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Abstract

We study quivers with potential (QPs) whose Jacobian algebras are finite-dimensional selfinjective. They are an analogue of the 'good QPs' studied by Bocklandt whose Jacobian algebras are 3-Calabi–Yau. We show that 2-representation-finite algebras are truncated Jacobian algebras of selfinjective QPs, which are factor algebras of Jacobian algebras by certain sets of arrows called cuts. We show that selfinjectivity of QPs is preserved under iterated mutation with respect to orbits of the Nakayama permutation. We give a sufficient condition for all truncated Jacobian algebras of a fixed QP to be derived equivalent. We introduce planar QPs which provide us with a rich source of selfinjective QPs.

Contents

1. Introduction

Quivers with potential (QPs) play important roles in the theory of cluster algebras/categories [BIRS11, DWZ08, DWZ10, KY11] and also appear in Seiberg duality in physics [BD02, Sei95]. They give rise to Ginzburg DG algebras [Gin06], which enjoy the 3-Calabi–Yau property and play a crucial role in the theory of generalized cluster categories [Ami09a, KVdB11]. The zeroth homology $\mathcal{P}(Q, W)$ of the Ginzburg DG algebra associated with a QP (Q, W) is called the *Jacobian algebra*. Important classes of algebras appearing in representation theory are

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known to be Jacobian algebras of certain QPs. For example, graded 3-Calabi–Yau algebras are Jacobian algebras of certain QPs [Boc08]. Also, every algebra A of global dimension two gives a QP (Q_A, W_A) whose Jacobian algebra is the 3-preprojective algebra $\Pi_3(A)$ of A (see [HI11, IO09, IO11, KVdB11]). In particular, cluster tilted algebras are Jacobian algebras of certain QPs [BIRS11].

Jacobian algebras of QPs are Iwanaga–Gorenstein of dimension at most one if they are finite dimensional [BIRS11]. A main subject in this paper is *selfinjective* QPs, which are QPs whose Jacobian algebras are finite-dimensional selfinjective algebras. It is known that a finite-dimensional algebra A of global dimension two has a cluster tilting module if and only if the 3-preprojective algebra $\Pi_3(A)$ is selfinjective [IO09]. In this case, A is called 2-*representation-finite*. 2-representation-finite algebras play a central role in three-dimensional Auslander–Reiten theory [HI11, Iya07a, Iya07b, Iya08, Iya11, IO09, IO11]. We shall give a structure theorem of 2-representation-finite algebras in terms of selfinjective QPs and their truncated Jacobian algebras, which is another main subject in this paper.

For a QP (Q, W), we call a set C of arrows of Q a *cut* if each cycle of Q appearing in W contains precisely one arrow in C. In this case, the factor algebra of the Jacobian algebra by the ideal generated by arrows in C is called the *truncated Jacobian algebra*. For example, algebras A of global dimension two are truncated Jacobian algebras of QPs (Q_A, W_A). This gives us a structure theory of tilted algebras in terms of cluster tilted algebras [BMR06, BFPPT10]. In [IO11], a family of 2-representation-finite algebras are constructed as truncated Jacobian algebras of certain QPs. Our first main result (Theorem 3.11) asserts that 2-representation-finite algebras are exactly truncated Jacobian algebras of selfinjective QPs. This structure theorem provides us with a rich source of 2-representation-finite algebras.

Notice that selfinjective QPs are an analogue of the 'good QPs' studied by Bocklandt [Boc08] whose Jacobian algebras are 3-Calabi–Yau. In a forthcoming paper, the authors will study finite-dimensional algebras called '2-representation-infinite' by using good QPs.

We study the relationship between selfinjectivity of QPs and mutation of QPs introduced by Derksen–Weyman–Zelevinsky [DWZ08]. Our second main result (Theorem 4.2) asserts that selfinjectivity of QPs is preserved under mutation along orbits of the Nakayama permutation. This provides us with a systematic method to construct selfinjective QPs in §§ 5 and 9.

We also study derived equivalence between truncated Jacobian algebras of a fixed QP. Although all truncated Jacobian algebras of a fixed QP are cluster equivalent in the sense of Amiot [Ami09b], they are not necessarily derived equivalent. So, it is natural to ask when all truncated Jacobian algebras of QPs are derived equivalent (e.g. [AO10]). Our third main result (Theorem 7.8) provides a sufficient condition. This is based on a combinatorial consideration of Galois coverings $\widetilde{Q}$ of quivers Q, which generalizes Riedtmann's translation quivers $\mathbf{Z}Q$ associated with quivers Q. Key results are a bijection (*cut–slice correspondence*) between certain sets of arrows in Q and certain full subquivers of $\widetilde{Q}$ called *slices* (Theorem 6.8) and transitivity of a certain combinatorial operation called *cut-mutation* (Theorem 6.18). This is relevant for us as cut-mutation is a combinatorial interpretation of 2-APR tilting [IO11] (Theorem 7.7).

In § 8, we introduce a CW complex associated with every QP, which we call the *canvas*. We call a QP *simply connected* if the canvas is simply connected. Simply connectedness is useful to show that all truncated Jacobian algebras are derived equivalent. In § 9, we introduce *planar QPs*, which are analogues of QPs associated with dimer models (e.g. [Bro11, Dav08, IU09]) and provide a rich source of simply connected QPs. We shall give three mutation-equivalence classes of planar QPs which are selfinjective. It is an open question whether they are all selfinjective planar QPs.

Conventions. Throughout this paper, we denote by K an algebraically closed field and by $D = \operatorname{Hom}_K(-, K)$ the K-dual. All modules are left modules. For a quiver Q, we denote by Q_0 the set of vertices and by Q_1 the set of arrows. We denote by $a : s(a) \to e(a)$ the start and end vertices. The composition ab of arrows (respectively, morphisms) a and b means first a and then b. In particular, the projective KQ-module $(KQ)e_i$ has a basis consisting of all paths ending at i.

2. Preliminaries

2.1 Quivers with potential

Let Q be a quiver. We denote by $\widehat{KQ}$ the *complete path algebra* and by $J_{\widehat{KQ}}$ the Jacobson radical of $\widehat{KQ}$ (see [BIRS11]). Then $\widehat{KQ}$ is a topological algebra with $J_{\widehat{KQ}}$-adic topology. We denote

$$\mathrm{com}_Q := \overline{[\widehat{KQ}, \widehat{KQ}]} \subset \widehat{KQ},$$

where $\overline{(\ )}$ denotes the closure. Then $\widehat{KQ}/\mathrm{com}_Q$ has a topological basis consisting of cyclic paths considered up to cyclic permutation. Thus, there is a unique continuous linear map

$$\sigma : \widehat{KQ}/\mathrm{com}_Q \to \widehat{KQ}$$

induced by $a_1 \cdots a_n \mapsto \sum_i a_i \cdots a_n a_1 \cdots a_{i-1}$ for all cycles $a_1 \cdots a_n$. For each $a \in Q_1$, define the continuous linear maps

$$L_{a^{-1}}, R_{a^{-1}} : J_{\widehat{KQ}} \to \widehat{KQ}$$

by $L_{a^{-1}}(ap) = p$, $R_{a^{-1}}(pa) = p$, and $L_{a^{-1}}\, p' = 0 = R_{a^{-1}}\, p''$ for all paths p, p', p'' such that p' and p'' do not start, respectively, end with a.

Composing these maps, we obtain the cyclic derivative

$$\partial_a = L_{a^{-1}} \circ \sigma = R_{a^{-1}} \circ \sigma : J_{\widehat{KQ}}/\mathrm{com}_Q \to \widehat{KQ}.$$

We also define

$$\partial_{(a,b)} = L_{b^{-1}} \circ R_{a^{-1}} \circ \sigma = R_{a^{-1}} \circ L_{b^{-1}} \circ \sigma : J^2_{\widehat{KQ}}/(J^2_{\widehat{KQ}} \cap \mathrm{com}_Q) \to \widehat{KQ}.$$

A *potential* is an element $W \in J^2_{\widehat{KQ}}/(J^2_{\widehat{KQ}} \cap \mathrm{com}_Q)$. We call (Q, W) a *quiver with potential* (*QP*) and define the *Jacobian ideal* and the *Jacobian algebra* by

$$\mathcal{J}(Q, W) := \overline{\langle \partial_a W \mid a \in Q_1 \rangle},$$
$$\mathcal{P}(Q, W) := \widehat{KQ}/\mathcal{J}(Q, W).$$

respectively. In this paper we allow Q to have loops.

2.2 3-preprojective algebras and generalized cluster categories

The following notion will play an important role in this paper.

Definition 2.1. Let A be a finite-dimensional K-algebra of global dimension at most two.

