p_2 - p_1}\left(e^{(p_2-p_1)\underline{u}} - 1\right), \quad t \in (0, T).$$

Recall that $\underline{u} \leq u \leq \underline{u} + C_0$ and $\underline{v} \leq v \leq \underline{v} + C_0$. If u and v, equivalently $\underline{u}$ and $\underline{v}$, blow up at the same time T, then the above inequality leads to a contradiction as we send t to T.

If $p_2 < p_1$ and $q_1 = q_2$, (2.3) is equivalent to $cd\underline{v} \leq e^{(p_2-p_1)\underline{u}(t)}d\underline{u} \leq Cd\underline{v}$, $t \in (0, T)$. Integrating from 0 to t, we obtain

$$c\underline{v}(t) \leq \frac{1}{(p_2 - p_1)}\left(e^{(p_2-p_1)\underline{u}(t)} - 1\right), \quad t \in (0, T).$$

As in the previous proof, we see that this inequality will lead to a contradiction if both u and v blow up at T.

In the case of $p_2 \geq p_1$ and $q_1 < q_2$, taking a similar procedure, we may prove the conclusion. The proof of Theorem 1.2 is completed. □

To complete the proof of Theorem 1.3, we recall the facts

$$\underline{u}(t) \leq u(x, t) \leq \overline{u}(t) \leq \underline{u}(t) + C_0, \quad x \in \mathbb{R}^N, \ t > 0,$$
$$\underline{v}(t) \leq v(x, t) \leq \overline{v}(t) \leq \underline{v}(t) + C_0, \quad x \in \mathbb{R}^N, \ t > 0.$$

So it will be enough to prove the following lemma.

LEMMA 2.1. (i) *If $p_2 > p_1 \geq 0$ and $q_1 > q_2 \geq 0$, then*

$$\lim_{t\to T} \frac{\underline{u}(t)}{|\log(T-t)|} = \frac{q_1 - q_2}{p_2 q_1 - p_1 q_2}, \quad \lim_{t\to T} \frac{\underline{v}(t)}{|\log(T-t)|} = \frac{p_2 - p_1}{p_2 q_1 - p_1 q_2}.$$

(ii) *If $p_2 > p_1 \geq 0$ and $q_1 = q_2 > 0$, then*

$$\lim_{t\to T} \frac{\underline{u}(t)}{\log|\log(T-t)|} = \frac{1}{p_2 - p_1}, \quad \lim_{t\to T} \frac{\underline{v}(t)}{|\log(T-t)|} = \frac{1}{q_2}.$$

(iii) *If $p_2 = p_1 > 0$ and $q_1 > q_2 \geq 0$, then*

$$\lim_{t\to T} \frac{\underline{u}(t)}{|\log(T-t)|} = \frac{1}{p_1}, \quad \lim_{t\to T} \frac{\underline{v}(t)}{\log|\log(T-t)|} = \frac{1}{q_1 - q_2}.$$

(iv) *If $p_2 = p_1 > 0$ and $q_1 = q_2 > 0$, then there exist constants $C, c > 0$ such that*

$$-c\log(T-t) - C \leq \underline{u}(t) \leq -C\log(T-t) + C, \quad t \in (0, T),$$
$$-c\log(T-t) - C \leq \underline{v}(t) \leq -C\log(T-t) + C, \quad t \in (0, T).$$

PROOF. (i) Since u and v simultaneously blow up, by (2.4) there exists T_0 such that

$$(q_1 - q_2)\underline{v} - C \le (p_2 - p_1)\underline{u} \le (q_1 - q_2)\underline{v} + C, \quad t \in (T_0, T). \tag{2.7}$$

As in the proof of Theorem 1.2, we see

$$\begin{aligned} c\exp\left(\frac{p_2q_1 - p_1q_2}{q_1 - q_2}\underline{u}\right) &\le \lambda\exp\left(p_1\underline{u}(t) + q_1\frac{p_2 - p_1}{q_1 - q_2}\underline{u}(t) - \frac{q_1C}{q_1 - q_2}\right) \\ &\le \underline{u}_t = \overline{u}_t \le \lambda e^{p_1\overline{u} + q_1\overline{v}} \\ &\le \lambda e^{(p_1+q_1)C_0}\exp\left(p_1\underline{u}(t) + q_1\frac{p_2 - p_1}{q_1 - q_2}\underline{u}(t) + \frac{q_1C}{q_1 - q_2}\right) \\ &\le C\exp\left(\frac{p_2q_1 - p_1q_2}{q_1 - q_2}\underline{u}\right), \quad t \in (T_0, T). \end{aligned}$$

We can then easily deduce the estimates of $\underline{u}$ from

$$-\log(T - t) - C \le \frac{p_2q_1 - p_1q_2}{q_1 - q_2}\underline{u}(t) \le -\log(T - t) + C, \quad t \in (0, T).$$

We use (2.7) again to get the blowup estimates of $\underline{v}$ from

$$-\log(T - t) - C \le \frac{p_2q_1 - p_1q_2}{p_2 - p_1}\underline{v}(t) \le -\log(T - t) + C, \quad t \in (0, T).$$

(ii) Similar to (i), it follows from (2.5) that for some $T_0 \in (0, T)$,

$$c\underline{v}(t) \le e^{(p_2 - p_1)\underline{u}(t)} \le C\underline{v}(t), \quad t \in (T_0, T). \tag{2.8}$$