(a) [KVdB11] For a presentation $A = \widehat{KQ}/\overline{\langle r_1, \dots, r_l \rangle}$ by a quiver Q and a minimal set $\{r_1, \dots, r_l\}$ of relations in $\widehat{KQ}$, we define the QP (Q_A, W_A) by:

- $Q_{A,0} = Q_0$;
- $Q_{A,1} = Q_1 \coprod C_A$ with $C_A := \{\rho_i : e(r_i) \to s(r_i) \mid 1 \leqslant i \leqslant l\}$;
- $W_A = \sum_{i=1}^{l} \rho_i r_i$.

(b) [HI11, IO09, IO11] We define the *complete* 3*-preprojective algebra* as the complete tensor algebra

$$\Pi_3(A) := \prod_{i \geqslant 0} \operatorname{Ext}^2_A(DA, A)^{\otimes_A i}$$

of the (A, A)-module $\operatorname{Ext}^2_A(DA, A)$.

The following relationship between (Q_A, W_A) and $\Pi_3(A)$ can be shown easily.

Proposition 2.2 (For example [KVdB11, Theorem 6.10]). *The K-algebras $\Pi_3(A)$ and $\mathcal{P}(Q_A, W_A)$ are isomorphic.*

Next let (Q, W) be a QP. We denote by $\Gamma(Q, W)$ the *complete Ginzburg DG algebra* (see [Gin06] and [KY11, § 2.6]). We have an isomorphism $H^0(\Gamma(Q, W)) \simeq \mathcal{P}(Q, W)$ of algebras [KY11, Lemma 2.8]. The *generalized cluster category* of (Q, W) is defined by

$$\mathcal{C}_{(Q,W)} := \operatorname{per} \Gamma(Q, W)/\mathcal{D}^{\mathrm{b}}\Gamma(Q, W),$$

where $\operatorname{per} \Gamma(Q, W)$ is the perfect derived category and $\mathcal{D}^{\mathrm{b}}\Gamma(Q, W)$ is the bounded derived category (see [KY11, § 7] for details).

We say that a K-linear triangulated category $\mathcal{C}$ is 2*-Calabi–Yau* (or 2*-CY*) if each morphism space is finite dimensional over K and there exists a functorial isomorphism

$$\operatorname{Hom}_{\mathcal{C}}(X, Y) \simeq D \operatorname{Hom}_{\mathcal{C}}(Y, X[2])$$

for any $X, Y \in \mathcal{C}$. In this case, we say that an object $T \in \mathcal{C}$ is *cluster tilting* if

$$\operatorname{add} T = \{X \in \mathcal{C} \mid \operatorname{Hom}_{\mathcal{C}}(T, X[1]) = 0\}.$$

Proposition 2.3 [Ami09a], [KY11, Theorem 7.21]. *If $\mathcal{P}(Q, W)$ is a finite-dimensional K-algebra, then $\mathcal{C}_{(Q,W)}$ is a 2-Calabi–Yau triangulated category with a cluster tilting object $\Gamma(Q, W)$ such that $\operatorname{End}_{\mathcal{C}_{(Q,W)}}(\Gamma(Q, W)) \simeq \mathcal{P}(Q, W)$.*

For a finite-dimensional K-algebra A of global dimension at most two, we define the *generalized cluster category* of A by

$$\mathcal{C}_A := \mathcal{C}_{(Q_A,W_A)}.$$

We have the following description of $\Gamma(Q_A, W_A)$ in terms of the *complete derived* 3*-preprojective DG algebra* $\mathbf{\Pi}_3(A)$ [KVdB11, § 4.1].

Proposition 2.4 [KVdB11, Theorem 6.10]. *$\mathbf{\Pi}_3(A)$ is quasi-isomorphic to $\Gamma(Q_A, W_A)$.*

In particular, $\mathcal{C}_A$ is independent of choice of the presentation of A. Moreover, we recover Proposition 2.2 by applying H^0 to the above quasi-isomorphism since we have $H^0(\mathbf{\Pi}_3(A)) = \Pi_3(A)$.

3. Truncated Jacobian algebras and selfinjective QPs

Let (Q, W) be a QP. To each subset $C \subset Q_1$ we associate a grading g_C on Q by

$$g_C(a) = \begin{cases} 1, & a \in C, \\ 0, & a \notin C. \end{cases}$$

Denote by Q_C the subquiver of Q with vertex set Q_0 and arrow set $Q_1 \backslash C$.

Definition 3.1. A subset $C \subset Q_1$ is called a *cut* if W is homogeneous of degree one with respect to g_C.

If C is a cut, then g_C induces a grading on $\mathcal{P}(Q, W)$ as well. The degree zero part of $\mathcal{P}(Q, W)$ is denoted by $\mathcal{P}(Q, W)_C$ and called the *truncated Jacobian algebra*. We have

$$\mathcal{P}(Q, W)_C = \mathcal{P}(Q, W)/\overline{\langle C\rangle} = \widehat{KQ_C}/\overline{\langle \partial_c W \mid c \in C\rangle}.$$

Definition 3.2. A cut C is called *algebraic* if the following conditions are satisfied.

- $\mathcal{P}(Q, W)_C$ is a finite-dimensional K-algebra with global dimension at most two.
- $\{\partial_c W\}_{c\in C}$ is a minimal set of generators of the ideal $\overline{\langle \partial_c W \mid c \in C\rangle}$ of $\widehat{KQ_C}$.

Truncating Jacobian algebras by algebraic cuts can be viewed as a kind of inverse to taking 3-preprojective algebras. More precisely, we have the following result.

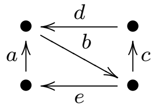
Proposition 3.3. (a) *Let A be a finite-dimensional K-algebra with* $\operatorname{gl.dim} A \leqslant 2$ *and* (Q_A, W_A) *be the associated QP. Then C_A is an algebraic cut of (Q_A, W_A) and $A \simeq \mathcal{P}(Q_A, W_A)_{C_A}$.*

(b) *Let C be an algebraic cut in a QP (Q, W) and set $A = \mathcal{P}(Q, W)_C$. Then (Q_A, W_A) and (Q, W) are isomorphic QPs.*

Proof. (a) Assume that A is as in Definition 2.1. The assertion follows from the observations $(Q_A)_{C_A} = Q$ and $\partial_{\rho_i} W_A = r_i$ for $1 \leqslant i \leqslant l$.

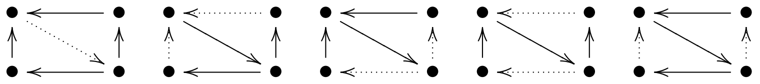
(b) By construction, there is an isomorphism $Q_A \simeq Q$ sending W_A to W. $\square$

Example 3.4. (a) Consider the following QP.



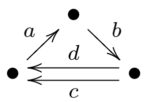
$$W = abe + bcd$$

It has five cuts.



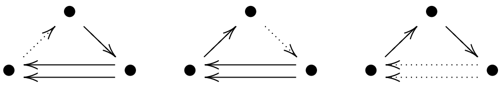
The last two cuts are not algebraic since $\mathcal{P}(Q, W)_C$ has global dimension three. The other cuts are algebraic.

(b) Next consider this QP.



$$W = abc + abd$$

It has three cuts.



The last cut is not algebraic since $\partial_c W = \partial_d W$ contradicts minimality. The other cuts are algebraic.

3.1 Selfinjective QPs

Let (Q, W) be a QP and let $\Lambda = \mathcal{P}(Q, W)$ be the Jacobian algebra. For each $i \in Q_0$, we denote by $P_i = \Lambda e_i$ the indecomposable projective left Λ-module, by I_i the maximal two-sided ideal of Λ associated with i, and by $S_i := \Lambda/I_i$ the simple Λ-bimodule. We denote by $(-)^* := \mathrm{Hom}_\Lambda(-, \Lambda) : \mathrm{proj}\,\Lambda \leftrightarrow \mathrm{proj}\,\Lambda^{\mathrm{op}}$ the duality between finitely generated projective modules. Then $P_i^* = e_i\Lambda$ is the indecomposable projective right Λ-module.

It is known that Jacobian algebras satisfy the following Iwanaga–Gorenstein property.

Proposition 3.5 [BIRS11]. *If $\mathcal{P}(Q, W)$ is finite dimensional, then*

$$\mathrm{inj.\,dim}_{\mathcal{P}(Q,W)}\, \mathcal{P}(Q, W) = \mathrm{inj.\,dim}_{\mathcal{P}(Q,W)^{\mathrm{op}}}\, \mathcal{P}(Q, W) \leqslant 1.$$

This motivates us to introduce the following class of QPs, which are the main subject of this paper.

Definition 3.6. (a) We say that a QP (Q, W) is *finite dimensional* if $\mathcal{P}(Q, W)$ is a finite-dimensional K-algebra.

(b) We say that a QP (Q, W) is *selfinjective* if $\mathcal{P}(Q, W)$ is a finite-dimensional selfinjective K-algebra.