Substituting into $ce^{p_2\underline{u}(t) + q_2\underline{v}(t)} \le \underline{v}_t(t) \le Ce^{p_2\underline{u}(t) + q_2\underline{v}(t)}$, $t \in (0, T)$, we get

$$c\underline{v}^{p_2/(p_2 - p_1)}(t)e^{q_2\underline{v}(t)} \le \underline{v}_t(t) \le C\underline{v}^{p_2/(p_2 - p_1)}(t)e^{q_2\underline{v}(t)}, \quad t \in (T_0, T).$$

Since $q_2 > 0$, we see that $\int_M^\infty s^{-p_2/(p_2 - p_1)}e^{-q_2s}\,ds$ is convergent for any fixed $M > 0$. Integrating the above inequalities from t to T yields that

$$c(T - t) \le \int_t^T \underline{v}^{-p_2/(p_2 - p_1)}e^{-q_2\underline{v}}\underline{v}_t\,dt = \int_{\underline{v}(t)}^\infty s^{-p_2/(p_2 - p_1)}e^{-q_2s}\,ds \le C(T - t), \tag{2.9}$$

for any $t \in (T_0, T)$. Notice that

$$\lim_{\underline{v}(t)\to\infty} \frac{\int_{\underline{v}(t)}^\infty s^{-p_2/(p_2 - p_1)}e^{-q_2s}\,ds}{\underline{v}^{-p_2/(p_2 - p_1)}e^{-q_2\underline{v}}} = \frac{1}{q_2}. \tag{2.10}$$

Therefore, combining (2.9) and (2.10), we easily deduce

$$c(T - t)^{-1} \le \underline{v}^{p_2/(p_2 - p_1)}e^{q_2\underline{v}} \le C(T - t)^{-1}, \quad t \to T.$$

Thus we have proved

$$-\log(T-t) - C \le q_2\underline{v} + \frac{p_2}{p_2-p_1}\log\underline{v} \le -\log(T-t) + C, \quad t \to T.$$

By using $\lim_{t\to T}\log\underline{v}(t)/\underline{v}(t) = 0$, we see

$$\lim_{t\to T}\frac{\underline{v}(t)}{|\log(T-t)|} = \frac{1}{q_2}. \tag{2.11}$$

On the other hand, using (2.8) again, we have

$$\lim_{t\to T}\frac{(p_2-p_1)\underline{u}(t)}{\log\underline{v}(t)} = 1.$$

It follows from (2.11) that

$$\lim_{t\to T}\frac{\underline{u}(t)}{\log|\log(T-t)|} = \frac{1}{p_2-p_1}.$$

(iii) The conclusions can be proved by arguments similar to those for (ii).

(iv) In this case, we use (2.6) to obtain $c_1e^{c_1\underline{v}(t)} \le \underline{v}_t(t) \le C_1e^{C_2\underline{v}(t)}$, $t \in (0,T)$. Integrating from t to T, we see

$$-c\log(T-t) - C \le \underline{v}(t) \le -C\log(T-t) + C, \quad t \in (0,T).$$

The estimates of $\underline{u}$ are immediately obtained by $c\underline{v} \le \underline{u} \le C\underline{v}$. □

PROOF OF THEOREM 1.3. Notice that

$$\lim_{t\to T}\frac{u(x,t)}{\underline{u}(t)} = \lim_{t\to T}\frac{v(x,t)}{\underline{v}(t)} = 1$$

uniformly in $\mathbb{R}^N$. It follows from Lemma 2.1 that Theorem 1.3 holds. □

3. Blowup properties for problem (1.2)

In this section, we consider problem (1.2). We first give the blowup criterion and then discuss the blowup properties. By an argument similar to that for problem (1.1), we define

$$\underline{u}(t) = \int_0^t u^{p_1}(x_0(s),s)v^{q_1}(x_0(s),s)\,ds, \quad \overline{u}(t) = \underline{u}(t) + C_0, \quad t > 0,$$

$$\underline{v}(t) = \int_0^t u^{p_2}(x_0(s),s)v^{q_2}(x_0(s),s)\,ds, \quad \overline{v}(t) = \underline{v}(t) + C_0, \quad t > 0,$$

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where $C_0 = \|u_0\|_\infty + \|v_0\|_\infty$. Then we have

$$\underline{u}_t - \Delta\underline{u} = \overline{u}_t - \Delta\overline{u} = u^{p_1}(x_0(t),t)v^{q_1}(x_0(t),t) = u_t - \Delta u,$$
$$\underline{v}_t - \Delta\underline{v} = \overline{v}_t - \Delta\overline{v} = u^{p_2}(x_0(t),t)v^{q_2}(x_0(t),t) = v_t - \Delta v.$$