Since

$$\sum_{a\in Q_1} (\partial_{(a,b)}W)a = \partial_b W \quad \text{and} \quad \sum_{b\in Q_1} b(\partial_{(a,b)}W) = \partial_a W,$$

we have the following complexes of left (respectively, right) Λ-modules:

$$P_i \xrightarrow{[b]} \bigoplus_{\substack{b\in Q_1\\ s(b)=i}} P_{e(b)} \xrightarrow{[\partial_{(a,b)}W]} \bigoplus_{\substack{a\in Q_1\\ e(a)=i}} P_{s(a)} \xrightarrow{[a]} P_i \to S_i \to 0, \tag{1}$$

$$P_i^* \xrightarrow{[a]} \bigoplus_{\substack{a\in Q_1\\ e(a)=i}} P^*_{s(a)} \xrightarrow{[\partial_{(a,b)}W]} \bigoplus_{\substack{b\in Q_1\\ s(b)=i}} P^*_{e(b)} \xrightarrow{[b]} P_i^* \to S_i \to 0, \tag{2}$$

which are exact everywhere except possibly at $\bigoplus P_{e(b)}$ and $\bigoplus P^*_{s(a)}$, respectively [BIRS11, § 4]. These sequences can be used to characterize selfinjective QPs.

Theorem 3.7. *The following conditions are equivalent for a QP (Q, W).*

(a) *(Q, W) is selfinjective.*

(b) *(Q, W) is finite dimensional and (1) is exact.*

(c) *(Q, W) is finite dimensional and (2) is exact.*

Proof. We only prove that (a) is equivalent to (b).

Condition (a) holds if and only if $\mathrm{Ext}^1_\Lambda(S_i, \Lambda) = 0$ for all $i \in Q_0$. This is equivalent to that we get an exact sequence by applying $\mathrm{Hom}_\Lambda(-, \Lambda)$ to the first part

$$\bigoplus_{\substack{a\in Q_1\\ e(a)=i}} P^*_{s(a)} \xrightarrow{[\partial_{(a,b)}W]} \bigoplus_{\substack{b\in Q_1\\ s(b)=i}} P^*_{e(b)} \xrightarrow{[b]} P_i^* \to S_i \to 0$$

of (2). This is equivalent to that (1) is exact. □

Immediately we have the following result.

PROPOSITION 3.8. *If a QP (Q, W) is selfinjective, then each arrow in Q appears in W.*

Proof. Assume that there is an arrow $b : i \to e(b)$ that does not appear in W. Then the middle map $[\partial_{(a,b)}W]$ in (1) is zero on the direct summand $P_{e(b)}$. This contradicts the exactness of (1) since $b : P_i \to P_{e(b)}$ is not surjective. □

3.2 Structure theorem of 2-representation-finite algebras

For a finite-dimensional algebra A, we say that $M \in \operatorname{mod} A$ is *cluster tilting* [Iya07a, Iya08] if

$$\begin{aligned} \operatorname{add} M &= \{X \in \operatorname{mod} A \mid \operatorname{Ext}^1_A(M, X) = 0\} \\ &= \{X \in \operatorname{mod} A \mid \operatorname{Ext}^1_A(X, M) = 0\}. \end{aligned}$$

It is known that the category add M is a higher analogue of the module category mod A in the sense that there exist 2-AR translation functors

$$\begin{tikzcd} \operatorname{add} M \arrow[r, shift left, "\tau_2 = D\operatorname{Ext}^2_A(-,A)"] & \operatorname{add} M \arrow[l, shift left, "\tau_2^- = \operatorname{Ext}^2_A(DA,-)"] \end{tikzcd}$$

which give 2-almost split sequences in add M.

We say that an algebra A is 2*-representation-finite* if there exists a cluster tilting A-module and gl. dim $A \leqslant 2$. In this case, it is known that $\Pi_3(A)$ regarded as an A-module gives a unique basic 2-cluster tilting A-module [Iya11].

We have the following criterion for 2-representation-finiteness.

PROPOSITION 3.9. *Let A be a K-algebra such that* gl. dim $A \leqslant 2$. *Then the following conditions are equivalent.*

(a) *A is 2-representation-finite.*

(b) *$\Pi_3(A)$ is a finite-dimensional selfinjective algebra.*

(c) *(Q_A, W_A) is a selfinjective QP.*

Proof. The equivalence of (a) and (b) is shown in [IO09]. The equivalence of (b) and (c) follows from Proposition 2.2. □

Propositions 3.3 and 3.9 give a strong motivation for studying selfinjective QPs and their algebraic cuts. To do the latter, the following result is very useful.

PROPOSITION 3.10. *Let (Q, W) be a selfinjective QP. Then every cut $C \subset Q_1$ is algebraic.*

Proof. We consider $\Lambda := \mathcal{P}(Q, W)$ to be graded by g_C. Let $A := \mathcal{P}(Q, W)_C$. For each $i \in Q_0$, set $P_i = \Lambda e_i$ and $\overline{P}_i = Ae_i$. Then P_i is a graded Λ-module and the part in degree zero is $(P_i)_0 = \overline{P}_i$. Fix $i \in Q_0$. By Proposition 3.7, we have an exact sequence

$$P_i \xrightarrow{[b]} \bigoplus_{\substack{b \in Q_1 \\ s(b)=i}} P_{e(b)} \xrightarrow{[\partial_{(a,b)}W]} \bigoplus_{\substack{a \in Q_1 \\ e(a)=i}} P_{s(a)} \xrightarrow{[a]} P_i \to S_i \to 0.$$

Note that $\partial_{(a,b)}W$ is homogeneous of degree $1 - g_C(a) - g_C(b)$. Thus, taking appropriate degree shifts we obtain the following exact sequence of graded Λ-modules and homogeneous morphisms of degree zero.

$$P_i(-1) \xrightarrow{[[b]\ [c]]} \begin{array}{c} \bigoplus\limits_{\substack{b\notin C\\ s(b)=i}} P_{e(b)}(-1) \\ \oplus \bigoplus\limits_{\substack{c\in C\\ s(c)=i}} P_{e(c)} \end{array} \xrightarrow{\left[\begin{smallmatrix}[\partial_{(d,b)}W] & [\partial_{(a,b)}W]\\ 0 & [\partial_{(a,c)}W]\end{smallmatrix}\right]} \begin{array}{c} \bigoplus\limits_{\substack{d\in C\\ e(d)=i}} P_{s(d)}(-1) \\ \oplus \bigoplus\limits_{\substack{a\notin C\\ e(a)=i}} P_{s(a)} \end{array} \xrightarrow{\left[\begin{smallmatrix}[d]\\ [a]\end{smallmatrix}\right]} P_i \to S_i \to 0.$$

Taking the degree zero part of the above sequence, we get an exact sequence

$$0 \to \bigoplus_{\substack{c\in C\\ s(c)=i}} \overline{P}_{e(c)} \xrightarrow{[\partial_{(a,c)}W]} \bigoplus_{\substack{a\notin C\\ e(a)=i}} \overline{P}_{s(a)} \xrightarrow{[a]} \overline{P}_i \to S_i \to 0.$$

Hence, gl. dim $A \leqslant 2$. Moreover, since the above sequence is exact it is also minimal and thus C is algebraic. □

We now give a structure theorem of 2-representation-finite algebras in terms of selfinjective QPs.

Theorem 3.11. (a) *For any selfinjective QP (Q, W) and cut C, the truncated Jacobian algebra $\mathcal{P}(Q, W)_C$ is 2-representation-finite.*

(b) *Every basic 2-representation-finite algebra appears in this way.*

In particular, truncated Jacobian algebras of selfinjective QPs are exactly basic 2-representation-finite algebras.

Proof. (a) By Proposition 3.10, we have that C is algebraic and so $A := \mathcal{P}(Q, W)_C$ has global dimension at most two. By Proposition 3.3(b), we have

$$\mathcal{P}(Q_A, W_A) \simeq \mathcal{P}(Q, W),$$

which is selfinjective. Therefore, by Proposition 3.9 A is 2-representation-finite.

(b) Up to Morita equivalence, every 2-representation-finite algebra is of the form

$$A = KQ/\langle r_1, \ldots, r_n\rangle = \widehat{KQ}/\overline{\langle r_1, \ldots, r_n\rangle}$$

for some minimal set of admissible relations $\{r_1, \ldots, r_n\}$. By Proposition 3.9, (Q_A, W_A) is selfinjective. Moreover, C_A is a cut and $A \simeq \mathcal{P}(Q_A, W_A)_{C_A}$ holds by Proposition 3.3(a). □

Theorem 3.11 reduces the description of 2-representation-finite algebras to truncated Jacobian algebras $\mathcal{P}(Q, W)_C$ of selfinjective QPs (Q, W) and their cuts C.

In order to construct 2-representation-finite algebras, we want to find selfinjective QPs. In the next section, we show that new selfinjective QPs can be constructed from a given one by mutation.

4. Mutation and selfinjectivity

4.1 Mutation of selfinjective QPs

Let (Q, W) be a QP and $\Lambda := \mathcal{P}(Q, W)$. When $k \in Q_0$ is not contained in any 2-cycle, we have a new QP $\mu_k(Q, W)$ called the *mutation* of (Q, W) at k (see [DWZ08]). Simple examples show that mutation does not preserve selfinjectivity of QPs. In this section, we define iterated mutation with respect to orbits of the Nakayama permutation (see also [BO11]) and show that it preserves selfinjectivity.

DEFINITION 4.1. Let (Q, W) be a selfinjective QP.

We denote by $\sigma : Q_0 \to Q_0$ the Nakayama permutation, so $D(e_i\Lambda) \simeq \Lambda e_{\sigma i}$ for any $i \in Q_0$.

For $k \in Q_0$, we denote by $(k) = \{k = \sigma^m k, \sigma k, \ldots, \sigma^{m-1}k\}$ the σ-orbit of k. We assume that:

- k is not contained in 2-cycles in Q;
- there are no arrows between $k, \sigma k, \ldots, \sigma^{m-1}k$ in Q.

In this case, we define the iterated mutation

$$\mu_{(k)}(Q, W) := \mu_{\sigma^{m-1}k} \circ \cdots \circ \mu_k(Q, W)$$

with respect to the orbit (k). This is well-defined since $\sigma^i k$ is not contained in 2-cycles in the quiver of $\mu_{\sigma^{i-1}k} \circ \cdots \circ \mu_k(Q, W)$ for any $0 \leqslant i < m$ by our assumption. Moreover, $\mu_{(k)}(Q, W)$ is independent of the choice of order of mutations $\mu_k, \ldots, \mu_{\sigma^{m-1}k}$.

The following key observation will be proved in the next section.

THEOREM 4.2. *Let (Q, W) be a selfinjective QP with the Nakayama permutation $\sigma : Q_0 \to Q_0$. Assume that $k \in Q_0$ satisfies the two conditions in Definition 4.1.*

(a) *The above mutation $\mu_{(k)}(Q, W)$ is again a selfinjective QP.*

(b) *The Nakayama permutation of $\mu_{(k)}(Q, W)$ is again given by σ.*

4.2 Selfinjective cluster tilting objects

Throughout this section, let $\mathcal{C}$ be a 2-CY triangulated category (see § 2.2). We consider a special class of cluster tilting objects in $\mathcal{C}$ and their mutation.

DEFINITION 4.3. We say that a cluster tilting object $T \in \mathcal{C}$ is *selfinjective* if $\mathrm{End}_{\mathcal{C}}(T)$ is a finite-dimensional selfinjective K-algebra.

We have the following criterion for selfinjectivity.

PROPOSITION 4.4 [IO09]. *Let $T = T_1 \oplus \cdots \oplus T_n \in \mathcal{C}$ be a basic cluster tilting object with indecomposable summands T_i.*

(a) *T is selfinjective if and only if $T \simeq T[2]$.*

(b) *In this case, we define a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ by $T_i[2] \simeq T_{\sigma i}$. Then σ gives the Nakayama permutation of $\mathrm{End}_{\mathcal{C}}(T)$.*

In the rest of this section, we shall show that selfinjectivity of cluster tilting objects is preserved under a certain kind of iterated cluster tilting mutation.

Let $T \in \mathcal{C}$ be a basic cluster tilting object and let $T = U \oplus V$ be a decomposition. We take triangles (called *exchange triangles*)

$$U^* \longrightarrow V' \xrightarrow{f} U \longrightarrow U^*[1] \quad \text{and} \quad U \xrightarrow{g} V'' \longrightarrow {}^*U \longrightarrow U[1] \tag{3}$$

with a minimal right $(\mathrm{add}\, V)$-approximation f of U and a minimal left $(\mathrm{add}\, V)$-approximation g of U. Let

$$\mu_U^+(T) := U^* \oplus V \quad \text{and} \quad \mu_U^-(T) := {}^*U \oplus V.$$

Notice that U is not assumed to be indecomposable. A typical case of cluster tilting mutation is given by $\mu_T^+(T) = T[-1]$ and $\mu_T^-(T) = T[1]$.

We have the following basic results.

Proposition 4.5 [IY08]. (a) *$\mu_U^+(T)$ and $\mu_U^-(T)$ are basic cluster tilting objects in $\mathcal{C}$.*

(b) *If U is indecomposable, then $\mu_U^+(T)$ and $\mu_U^-(T)$ are isomorphic, which we denote by $\mu_U(T)$.*

We have the following general result.

Proposition 4.6. *Let $T = U \oplus V$ be a basic selfinjective cluster tilting object. If $U \simeq U[2]$, then $\mu_U^+(T)$ and $\mu_U^-(T)$ are selfinjective cluster tilting objects in $\mathcal{C}$.*

Proof. Clearly, we have $V \simeq V[2]$. Let $a : U \to U[2]$ be an isomorphism and

$$\begin{tikzcd} U^* \arrow[r] & V' \arrow[r, "f"] & U \arrow[r] & U^*[1] \end{tikzcd}$$

be an exchange triangle. Then $fa : V' \to U[2]$ is a minimal right $(\mathrm{add}\, V)$-approximation. Since $V \simeq V[2]$, we have that $f[2] : V'[2] \to U[2]$ is also a minimal right $(\mathrm{add}\, V)$-approximation. Thus, we have a commutative diagram

$$\begin{tikzcd} U^* \arrow[r] & V' \arrow[r, "f"] \arrow[d, "b"] & U \arrow[r] \arrow[d, "a"] & U^*[1] \\ U^*[2] \arrow[r] & V'[2] \arrow[r, "{f[2]}"] & U[2] \arrow[r] & U^*[3] \end{tikzcd}$$

of triangles with isomorphisms a and b. By an axiom of triangulated categories, there exists an isomorphism $c : U^* \to U^*[2]$ which keeps the above diagram commutative. In particular, we have $U^* \simeq U^*[2]$ and so $\mu_U^+(T) \simeq \mu_U^+(T)[2]$. Thus, $\mu_U^+(T)$ is a selfinjective cluster tilting object by Proposition 4.4. □

Now let us observe $\mu_U^+(T)$ and $\mu_U^-(T)$ more explicitly. We take decompositions $U = T_1 \oplus \cdots \oplus T_m$ and $V = T_{m+1} \oplus \cdots \oplus T_n$ into indecomposable summands. For $i = 1, \ldots, m$, we take triangles

$$\begin{tikzcd} T_i^\vee \arrow[r] & V_i' \arrow[r, "f_i"] & T_i \arrow[r] & T_i^\vee[1] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T_i \arrow[r, "g_i"] & V_i'' \arrow[r] & {}^\vee T_i \arrow[r] & T_i[1] \end{tikzcd}$$

with a minimal right $(\mathrm{add}\, V)$-approximation f_i of T_i and a minimal left $(\mathrm{add}\, V)$-approximation g_i of T_i. Then by definition we have decompositions

$$\mu_U^+(T) \simeq T_1^\vee \oplus \cdots \oplus T_m^\vee \oplus T_{m+1} \cdots \oplus T_n \quad \text{and} \quad \mu_U^-(T) \simeq {}^\vee T_1 \oplus \cdots \oplus {}^\vee T_m \oplus T_{m+1} \cdots \oplus T_n \tag{4}$$

into indecomposable summands.

Proposition 4.7. *In Proposition 4.6, the Nakayama permutations of $\mu_U^+(T)$ and $\mu_U^-(T)$ with respect to the decompositions in (4) coincide with that of T.*

Proof. Using an isomorphism $T_{\sigma i} \to T_i[2]$, we get isomorphisms $T_{\sigma i}^\vee \to T_i^\vee[2]$ and ${}^\vee T_{\sigma i} \to {}^\vee T_i[2]$ by a similar argument as in the proof of Proposition 4.6. Thus, the assertion follows. □

Although we do not assume that U is indecomposable, sometimes we can decompose μ_U as the iterated cluster tilting mutation with respect to indecomposable summands.

Proposition 4.8. *If there are no arrows between vertices $T_1, \ldots, T_m$ in the quiver of $\mathrm{End}_{\mathcal{C}}(T)$, then we have $\mu_U^+(T) \simeq \mu_U^-(T) \simeq \mu_{T_m} \circ \cdots \circ \mu_{T_1}(T)$.*

Proof. Inductively, we shall show that $\mu_{T_k} \circ \cdots \circ \mu_{T_1}(T)$ is isomorphic to

$$T^{(k)} := T_1^\vee \oplus \cdots \oplus T_k^\vee \oplus T_{k+1} \oplus \cdots \oplus T_n$$

for any $1 \leqslant k \leqslant m$. We assume that this is true for $k = i-1$, and we shall show that $\mu_{T_i}(T^{(i-1)}) \simeq T^{(i)}$. We only have to show that a right (add V)-approximation of T_i is also a right (add $T^{(i-1)}/T_i$)-approximation. This is true since any morphism $T_{i+1} \oplus \cdots \oplus T_n \to T_i$ factors through f_i by our assumption, and we have a triangle $T_k^\vee \to V_k' \to T_k \to T_k^\vee[1]$, which implies that any morphism $T_k^\vee \to T_i$ factors through $V_k' \in \operatorname{add} V$.