Moreover, $\underline{u}(x,0) = 0 \le u_0(x) \le \overline{u}(x,0)$ and $\underline{v}(x,0) = 0 \le v_0(x) \le \overline{v}(x,0)$. Using the maximum principle, we have $\underline{u} \le u \le \overline{u}$ and $\underline{v} \le v \le \overline{v}$ as long as (u,v) exists. In particular, $\underline{u}(t) \le u(x_0(t),t) \le \overline{u}(t)$ and $\underline{v}(t) \le v(x_0(t),t) \le \overline{v}(t)$. Hence

$$\underline{u}_t \ge \underline{u}^{p_1}\underline{v}^{q_1}, \quad \underline{v}_t \ge \underline{u}^{p_2}\underline{v}^{q_2}, \quad \overline{u}_t \le \overline{u}^{p_1}\overline{v}^{q_1}, \quad \overline{v}_t \le \overline{u}^{p_2}\overline{v}^{q_2}, \quad t \in (0,T). \tag{3.1}$$

PROOF OF THEOREM 1.4. Since the nonlinearities of (3.1) are not necessarily locally Lipschitz, it is not clear whether the comparison principle is applicable in all cases. However, by $\overline{u}(0) = \overline{v}(0) = C_0 > 0$, we may apply the comparison principle for $(\overline{u},\overline{v})$ and $(\overline{w},\overline{s})$ (for the construction of $(\overline{w},\overline{s})$, see the following arguments).

Case (i): $p_1 < 1, q_2 < 1$ and $D < 0$. Take $\overline{w}(x,t) = (C+t)^p, \overline{s}(x,t) = (C+t)^q$, where $C > 0$ is to be chosen, $p = -2(q_1 - q_2 + 1)/D$ and $q = -2(p_2 - p_1 + 1)/D$. It follows that $p, q > 0$, $p(1-p_1) - qq_1 = 2$ and $q(1-q_2) - pp_2 = 2$. After a simple calculation, we obtain

$$\overline{w}_t = p(C+t)^{p-1} > (C+t)^{p-2} = \overline{w}^{p_1}\overline{s}^{q_1}, \quad t > 0,$$
$$\overline{s}_t = q(C+t)^{q-1} > (C+t)^{q-2} = \overline{w}^{p_2}\overline{s}^{q_2}, \quad t > 0,$$

for C sufficiently large. If we further take $C > 0$ such that $C^p \ge \overline{u}(0) = C_0$, $C^q \ge \overline{v}(0) = C_0$, we have $(\overline{u},\overline{v}) \le (\overline{w},\overline{s})$ for $t \ge 0$ by (3.1) and the comparison principle. This shows that (u,v) exists globally.

Case (ii): $p_1 < 1, q_2 < 1, D = 0$. Given large $C > C_0$, we take $\overline{w}(x,t) = Ce^{pt}$, $\overline{s}(x,t) = Ce^{qt}$, where p is to be chosen later and $q = (1-p_1)q_1^{-1}p = p_2(1-q_2)^{-1}p$. Then

$$\overline{w}_t = pCe^{pt} > C^{p_1+q_1}e^{(pp_1+qq_1)t} = \overline{w}^{p_1}\overline{s}^{q_1}, \quad t > 0,$$
$$\overline{s}_t = qCe^{qt} > C^{p_2+q_2}e^{(pp_2+qq_2)t} = \overline{w}^{p_2}\overline{s}^{q_2}, \quad t > 0,$$

for p large enough. Therefore, arguments analogous with those for Case (i) imply that the solution (u,v) of (1.2) is global.

Case (iii): $p_1 > 1$. Without loss of generality, we suppose that $T > 1$. Using (3.1) and noticing that $\underline{v}(t)$ is nondecreasing in t, we have $\underline{u}_t(t) \ge \underline{u}^{p_1}(t)\underline{v}^{q_1}(1), t > 1$. Combining this inequality with $\underline{u}(1) > 0$, we see that $(\underline{u},\underline{v})$ blows up in a finite time. Therefore the solution (u,v) of (1.2) also blows up in a finite time.

Case (iv): $q_2 > 1$. The arguments are similar to those for Case (iii), so we omit them here.

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Case (v): $p_1 \le 1$, $q_2 \le 1$ and $D > 0$. In this case, it follows from $p_2 q_1 > 0$ that $p_2 > p_1 - 1$, $q_1 > q_2 - 1$ and $D > 0$. By Lemma 3.1 (i), we have

$$\underline{u}_t \ge \underline{u}^{p_1} \underline{v}^{q_1} \ge c\underline{u}^{(p_2 q_1 - p_1 q_2 + p_1 + q_1)/(q_1 - q_2 + 1)}, \quad t \in (T_0, T),$$

namely,

$$-\frac{q_1 - q_2 + 1}{D} \left(\underline{u}^{-D/(q_1 - q_2 + 1)}\right)_t \ge c, \quad t \in (T_0, T).$$

Integrating this inequality from T_0 to t, we obtain

$$\underline{u}^{-D/(q_1 - q_2 + 1)}(T_0) - \underline{u}^{-D/(q_1 - q_2 + 1)}(t) \ge c(t - T_0).$$

Since $q_1 > q_2 - 1$ and $D > 0$, the above inequality cannot hold for all time. Therefore $(\underline{u}, \underline{v})$ blows up in a finite time and so does (u, v).

It follows from $\underline{u} \le u$, $\underline{v} \le v$ that the blowup set is $\mathbb{R}^N$. □

The following lemma will play a key role in establishing simultaneous blowup. From now on, we take $T_0 = T/2$.

LEMMA 3.1. *There exist constants $C > c > 0$, such that*

(i) *if $p_2 > p_1 - 1$, $q_1 > q_2 - 1$, $D > 0$, then*

$$c\underline{u}^{p_2 - p_1 + 1} \le \underline{v}^{q_1 - q_2 + 1} \le C\underline{u}^{p_2 - p_1 + 1}, \quad t \in (T_0, T)$$
$$c\overline{u}^{p_2 - p_1 + 1} \le \overline{v}^{q_1 - q_2 + 1} \le C\overline{u}^{p_2 - p_1 + 1}, \quad t \in (T_0, T);$$

(ii) *if $p_2 > p_1 - 1 > 0$, $q_1 = q_2 - 1 > 0$, then*

$$c\underline{u}^{p_2 - p_1 + 1} \le \log \underline{v} \le C\underline{u}^{p_2 - p_1 + 1}, \quad c\overline{u}^{p_2 - p_1 + 1} \le \log \overline{v} \le C\overline{u}^{p_2 - p_1 + 1}, \quad t \in (T_0, T);$$

(iii) *if $p_2 = p_1 - 1 > 0$, $q_1 > q_2 - 1 > 0$, then*

$$c\underline{v}^{q_1 - q_2 + 1} \le \log \underline{u} \le C\underline{v}^{q_1 - q_2 + 1}, \quad c\overline{v}^{q_1 - q_2 + 1} \le \log \overline{u} \le C\overline{v}^{q_1 - q_2 + 1}, \quad t \in (T_0, T);$$

(iv) *if $p_2 = p_1 - 1 > 0$, $q_1 = q_2 - 1 > 0$, then*

$$c\underline{u} \le \log \underline{v} \le C\underline{u}, \quad c\overline{u} \le \log \overline{v} \le C\overline{u}, \quad t \in (T_0, T).$$

PROOF. (i) We prove that $\underline{v}^{q_1 - q_2 + 1} \le C\underline{u}^{p_2 - p_1 + 1}$ holds for all $t \in (T_0, T)$. Let

$$J(t) = \bar{C} \frac{\underline{u}^{p_2 - p_1 + 1}}{p_2 - p_1 + 1} - \frac{\underline{v}^{q_1 - q_2 + 1}}{q_1 - q_2 + 1},$$

where $\bar{C}$ is to be determined. Notice that

$$\underline{v}_t = \overline{v}_t \le \overline{u}^{p_2} \overline{v}^{q_2} = (\underline{u} + C_0)^{p_2} (\underline{v} + C_0)^{q_2}.$$

By (3.1), we easily obtain

$$\begin{aligned} J'(t) &= \bar{C}\underline{u}^{p_2-p_1}\underline{u}_t - \underline{v}^{q_1-q_2}\underline{v}_t \\ &\geq \bar{C}\underline{u}^{p_2}\underline{v}^{q_1} - \underline{v}^{q_1-q_2}(\underline{u}+C_0)^{p_2}(\underline{v}+C_0)^{q_2} \\ &= \bar{C}\underline{u}^{p_2}\underline{v}^{q_1}\left(1-\frac{1}{\bar{C}}\left(\frac{\underline{u}+C_0}{\underline{u}}\right)^{p_2}\left(\frac{\underline{v}+C_0}{\underline{v}}\right)^{q_2}\right). \end{aligned}$$

On the other hand, note that $\underline{u}$ and $\underline{v}$ are nondecreasing in t. Taking $\bar{C}$ with

$$\bar{C} \geq \max\left(\left(\frac{\underline{u}(T_0)+C_0}{\underline{u}(T_0)}\right)^{p_2}\left(\frac{\underline{v}(T_0)+C_0}{\underline{v}(T_0)}\right)^{q_2}, \left(\frac{p_2-p_1+1}{q_1-q_2+1}\right)\frac{\underline{v}^{q_1-q_2+1}(T_0)}{\underline{u}^{p_2-p_1+1}(T_0)}\right),$$

we see that $J'(t) > 0$ for $t \in (T_0, T)$ and $J(T_0) > 0$, which implies $J(t) > 0$, $t \in (T_0, T)$. Therefore, there exists some constant $C > 0$ such that $\underline{v}^{q_1-q_2+1} \leq C\underline{u}^{p_2-p_1+1}$, $t \in (T_0, T)$. The other inequalities can be proved by similar arguments.

The proof of (ii)–(iv) is similar to (i), so we omit it. The proof of the lemma is completed. □

PROOF OF THEOREM 1.5. The assumption of Theorem 1.5 ensures that (u, v) blows up in a finite time. The simultaneous blowup can be directly obtained by Lemma 3.1. It remains to prove the non-simultaneous blowup.