Consequently, we have $\mu_{T_m} \circ \cdots \circ \mu_{T_1}(T) \simeq T_1^\vee \oplus \cdots \oplus T_m^\vee \oplus T_{m+1} \oplus \cdots \oplus T_n \simeq \mu_U^+(T)$. ☐

The following result shows compatibility of cluster tilting mutation and QP mutation.

PROPOSITION 4.9 [BIRS11, Theorem 5.2]. *Let $\mathcal{C}$ be a 2-CY triangulated category and $T = T_1 \oplus \cdots \oplus T_n \in \mathcal{C}$ a basic cluster tilting object with indecomposable summands T_i. If $\operatorname{End}_{\mathcal{C}}(T) \simeq \mathcal{P}(Q, W)$ for a QP (Q, W), $k \in Q_0$ is not contained in 2-cycles in Q and the glueing condition is satisfied at k, then $\operatorname{End}_{\mathcal{C}}(\mu_{T_k}(T)) \simeq \mathcal{P}(\mu_k(Q, W))$.*

Now we are ready to prove Theorem 4.2.

Let $\mathcal{C} := \mathcal{C}_{(Q,W)}$ be a generalized cluster category in § 2.2. Then there exists a cluster tilting object $T \in \mathcal{C}$ such that $\operatorname{End}_{\mathcal{C}}(T) \simeq \mathcal{P}(Q, W)$ by Proposition 2.3. Let U be the summand of T corresponding to the vertices $k, \sigma k, \ldots, \sigma^{m-1}k$. By Proposition 4.4(b), we have $U \simeq U[2]$. By Proposition 4.6, we know that $T' := \mu_U^+(T)$ is a selfinjective cluster tilting object. Thus, $\operatorname{End}_{\mathcal{C}}(T')$ is a selfinjective algebra. On the other hand, T' is isomorphic to $\mu_{T_{\sigma^{m-1}k}} \circ \cdots \circ \mu_{T_k}(T)$ by Proposition 4.8. Since (Q, W) is selfinjective, the glueing condition is satisfied at any vertex by [BIRS11, Theorem 4.9]. Since there are no arrows between the vertices $k, \sigma k, \ldots, \sigma^{m-1}k$, it is easy to see that $\mu_{\sigma^{i-1}k} \circ \cdots \circ \mu_k(Q, W)$ satisfies the glueing condition at the vertices $\sigma^i k, \ldots, \sigma^{m-1}k$. Applying Proposition 4.9 repeatedly, we have

$$\operatorname{End}_{\mathcal{C}}(T') \simeq \mathcal{P}(\mu_{\sigma^{m-1}k} \circ \cdots \circ \mu_k(Q, W)),$$

so we have the assertion.

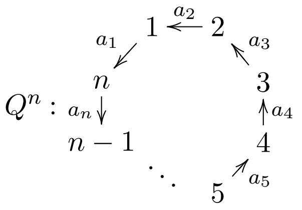
The statement for Nakayama permutation follows from Proposition 4.7. ☐

5. Examples of selfinjective QPs

5.1 Examples from cluster categories

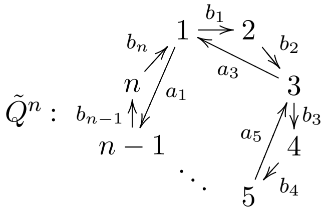
Cluster tilted algebras have been shown to be Jacobian algebras of QPs [BIRS11, KVdB11]. The selfinjective ones have been classified by Ringel [Rin08]. The classification consists of the two families described below.

For each $n > 3$, define the following QP.



$$W^n = a_n \cdots a_1$$

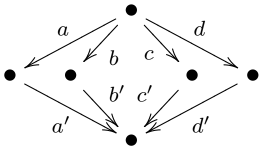
If n is even, then also define the following QP.



$$\tilde{W}^n = a_{n-1}a_{n-3}\cdots a_1 - \sum a_i b_{i-2} b_{i-1}$$

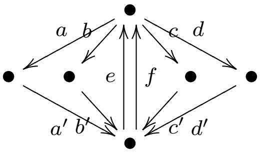
These QPs are selfinjective with Nakayama automorphism induced by $i \mapsto i-2$. Notice that mutating at the even vertices in $(\tilde{Q}^n, \tilde{W}^n)$ gives (Q^n, W^n). This is not surprising as the result must again be cluster tilted by [BIRS11] and selfinjective by Theorem 4.2. By [Rin08], there is only one alternative.

Next we consider cluster categories associated with canonical algebras. Selfinjective cluster tilting objects are classified in [HILO] (see also [BG09]). Let us give one example. It is shown in [IO09] that the tubular algebra of type (2, 2, 2, 2) given below is 2-representation-finite.



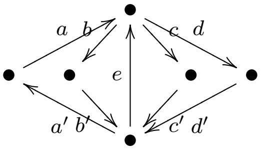
$$aa' + bb' + cc' = 0 = aa' + \lambda bb' + dd'$$

In particular, we have the following selfinjective QP.



$$W = aa'e + bb'e + cc'e + aa'f + \lambda bb'f + dd'f$$

The Nakayama permutation is the identity. Mutating at the leftmost vertex, we have the following selfinjective QP with $\lambda' = \lambda/(\lambda - 1)$.



$$W = bb'e + cc'e + dd'e + \lambda' bb'a'a + dd'a'a$$

5.2 Examples from pairs of Dynkin quivers

In this section, we construct a new class of selfinjective QPs.

Let Q^1 and Q^2 be finite quivers without oriented cycles. Define $Q = Q^1 \tilde{\otimes} Q^2$ by

$$Q_0 = Q_0^1 \times Q_0^2$$

and

$$Q_1 = (Q_0^1 \times Q_1^2) \coprod (Q_1^1 \times Q_0^2) \coprod (Q_1^1 \times Q_1^2),$$

where

$$s(a, y) = (s(a), y), \quad s(x, b) = (x, s(b)), \quad s(a, b) = (e(a), e(b)),$$
$$e(a, y) = (e(a), y), \quad e(x, b) = (x, e(b)), \quad e(a, b) = (s(a), s(b))$$

for any $x \in Q^1_0$, $y \in Q^2_0$, $a \in Q^1_1$, and $b \in Q^2_1$. Also, define the potential

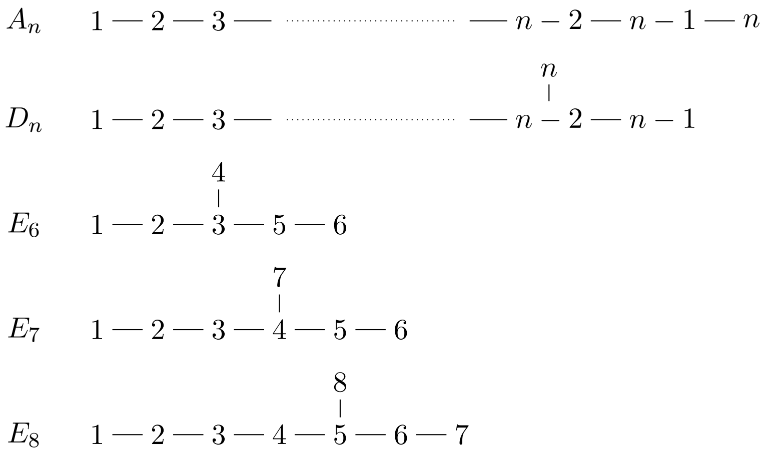
$$W = W^{\tilde{\otimes}}_{Q^1,Q^2} = \sum_{\substack{a \in Q^1_1 \\ b \in Q^2_1}} (s(a), b)(a, e(b))(a, b) - (a, s(b))(e(a), b)(a, b).$$

By the results in [Kel10], $C = Q^1_1 \times Q^2_1$ is an algebraic cut and there is an algebra isomorphism

$$\mathcal{P}(Q, W)_C \simeq KQ^1 \otimes KQ^2$$

mapping $(x, y) \mapsto e_x \otimes e_y$, $(a, y) \mapsto a \otimes e_y$ and $(x, b) \mapsto e_x \otimes b$. Hence, $\Pi_3(KQ^1 \otimes KQ^2) \simeq \mathcal{P}(Q, W)$. As is common, we denote Q_C by $Q^1 \otimes Q^2$.

Let us consider Dynkin diagrams.



We define a canonical involution ω of each Dynkin diagram as follows.

- For A_n, we put $\omega(i) = n + 1 - i$.
- For D_n with odd n, we put $\omega(n-1) = n$, $\omega(n) = n - 1$, and $\omega(i) = i$ for other i.
- For E_6, we put $\omega(1) = 6$, $\omega(2) = 5$, $\omega(5) = 2$, $\omega(6) = 1$, and $\omega(i) = i$ for other i.
- For other types, we put $\omega = \mathrm{id}$.

We call a Dynkin quiver *stable* if it is stable under ω.