In the case of $p_2 < p_1 - 1$ and $q_1 \geq q_2 - 1$, we assume both $\underline{u}$ and $\underline{v}$ blow up at a finite time T to draw a contradiction. Notice that

$$\begin{aligned} \underline{u}^{p_1}\underline{v}^{q_1} &\leq \underline{u}_t \leq (\underline{u}+C_0)^{p_1}(\underline{v}+C_0)^{q_1}, \quad t \in (0, T), \\ \underline{u}^{p_2}\underline{v}^{q_2} &\leq \underline{v}_t \leq (\underline{u}+C_0)^{p_2}(\underline{v}+C_0)^{q_2}, \quad t \in (0, T). \end{aligned}$$

It follows that

$$\frac{\underline{u}^{p_1}\underline{v}^{q_1}}{(\underline{u}+C_0)^{p_2}(\underline{v}+C_0)^{q_2}} \leq \frac{d\underline{u}}{d\underline{v}} \leq \frac{(\underline{u}+C_0)^{p_1}(\underline{v}+C_0)^{q_1}}{\underline{u}^{p_2}\underline{v}^{q_2}}, \quad t \in (0, T).$$

By the assumption that $\underline{u}$ and $\underline{v}$ simultaneously blow up, we see there exists $T_1 < T$ such that

$$c\underline{u}^{p_1-p_2}(t)\underline{v}^{q_1-q_2}(t) \leq \frac{d\underline{u}}{d\underline{v}} \leq C\underline{u}^{p_1-p_2}(t)\underline{v}^{q_1-q_2}(t), \quad t \in (T_1, T),$$

namely,

$$c\underline{v}^{q_1-q_2}(t)d\underline{v} \leq \underline{u}^{p_2-p_1}(t)d\underline{u} \leq C\underline{v}^{q_1-q_2}(t)d\underline{v}, \quad t \in (T_1, T). \tag{3.2}$$

In the case of $p_2 < p_1 - 1$ and $q_1 > q_2 - 1$, we integrate (3.2) on $[T_1, t]$ to obtain

$$\frac{c}{q_1-q_2+1}\left(\underline{v}^{q_1-q_2+1}(t)-\underline{v}^{q_1-q_2+1}(T_1)\right) \leq \frac{1}{p_2-p_1+1}\left(\underline{u}^{p_2-p_1+1}(t)-\underline{u}^{p_2-p_1+1}(T_1)\right)$$

for all $t \in (T_1, T)$. By $p_2 < p_1 - 1$, $q_1 > q_2 - 1$, we see the left-hand side approaches $+\infty$, while the right-hand side is finite as $t \to T$. This is a contradiction.

If $p_2 < p_1 - 1$ and $q_1 = q_2 - 1$, (3.2) is equivalent to

$$c\underline{v}^{-1}(t)d\underline{v} \le \underline{u}^{p_2-p_1}(t)d\underline{u} \le C\underline{v}^{-1}(t)d\underline{v}, \quad t \in (T_1, T).$$

As in the previous proof, we integrate the above inequalities to obtain

$$c\left(\log \underline{v}(t) - \underline{v}(T_1)\right) \le \frac{1}{p_2 - p_1 + 1}\left(\underline{u}^{p_2-p_1+1}(t) - \underline{u}^{p_2-p_1+1}(T_1)\right), \quad t \in (T_1, T).$$

This is a contradiction as $t \to T$. So only non-simultaneous blowup occurs.

For the case $p_2 \ge p_1 - 1$ and $q_1 < q_2 - 1$, we can prove the conclusion by similar arguments. □

Recall that $\underline{u}(t) \le u(x,t) \le \underline{u}(t) + C_0$ and $\underline{v}(t) \le v(x,t) \le \underline{v}(t) + C_0$ for all $(x,t) \in \mathbb{R}^N \times \{t > 0\}$. To prove Theorem 1.6, it is sufficient to show the following lemma.

LEMMA 3.2. (i) *If $p_2 > p_1 - 1$, $q_1 > q_2 - 1$ and $D > 0$, then*

$$\lim_{t\to T}(T-t)^{\alpha}\underline{u} = \left(\frac{1}{\alpha}\left(\frac{\alpha}{\beta}\right)^{q_1/(q_1-q_2+1)}\right)^{-\alpha},$$

$$\lim_{t\to T}(T-t)^{\beta}\underline{v} = \left(\frac{1}{\beta}\left(\frac{\beta}{\alpha}\right)^{p_2/(p_2-p_1+1)}\right)^{-\beta}.$$

(ii) *If $p_2 > p_1 - 1 > 0$ and $q_1 = q_2 - 1 > 0$, then*

$$\lim_{t\to T}|\log(T-t)|^{-1/(p_2-p_1+1)}\underline{u} = \left(\frac{1}{q_1}(p_2 - p_1 + 1)\right)^{1/(p_2-p_1+1)},$$

$$\lim_{t\to T}(T-t)\underline{v}^{q_1}(\log \underline{v})^{p_2/(p_2-p_1+1)} = \frac{1}{q_1}(p_2 - p_1 + 1)^{-p_2/(p_2-p_1+1)}.$$

(iii) *If $p_2 = p_1 - 1 > 0$ and $q_1 > q_2 - 1 > 0$, then*

$$\lim_{t\to T}(T-t)\underline{u}^{p_2}(\log \underline{u})^{q_1/(q_1-q_2+1)} = \frac{1}{p_2}(q_1 - q_2 + 1)^{-q_1/(q_1-q_2+1)},$$

$$\lim_{t\to T}|\log(T-t)|^{-1/(q_1-q_2+1)}\underline{v} = \left(\frac{1}{p_2}(q_1 - q_2 + 1)\right)^{1/(q_1-q_2+1)}.$$

(iv) *If $p_2 = p_1 - 1 > 0$ and $q_1 = q_2 - 1 > 0$, then*

$$\lim_{t\to T}|\log(T-t)|^{-1}\log \underline{u} = \lim_{t\to T}|\log(T-t)|^{-1}\log \underline{v} = 1/(p_2 + q_1).$$

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PROOF. We will use the notation $f \sim g$ if $\lim_{t\to T} f(t)/g(t) = 1$. Under the assumption of Theorem 1.6, by Lemma 3.1, u and v blow up simultaneously at a finite time T. Then $\underline{u}(t) \le u(x,t) \le \underline{u}(t) + C_0$ and $\underline{v}(t) \le v(x,t) \le \underline{v}(t) + C_0$ imply that

$$\begin{aligned} \underline{u}_t &= u^{p_1}(x_0(t),t)v^{q_1}(x_0(t),t) \sim \underline{u}^{p_1}\underline{v}^{q_1}, \\ \underline{v}_t &= u^{p_2}(x_0(t),t)v^{q_2}(x_0(t),t) \sim \underline{u}^{p_2}\underline{v}^{q_2}, \end{aligned} \tag{3.3}$$

as $t \to T$. Therefore we easily get

$$\frac{\underline{u}_t(t)}{\underline{v}_t(t)} \sim \frac{\underline{v}^{q_1-q_2}(t)}{\underline{u}^{p_2-p_1}(t)}, \quad t \to T. \tag{3.4}$$

(i) $p_2 > p_1 - 1$, $q_1 > q_2 - 1$ and $D > 0$. It follows from (3.4) that

$$\frac{\underline{u}^{p_2-p_1+1}(t)}{p_2-p_1+1} \sim \frac{\underline{v}^{q_1-q_2+1}(t)}{q_1-q_2+1}, \quad t \to T.$$

Substituting this into (3.3), we obtain

$$\begin{aligned} \underline{u}_t(t) &\sim \left(\frac{q_1-q_2+1}{p_2-p_1+1}\right)^{q_1/(q_1-q_2+1)} \underline{u}^{(q_1p_2-p_1q_2+p_1+q_1)/(q_1-q_2+1)}(t), \quad t \to T, \\ \underline{v}_t(t) &\sim \left(\frac{p_2-p_1+1}{q_1-q_2+1}\right)^{p_2/(p_2-p_1+1)} \underline{v}^{(q_1p_2-p_1q_2+p_2+q_2)/(p_2-p_1+1)}(t), \quad t \to T. \end{aligned}$$

Integrating these equivalences from t to T, we can get the conclusion (i).

(ii) $p_2 > p_1 - 1 > 0$ and $q_1 = q_2 - 1 > 0$. By (3.4), we have

$$\frac{1}{p_2-p_1+1}\underline{u}^{p_2-p_1+1}(t) \sim \log \underline{v}(t), \quad t \to T. \tag{3.5}$$

Substituting (3.5) into (3.3), we obtain

$$\underline{v}_t(t) \sim (p_2-p_1+1)^{p_2/(p_2-p_1+1)}(\log \underline{v}(t))^{p_2/(p_2-p_1+1)}\underline{v}^{q_2}(t), \quad t \to T.$$

By an integration from t (t is sufficiently close to T) to T, we see

$$\begin{aligned} \int_{\underline{v}(t)}^{+\infty} (\log s)^{-p_2/(p_2-p_1+1)} s^{-q_2}\, ds &= \int_t^T (\log \underline{v})^{-p_2/(p_2-p_1+1)}\underline{v}^{-q_2}\underline{v}_t\, dt \\ &\sim (p_2-p_1+1)^{p_2/(p_2-p_1+1)}(T-t). \end{aligned}$$

Having $q_2 > 1$ implies that the integral on the left-hand side is convergent. Notice that

$$\lim_{\underline{v}(t)\to+\infty} \frac{\int_{\underline{v}}^{+\infty} s^{-p_2/(p_2-p_1+1)} s^{-q_2}\, ds}{\underline{v}^{1-q_2}(\log \underline{v})^{-p_2/(p_2-p_1+1)}} = \frac{1}{q_2-1}. \tag{3.6}$$

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Then we easily get, as $t \to T$,

$$\underline{v}^{1-q_2}(\log \underline{v})^{-p_2/(p_2-p_1+1)} \sim (q_2-1)(p_2-p_1+1)^{p_2/(p_2-p_1+1)}(T-t). \tag{3.7}$$

Since $q_1 = q_2 - 1$, we have established the estimates of $\underline{v}$. Combining (3.5) and (3.7), we get $\underline{v}^{q_1} \sim q_1^{-1}(T-t)^{-1}\underline{u}^{-p_2}$, as $t \to T$. Substituting it into (3.3) and integrating from t to T, we may deduce the asymptotic behaviour of $\underline{u}$:

$$\lim_{t\to T} |\log(T-t)|^{-1/(p_2-p_1+1)}\underline{u}(t) = \left(\frac{1}{q_1}(p_2-p_1+1)\right)^{1/(p_2-p_1+1)}$$

(iii) Using arguments similar to those for (ii), we can prove the conclusion.