The *Coxeter number* h of each Dynkin diagram is given as follows.

A_n	D_n	E_6	E_7	E_8
$n+1$	$2(n-1)$	12	18	30

PROPOSITION 5.1. *Let Q^1 and Q^2 be Dynkin quivers with canonical involutions ω_1 and ω_2, respectively. Assume that they are stable and have the same Coxeter numbers. Then $(Q^1 \tilde{\otimes} Q^2, W^{\tilde{\otimes}}_{Q^1,Q^2})$ is a selfinjective QP with Nakayama permutation induced by $\omega_1 \times \omega_2$.*

Proof. Under the assumptions, $KQ^1 \otimes KQ^2$ is 2-representation-finite by [HI11]. Since

$$\mathcal{P}(Q^1 \tilde{\otimes} Q^2, W^{\tilde{\otimes}}_{Q^1,Q^2}) \simeq \Pi_3(KQ^1 \otimes KQ^2),$$

Proposition 3.9 implies that $(Q^1 \tilde{\otimes} Q^2, W^{\tilde{\otimes}}_{Q^1,Q^2})$ is selfinjective. Moreover, by the results in [HI11] the Nakayama permutation is induced by $\omega_1 \times \omega_2$. □

We use the class of selfinjective QPs in Proposition 5.1 to construct another one by mutation.

Let Q^1 and Q^2 be Dynkin quivers with alternating orientation (i.e. each vertex is either a sink or a source). Write $Q_0^i = X_i \coprod Y_i$, where X_i consists of sources and Y_i of sinks, i.e. for each arrow $x \xrightarrow{a} y \in Q_1^i$ it holds that $x \in X_i$ and $y \in Y_i$.

Recall from [Kel10] that the square product $Q^1 \square Q^2$ is the quiver obtained from $Q^1 \otimes Q^2$ by replacing each arrow a satisfying $s(a) \in Y_1 \times X_2$ or $e(a) \in Y_1 \times X_2$ with an arrow a^* of opposite orientation. Let $W^{\square}_{Q^1,Q^2}$ be the potential defined by the sum of all cycles of the form

$$(x, b)(a, y')(y, b)^*(a, x')^*,$$

where $x \xrightarrow{a} y \in Q_1^1$ and $x' \xrightarrow{b} y' \in Q_1^2$.

THEOREM 5.2. *Let Q^1 and Q^2 be Dynkin quivers with alternating orientation such that their Coxeter numbers coincide. Let ω_1 and ω_2 be the corresponding canonical involutions. Then $(Q^1 \square Q^2, W^{\square}_{Q^1,Q^2})$ is a selfinjective QP with Nakayama permutation induced by $\omega_1 \times \omega_2$.*

Proof. Let $Y_1 \times X_2 = \{i_1, \ldots, i_s\}$ and set

$$(Q, W) = \mu_{i_s} \circ \cdots \circ \mu_{i_1}(Q^1 \square Q^2, W^{\square}_{Q^1,Q^2}).$$

We proceed to show that $(Q, W) \simeq (Q^1 \tilde{\otimes} Q^2, W^{\tilde{\otimes}}_{Q^1,Q^2})$.

The paths of length two through $(y, x') \in Y_1 \times X_2$ are exactly those of the form $(y, b)^*(a, x')^*$, where $x \xrightarrow{a} y \in Q_1^1$ and $x' \xrightarrow{b} y' \in Q_1^2$ for some $x \in X_1$ and $y' \in Y_2$. Hence,

$$W = \sum_{a,b} (x, b)(a, y')[(y, b)^*(a, x')^*] + (a, x')^{**}(y, b)^{**}[(y, b)^*(a, x')^*].$$

Now consider the isomorphism $\widehat{KQ} \simeq K(\widehat{Q^1 \tilde{\otimes} Q^2})$ defined by $[(y, b)^*(a, x')^*] \mapsto (a, b)$, $(a, x')^{**} \mapsto (a, x')$, $(y, b)^{**} \mapsto -(y, b)$, and identity on all other arrows. It maps the potential W to

$$\sum (x, b)(a, y')(a, b) - (a, x')(y, b)(a, b).$$

Thus, the claim is proved.

Now observe that if the common Coxeter number is even, then every alternating orientation is in fact stable. Hence, $(Q^1 \tilde{\otimes} Q^2, W^{\tilde{\otimes}}_{Q^1,Q^2})$ is selfinjective by Proposition 5.1. Moreover, $Y_1 \times X_2$ is invariant under the Nakayama permutation which is induced by $\omega_1 \times \omega_2$. Hence,

$$(Q^1 \square Q^2, W^{\square}_{Q^1,Q^2}) \simeq \mu_{i_1} \circ \cdots \circ \mu_{i_s}(Q, W)$$

is selfinjective with Nakayama permutation induced by $\omega_1 \times \omega_2$ by Theorem 4.2.

It remains to consider the case when the Coxeter number is odd. It is covered by the following proposition. □

PROPOSITION 5.3. *Let Q be a Dynkin quiver of type $\mathbb{A}_{2s}$ with alternating orientation. Then $(Q \square Q, W^{\square}_{Q,Q})$ is a selfinjective QP with Nakayama permutation induced by $\omega \times \omega$.*

Proof. We label the arrows in Q as follows.

$$\begin{tikzcd} 1 \arrow[r, "a_1"] & 2 & 3 \arrow[l, "b_3"'] \arrow[r, "a_3"] & \cdots \arrow[r, "a_{2s-1}"] & 2s \end{tikzcd}$$

Thus, $Q \square Q$ is the quiver:

$$\begin{tikzcd}
(1,2s) \arrow[r, "{(a_1,2s)}"] & (2,2s) \arrow[d, "{(2,a_{2s-1})^*}"] & (3,2s) \arrow[l, "{(b_3,2s)}"'] \arrow[r, "{(a_3,2s)}"] & \cdots \arrow[r, "{(a_{2s-1},2s)}"] & (2s,2s) \arrow[d, "{(2s,a_{2s-1})^*}"] \\
\vdots \arrow[u, "{(1,a_{2s-1})}"'] & \vdots \arrow[d, "{(2,a_3)^*}"] & \vdots \arrow[u, "{(3,a_{2s-1})}"'] & \iddots & \vdots \arrow[d, "{(2s,a_3)^*}"] \\
(1,3) \arrow[u, "{(1,a_3)}"'] \arrow[d, "{(1,b_3)}"] & (2,3) \arrow[l, "{(a_1,3)^*}"'] \arrow[r, "{(b_3,3)^*}"] & (3,3) \arrow[u, "{(3,a_3)}"'] \arrow[d, "{(3,b_3)}"] & \cdots \arrow[l, "{(a_3,3)^*}"'] & (2s,3) \arrow[l, "{(a_{2s-1},3)^*}"'] \\
(1,2) \arrow[r, "{(a_1,2)}"] & (2,2) \arrow[u, "{(2,b_3)^*}"'] \arrow[d, "{(2,a_1)^*}"] & (3,2) \arrow[l, "{(b_3,2)}"'] \arrow[r, "{(a_3,2)}"] & \cdots \arrow[r, "{(a_{2s-1},2)}"] & (2s,2) \arrow[u, "{(2s,b_3)^*}"'] \arrow[d, "{(2s,a_1)^*}"] \\
(1,1) \arrow[u, "{(1,a_1)}"'] & (2,1) \arrow[l, "{(a_1,1)^*}"'] \arrow[r, "{(b_3,1)^*}"] & (3,1) \arrow[u, "{(3,a_1)}"'] & \cdots \arrow[l, "{(a_3,1)^*}"'] & (2s,1) \arrow[l, "{(a_{2s-1},1)^*}"']
\end{tikzcd}$$

The potential $W^{\square}_{Q,Q}$ is the sum of all cycles of length four that correspond to the small squares appearing above. Consider the automorphism $K\widehat{(Q \square Q)} \simeq K\widehat{(Q \square Q)}$ defined by multiplying all arrows of the form $(2k-1, a_{2i+1})$ and $(a_{2i+1}, 2k)$ by -1 and all other arrows by 1. It sends $W^{\square}_{Q,Q}$ to the potential W given as the sum of all clockwise oriented small squares minus the sum of all counterclockwise oriented small squares. Thus, it suffices to show that $(Q \square Q, W)$ is selfinjective with Nakayama permutation induced by $\omega \times \omega$.

Set $\Lambda = \mathcal{P}(Q \square Q, W)$. We say that two paths p and p' in $Q \square Q$ are congruent (written $p \equiv p'$) if $p - p' \in \langle \partial_a W \mid a \in (Q \square Q)_1 \rangle$. Similarly, we write $p \equiv 0$ if $p \in \langle \partial_a W \mid a \in (Q \square Q)_1 \rangle$. Since the relations $\partial_a W$ are all commutativity or zero relations, the set B of non-zero congruence classes of paths forms a basis in Λ. For each $(i,j) \in (Q \square Q)_0$, let $B_{(i,j)}$ and ${}_{(i,j)}B$ be the subsets of B corresponding to paths ending and starting at (i,j), respectively. These sets form bases of $\Lambda e_{(i,j)}$ and $D(e_{(i,j)}\Lambda)$, respectively. We proceed to define a bijection ${}_{(i,j)}B \simeq B_{\sigma(i,j)}$ which induces an isomorphism $D(e_{(i,j)}\Lambda) \simeq \Lambda e_{\sigma(i,j)}$, thus completing the proof.