(iv) $p_2 = p_1 - 1 > 0$ and $q_1 = q_2 - 1 > 0$. In this case, we have $\log \underline{u}(t) \sim \log \underline{v}(t)$, as $t \to T$. Integrating (3.3) from t to T, we obtain

$$\frac{1}{p_2}\underline{u}^{-p_2}(t) \sim \int_t^T \underline{v}^{q_1}(s)\,ds, \quad \frac{1}{q_1}\underline{v}^{-q_1}(t) \sim \int_t^T \underline{u}^{p_2}(s)\,ds, \quad t \to T.$$

Therefore $p_2 > 0$ implies that $\lim_{t\to T}\int_t^T \underline{u}^{p_2}(s)\,ds = 0$, and $q_1 > 0$ suggests that $\lim_{t\to T}\int_t^T \underline{v}^{q_1}(s)\,ds = 0$. Moreover, we also see

$$\left(\int_t^T \underline{u}^{p_2}(s)\,ds \int_t^T \underline{v}^{q_1}(s)\,ds\right)' \sim -\frac{p_2+q_1}{p_2 q_1}, \quad t \to T. \tag{3.8}$$

Then, integrating (3.8), we obtain

$$\int_t^T \underline{u}^{p_2}(s)\,ds \int_t^T \underline{v}^{q_1}(s)\,ds \sim \frac{p_2+q_1}{p_2 q_1}(T-t), \quad t \to T.$$

That is,

$$\underline{u}^{-p_2}(t)\underline{v}^{-q_1}(t) \sim (p_2+q_1)(T-t), \quad t \to T.$$

Combining this with (3.3), we easily deduce that

$$\underline{u}'(t) \sim \frac{1}{p_2+q_1}\underline{u}^{p_1-p_2}(T-t)^{-1} \sim \frac{1}{p_2+q_1}(T-t)^{-1}\underline{u}(t), \quad t \to T,$$
$$\underline{v}'(t) \sim \frac{1}{p_2+q_1}\underline{v}^{q_2-q_1}(T-t)^{-1} \sim \frac{1}{p_2+q_1}(T-t)^{-1}\underline{v}(t), \quad t \to T.$$

Therefore we obtain

$$\log \underline{u}(t) \sim \frac{1}{p_2+q_1}|\log(T-t)| \sim \log \underline{v}(t), \quad t \to T,$$

which implies that the conclusion of (iv) holds. □

4. Neumann problem for several nonlocal models

In this section, we investigate the following four diffusion systems with nonlocal sources. The first is

$$\begin{aligned} u_t &= \Delta u + \lambda \int_\Omega e^{p_1 u + q_1 v}\,dx, \quad (x,t) \in \Omega \times (0,T), \\ v_t &= \Delta v + \mu \int_\Omega e^{p_2 u + q_2 v}\,dx, \quad (x,t) \in \Omega \times (0,T). \end{aligned} \tag{4.1}$$

Hereafter, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$. If we use the mean value theorem for integrals, we may consider the following system with localised reaction terms:

$$\begin{aligned} u_t &= \Delta u + \lambda e^{p_1 u(x_1(t),t) + q_1 v(x_1(t),t)}, \\ v_t &= \Delta v + \mu e^{p_2 u(x_2(t),t) + q_2 v(x_2(t),t)}, \end{aligned} \tag{4.2}$$

for $(x,t) \in \Omega \times (0,T)$. Next, we study the power-type system with space-integrals

$$\begin{aligned} u_t &= \Delta u + \int_\Omega u^{p_1} v^{q_1}\,dx, \quad (x,t) \in \Omega \times (0,T), \\ v_t &= \Delta v + \int_\Omega u^{p_2} v^{q_2}\,dx, \quad (x,t) \in \Omega \times (0,T). \end{aligned} \tag{4.3}$$

Finally, we investigate the system

$$\begin{aligned} u_t &= \Delta u + u^{p_1}(x_1(t),t) v^{q_1}(x_1(t),t), \\ v_t &= \Delta v + u^{p_2}(x_2(t),t) v^{q_2}(x_2(t),t), \end{aligned} \tag{4.4}$$

$(x,t) \in \Omega \times (0,T)$. We investigate all these models coupled with homogeneous Neumann boundary values and Cauchy data,

$$\begin{aligned} &\frac{\partial u}{\partial \nu}(x,t) = \frac{\partial v}{\partial \nu}(x,t) = 0, && (x,t) \in \partial\Omega \times (0,T), \\ &u(x,0) = u_0(x), \quad v(x,0) = v_0(x), && x \in \Omega, \end{aligned} \tag{4.5}$$

where ν is the unit outward normal of $\partial\Omega$, while $u_0(x), v_0(x) \in C^{2+\alpha}(\bar{\Omega})$, $(0 < \alpha < 1)$, are nonnegative nontrivial functions and the consistency conditions hold:

$$\frac{\partial u_0}{\partial \nu} = \frac{\partial v_0}{\partial \nu} = 0$$

on the boundary $\partial\Omega$. We also assume $x_1, x_2 : \mathbb{R}^+ \to \Omega$ are Hölder continuous.

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As mentioned in the introduction, some authors have studied the above problems with homogeneous Dirichlet boundary conditions (see [10] for (4.3), [21] for (4.2) and (4.4) with $x_1(t) = x_2(t) = 0$, and [16] for some special cases). In these papers, the establishment of blowup estimates is based on the fact that $-\Delta u \le C$, $-\Delta v \le C$ (which is essentially due to Souplet [18]). For the Neumann boundary problem, however, the estimates of $-\Delta u$, $-\Delta v$ are not straightforward. Recently, using Green's function, Xiang *et al.* [20] considered systems (4.1)–(4.4) (subject to (4.5) for each case) with $p_1 = q_2 = 0$.