Consider the four cuts

$$\begin{aligned} R &= \{(a_{2i+1}, 2k), (b_{2j+1}, 2k-1)^*\}_{i,j,k}, & U &= \{(2k-1, a_{2i+1}), (2k, b_{2j+1})^*\}_{i,j,k}, \\ L &= \{(a_{2i+1}, 2k-1)^*, (b_{2j+1}, 2k)\}_{i,j,k}, & D &= \{(2k, a_{2i+1})^*, (2k-1, b_{2j+1})\}_{i,j,k}. \end{aligned}$$

We define a $\mathbf{Z}^4$ grading on $(Q \square Q, W)$ by $\deg(a) = (g_R(a), g_U(a), g_L(a), g_D(a))$. Moreover, we associate to each path $p = a_1 \cdots a_d$ in Q an element $s(p) = (s_1, \ldots, s_d) \in \{\pm 1\}^d$ by setting $s_i = 1$ if $a_i \in R \cup U$ and $s_i = -1$ if $a_i \in L \cup D$.

At each vertex $x \in (Q \square Q)_0$, there are at most two arrows leaving that vertex. They both lie in either $R \cup L$ or $U \cup D$. The same is true for arrows entering x. Hence, each path p is determined by $s(p)$ together with either the starting point or the end point of p. Moreover, if $s(p) = (u, v, -u)$ for some $u, v \in \{\pm 1\}$, then either $p \equiv 0$ or there is $p' \equiv p$ such that $s(p') = (-u, v, u)$.

We conclude that for each path $p \not\equiv 0$ there is a path $p' \equiv p$ such that $s(p')_i \geqslant s(p')_{i+2}$ for all i. This condition determines $s(p')$ by $\deg(p)$ together with either the starting point or the end point of p. Thus, p is determined up to $\equiv$ by $\deg(p)$ together with either the starting point or the end point of p.

For each $(i,j) \in (Q \,\Box\, Q)_0$, set $d_{ij} = (2s-i, 2s-j, i-1, j-1) \in \mathbf{Z}^4$. For $a, b \in \mathbf{Z}^4$, we write $a \leqslant b$ if $a_i \leqslant b_i$ for all $1 \leqslant i \leqslant 4$. Let p be a path starting at $(i,j) \in (Q \,\Box\, Q)_0$. We claim that $p \equiv 0$ if and only if $\deg(p) \nleqslant d_{ij}$. We only treat the case when $\deg(p)_1 > 2s-i$ and $i+j$ is even as the other cases are similar. By the arguments above, there is a path $p' \equiv p$ such that $s(p')_{2k} = 1$ for $1 \leqslant k \leqslant 2s-i$, $s(p')_{4s-2i+2} = -1$, and $s(p')_{4s-2i+4} = 1$. Then the arrows in positions $4s-2i+2$, $4s-2i+3$, and $4s-2i+4$ in p' form one of the zero relations in $\{\partial_a W \mid a \in (Q \,\Box\, Q)_1\}$. Hence, $p \equiv 0$. On the other hand, if none of the inequalities above hold, then it is impossible to reach any of the zero relations in $\{\partial_a W \mid a \in (Q \,\Box\, Q)_1\}$.

Observe that for each $(i,j) \in (Q \,\Box\, Q)_0$ there is a path p_{ij} starting at (i,j) of degree d_{ij}. In particular, p_{ij} ends at $(2s-i+1, 2s-j+1) = (\omega \times \omega)(i,j)$. Now let p be a path starting at (i,j) satisfying $p \not\equiv 0$. Then $\deg(p) \leqslant \deg(p_{ij})$ and so there is a path q ending at $(\omega \times \omega)(i,j)$ such that $pq \equiv p_{ij}$. Thus, $\deg(q) = d_{ij} - \deg(p)$ and so q is determined uniquely up to $\equiv$ by p. Similarly, one can show that for each path $q \not\equiv 0$ ending at $(\omega \times \omega)(i,j)$ there is a path p starting at (i,j) unique up to $\equiv$ such that $pq \equiv p_{ij}$.

We define the bijection $\phi : {}_{(i,j)}B \to B_{\sigma(i,j)}$ by $\phi(p) = q$, where $pq \equiv p_{ij}$. It is easy to see that ϕ induces a Λ-module isomorphism $D(e_{(i,j)}\Lambda) \simeq \Lambda e_{\sigma(i,j)}$. $\square$

6. Covering theory of truncated quivers

In the previous sections, we have seen how mutation can be used to construct and organize selfinjective QPs. To construct 2-representation-finite algebras, we also need to consider their cuts. In this section, we will be working in the setting when Q is a quiver and C is an arbitrary set of arrows in Q. In that case, we call the pair (Q, C) a *truncated quiver*. Later, we will specialize to the setting of QPs and their cuts. Our results in this section generalize those in [IO11] given for 'quivers of type A'.

First recall some classical notions. The *double* of Q is the quiver $\overline{Q}$ defined by $\overline{Q}_0 = Q_0$ and $\overline{Q}_1 = Q_1 \coprod \{a^{-1} \mid a \in Q_1\}$ with $s(a^{-1}) = e(a)$ and $e(a^{-1}) = s(a)$. A *walk* in Q is a path in $\overline{Q}$. For each walk $p = a_1^{s_1} \cdots a_n^{s_n}$ in Q, its inverse walk is defined as $p^{-1} = a_n^{-s_n} \cdots a_1^{-s_1}$. Let $\sim$ be the equivalence relation on walks generated by $app^{-1}b \sim ab$ for all walks a, p, and b such that the end point of a coincides with the starting point of p and b. We call a walk *reduced* if it is shortest in its equivalence class.

The grading g_C on Q extends to $\overline{Q}$ by $g_C(a^{-1}) = -g_C(a)$. This grading is invariant under $\sim$.

6.1 Cut–slice correspondence

We now give a general construction of a Galois covering of an arbitrary quiver Q with the Galois group $\mathbf{Z}$.

Definition 6.1. Let (Q, C) be a truncated quiver.

(a) We define a new quiver $\mathbf{Z}(Q, C)$ as follows:
  - $\mathbf{Z}(Q, C)_0 := Q_0 \times \mathbf{Z}$;
  - $\mathbf{Z}(Q, C)_1$ consists of arrows $(a, \ell) : (x, \ell) \to (y, \ell)$ for any arrow $a : x \to y$ in $Q_1 \backslash C$ and $(a, \ell) : (x, \ell) \to (y, \ell - 1)$ for any arrow $a : x \to y$ in C.

(b) We define a morphism $\pi : \mathbf{Z}(Q, C) \to Q$ of quivers by $\pi(x, \ell) := x$ for any $x \in Q_0$ and $\pi(a, \ell) := a$ for any $a \in Q_1$.

(c) We define an automorphism $\tau : \mathbf{Z}(Q, C) \to \mathbf{Z}(Q, C)$ by $\tau(x, \ell) := (x, \ell + 1)$ for any $x \in Q_0$ and $\tau(a, \ell) := (a, \ell + 1)$ for any $a \in Q_1$.

Example 6.2. (a) Let Q be a quiver and $\overline{Q}$ be the double of Q. Then the quiver $\mathbf{Z}(\overline{Q}, Q_1)$ coincides with the translation quiver $\mathbf{Z}Q$ constructed by Riedtmann [Rie80].

(b) Consider the following truncated quiver (Q, C) with $C = \{b\}$.

$$\begin{tikzcd}
1 \arrow[r, shift left, "a"] \arrow[r, shift right, dotted, "b"'] & 2 \arrow[r, "c"] & 3
\end{tikzcd}$$

Then $\mathbf{Z}(Q, C)$ is given by the following quiver.

$$\begin{tikzcd}
 & \cdots & \\
(1,-1) \arrow[ur, "{(b,-1)}"] \arrow[r, "{(a,-1)}"] & (2,-1) \arrow[r, "{(c,-1)}"] & (3,-1) \\
(1,0) \arrow[ur, "{(b,0)}"] \arrow[r, "{(a,0)}"] & (2,0) \arrow[r, "{(c,0)}"] & (3,0) \\
(1,1) \arrow[ur, "{(b,1)}"] \arrow[r, "{(a,1)}"] & (2,1) \arrow[r, "{(c,1)}"] & (3,1) \\
 & \cdots &
\end{tikzcd}$$

The following observation is immediate.

LEMMA 6.3. *For any $x \in Q_0$ and $\widetilde{x} \in \pi^{-1}(x)$, we have that π induces a bijection between arrows starting (respectively, ending) at $\widetilde{x}$ in $\mathbf{Z}(Q, C)$ and those of x in Q.*

The following easy observation is useful.