We will use the ideas for dealing with the Cauchy problems (1.1) and (1.2) to investigate the Neumann problem. Using procedures similar to those for the Cauchy problem, we may define two functions $G_1(t)$, $G_2(t)$ by integrating the reaction terms on $(0, t)$. For example, we define

$$G_1(t) = \lambda \int_0^t \int_\Omega e^{p_1 u(x,\tau)+q_1 v(x,\tau)}\, dx\, d\tau, \quad t > 0,$$
$$G_2(t) = \mu \int_0^t \int_\Omega e^{p_2 u(x,\tau)+q_2 v(x,\tau)}\, dx\, d\tau, \quad t > 0,$$

for problem (4.1). Then, we take $\bar{G}_1(t) = G_1(t) + C_0$, $\bar{G}_2(t) = G_2(t) + C_0$, where $C_0 = \|u_0\|_\infty + \|v_0\|_\infty$. The arguments are based on the following lemma.

LEMMA 4.1. *Assume* (u, v) *is a solution of* (4.1) (*respectively* (4.2), (4.3), (4.4)) *with* (4.5). *Then*

$$G_1(t) \le u(x,t) \le \bar{G}_1(t), \quad G_2(t) \le v(x,t) \le \bar{G}_2(t), \quad (x,t) \in \bar{\Omega} \times (0,T).$$

Lemma 4.1 can be proved by the standard comparison principle (see Lemma 3.2 in [20]).

We will find $G_1(t)$, $G_2(t)$ (respectively $\bar{G}_1(t)$, $\bar{G}_2(t)$) have properties similar to those of functions $\underline{u}$, $\underline{v}$ (respectively $\overline{u}$, $\overline{v}$) defined in Sections 2 and 3 for the Cauchy problem (1.1) and (1.2). Therefore, using arguments similar to those in Sections 2 and 3, we have the following theorems.

THEOREM 4.2. *For problems* (4.1) *and* (4.5) *and* (4.2) *and* (4.5), *the conclusions of Theorems* 1.1–1.3 *hold except that* $\mathbb{R}^N$ *is replaced with* $\bar{\Omega}$.

THEOREM 4.3. *For problem* (4.4) *and* (4.5), *the conclusions of Theorems* 1.4–1.6 *hold except that the blowup set is replaced with* $\bar{\Omega}$.

THEOREM 4.4. *For problem* (4.3) *and* (4.5), *we have*:

(1) *The conclusions of Theorems* 1.4 *and* 1.5 *hold except that the blowup set is replaced with* $\bar{\Omega}$.

(2) *The following asymptotic behaviour holds*:

(i) *If* $p_2 > p_1 - 1, q_1 > q_2 - 1$ *and* $D > 0$, *then*

$$\lim_{t\to T}(T-t)^{\alpha}u(x,t) = \left(\frac{|\Omega|}{\alpha}\left(\frac{\alpha}{\beta}\right)^{q_1/(q_1-q_2+1)}\right)^{-\alpha},$$

$$\lim_{t\to T}(T-t)^{\beta}v(x,t) = \left(\frac{|\Omega|}{\beta}\left(\frac{\beta}{\alpha}\right)^{p_2/(p_2-p_1+1)}\right)^{-\beta};$$

(ii) *If* $p_2 > p_1 - 1 > 0$ *and* $q_1 = q_2 - 1 > 0$, *then*

$$\lim_{t\to T}|\log(T-t)|^{-1/(p_2-p_1+1)}u(x,t) = \frac{1}{q_1|\Omega|}(p_2-p_1+1)^{1/(p_2-p_1+1)},$$

$$\lim_{t\to T}(T-t)v^{q_1}(x,t)(\log v(x,t))^{p_2/(p_2-p_1+1)} = \left(\frac{1}{q_1}(p_2-p_1+1)\right)^{-p_2/(p_2-p_1+1)};$$

(iii) *If* $p_2 = p_1 - 1 > 0$ *and* $q_1 > q_2 - 1 > 0$, *then*

$$\lim_{t\to T}(T-t)u^{p_2}(x,t)(\log u(x,t))^{q_1/(q_1-q_2+1)} = \left(\frac{1}{p_2}(q_1-q_2+1)\right)^{-q_1/(q_1-q_2+1)},$$

$$\lim_{t\to T}|\log(T-t)|^{-1/(q_1-q_2+1)}v(x,t) = \frac{1}{p_2|\Omega|}(q_1-q_2+1)^{1/(q_1-q_2+1)};$$

(iv) *If* $p_2 = p_1 - 1 > 0$ *and* $q_1 = q_2 - 1 > 0$, *then*

$$\lim_{t\to T}|\log(T-t)|^{-1}\log u(x,t) = \lim_{t\to T}|\log(T-t)|^{-1}\log v(x,t) = 1/(p_2+q_1).$$

All these limits hold uniformly in $\bar{\Omega}$.

REMARK. Theorems 4.2–4.4 extend the results in [13, 20] to more general cases. Moreover, their proofs are completely different from those of [13, 20].

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