LEMMA 6.4. *Let p be a walk in Q with $s(p) = x_0$ and $e(p) = y_0$. Let $\widetilde{x}_0 \in \pi^{-1}(x_0)$.*

(a) *There exists a unique walk $\widetilde{p}$ in $\widetilde{Q}$ such that $s(\widetilde{p}) = \widetilde{x}_0$ and $\pi(\widetilde{p}) = p$.*

(b) *If p is a cyclic walk, then we have $e(\widetilde{p}) = \tau^{-g_C(p)}(s(\widetilde{p}))$.*

We say that $\widetilde{p}$ is a *lift* of p.

Proof. (a) This is immediate from Lemma 6.3.

(b) Let $a : x \to y$ be an arrow in Q and $\widetilde{a} : (x, \ell) \to (y, \ell')$ be a lift of a. If $a \notin C$, then we have $\ell' = \ell$. If $a \in C$, then we have $\ell' = \ell - 1$. Thus, the assertion follows. □

The quiver $\mathbf{Z}(Q, C)$ depends on C. So, it is natural to ask when $\mathbf{Z}(Q, C)$ and $\mathbf{Z}(Q, C')$ are isomorphic for subsets C and C' of Q_1. The key notion is the following definition.

DEFINITION 6.5. We say that two subsets $C, C' \subset Q_1$ are *compatible* if for any cyclic walk p in Q the equality $g_C(p) = g_{C'}(p)$ holds. In that case, we write $C \sim C'$.

PROPOSITION 6.6. *For subsets C and C' of Q_1, the following conditions are equivalent.*

(a) *C and C' are compatible.*

(b) *There is an isomorphism $f : \mathbf{Z}(Q, C) \to \mathbf{Z}(Q, C')$ of quivers such that the following diagrams commute.*

$$\begin{tikzcd}
\mathbf{Z}(Q,C) \arrow[rr, "f"] \arrow[dr, "\pi"'] & & \mathbf{Z}(Q,C') \arrow[dl, "\pi"] \\
 & Q &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Z}(Q,C) \arrow[r, "f"] \arrow[d, "\tau"'] & \mathbf{Z}(Q,C') \arrow[d, "\tau"] \\
\mathbf{Z}(Q,C) \arrow[r, "f"] & \mathbf{Z}(Q,C')
\end{tikzcd}$$

Proof. (b) $\Rightarrow$ (a) Let p be a cyclic walk in Q starting at x. Fix $\widetilde{x} \in \pi^{-1}(x)$ and $\widetilde{x}' := f(\widetilde{x})$. Let $\widetilde{p}$ (respectively, $\widetilde{p}'$) be a lift of p to $\mathbf{Z}(Q, C)$ (respectively, $\mathbf{Z}(Q, C')$) starting at x (respectively, $\widetilde{x}'$). Then we have $\widetilde{p}' = f(\widetilde{p})$. Thus, we have

$$\tau^{-g_{C'}(p)}(\widetilde{x}') \overset{\text{Lemma 6.4}}{=} e(\widetilde{p}') = f(e(\widetilde{p})) \overset{\text{Lemma 6.4}}{=} f(\tau^{-g_C(p)}(\widetilde{x})) = \tau^{-g_C(p)}(f(\widetilde{x})) = \tau^{-g_C(p)}(\widetilde{x}').$$

Consequently, we have $g_C(p) = g_{C'}(p)$.

(a) $\Rightarrow$ (b) Without loss of generality, we can assume that Q is connected. Fix vertices $x \in \mathbf{Z}(Q, C)_0$ and $x' \in \mathbf{Z}(Q, C')_0$ such that $\pi(x) = \pi(x')$.

First we define a map $f : \mathbf{Z}(Q, C)_0 \to \mathbf{Z}(Q, C')_0$. For any $y \in \mathbf{Z}(Q, C)_0$, we take a walk $p : x \to y$ in $\mathbf{Z}(Q, C)$. We take a lift $p' : x' \to y'$ of $\pi(p)$ to $\mathbf{Z}(Q, C')$ starting at x'. We shall show that $f(y) := y'$ is independent of choice of p. Let $q : x \to y$ be another walk and let $q' : x' \to y''$ be a lift of $\pi(q)$ to $\mathbf{Z}(Q, C')$ starting at x'. Then both $p^{-1}q : y \to y$ and $p'^{-1}q' : y' \to y''$ are lifts of a cyclic walk $\pi(p^{-1}q) : \pi(y) \to \pi(y)$ to $\mathbf{Z}(Q, C)$ and $\mathbf{Z}(Q, C')$, respectively. By Lemma 6.4, we have $y = \tau^{-g_C(\pi(p^{-1}q))}(y)$ and so $g_C(\pi(p^{-1}q)) = 0$. Thus, we have

$$y'' \overset{\text{Lemma 6.4}}{=} \tau^{-g_{C'}(\pi(p^{-1}q))}(y') \overset{C \sim C'}{=} \tau^{-g_C(\pi(p^{-1}q))}(y') = y'.$$

Thus, we have a well-defined map $f : \mathbf{Z}(Q, C)_0 \to \mathbf{Z}(Q, C')_0$, which is clearly a bijection.

Next we define a map $f : \mathbf{Z}(Q, C)_1 \to \mathbf{Z}(Q, C')_1$. For any arrow $a : y \to z$ in $\mathbf{Z}(Q, C)$, we define $f(a)$ as a lift of $\pi(a)$ starting at $f(y)$ to $\mathbf{Z}(Q, C')$. Clearly, f gives an isomorphism $\mathbf{Z}(Q, C) \to \mathbf{Z}(Q, C')$ of quivers. □

The next aim of this section is to give a one-to-one correspondence between subsets of Q_1 which are compatible with C and certain full subquivers of $\mathbf{Z}(Q, C)$ defined as follows.

DEFINITION 6.7. A *slice* is a full subquiver S of $\mathbf{Z}(Q, C)$ satisfying the following conditions.

- Any τ-orbit in $\mathbf{Z}(Q, C)_0$ contains precisely one vertex which belongs to S.
- For any arrow $a : x \to y$ in $\mathbf{Z}(Q, C)$ with $x \in S_0$, we have $y \in S_0 \cup \tau^{-1} S_0$.

For example, the full subquiver of $\mathbf{Z}(Q, C)$ with the set $Q_0 \times \{0\}$ of vertices is a slice. If S is a slice, then clearly so is $\tau^\ell S$ for any $\ell \in \mathbf{Z}$.

THEOREM 6.8 (Cut–slice correspondence). *Let (Q, C) be a truncated quiver such that Q is connected. Then we have a bijection*

$$\{S \subset \mathbf{Z}(Q, C) \mid S \text{ is a slice}\}/\tau^{\mathbf{Z}} \to \{C' \subset Q_1 \mid C' \sim C\}$$

induced by $S \mapsto C_S := Q_1 \backslash \pi(S_1)$. Moreover, π induces an isomorphism $S \to Q_{C_S}$ of quivers.

Proof. (i) We shall show that C_S is compatible with C.

We only have to construct an isomorphism $\mathbf{Z}(Q, C_S) \to \mathbf{Z}(Q, C)$ of quivers satisfying the conditions in Proposition 6.6(b). For each $x \in Q_0$, we denote by $\widetilde{x}$ the unique lift of x in S_0. Define a bijection $f : \mathbf{Z}(Q, C_S)_0 \to \mathbf{Z}(Q, C)_0$ by $f(x, \ell) := \tau^\ell \widetilde{x}$.

We shall show that f gives a quiver isomorphism. Arrows of $\mathbf{Z}(Q, C_S)$ have the form $(a, \ell) : (x, \ell) \to (y, \ell)$ with $a \in Q_{C_S}$ or $(a, \ell) : (x, \ell) \to (y, \ell - 1)$ with $a \in C_S$. If $a \in Q_{C_S}$, then we have a lift $\widetilde{a} : \widetilde{x} \to \widetilde{y}$ in S_1. Thus, we can define $f(a, \ell) := \tau^\ell \widetilde{a} : \tau^\ell \widetilde{x} \to \tau^\ell \widetilde{y}$. If $a \in C_S$, then we have a lift $\widetilde{a} : \widetilde{x} \to \tau^{-1} \widetilde{y}$ in S_1. Thus, we can define $f(a, \ell) := \tau^\ell \widetilde{a} : \tau^\ell \widetilde{x} \to \tau^{\ell - 1} \widetilde{y}$.

(ii) We shall show that the map $S \mapsto C_S$ is surjective.

Let C' be a subset of Q_1 which is compatible with C. Then we have an isomorphism $f : \mathbf{Z}(Q, C) \to \mathbf{Z}(Q, C')$ given in Proposition 6.6. Let S' be the full subquiver of $\mathbf{Z}(Q, C')$ with the set of vertices $Q_0 \times \{0\}$. Then S' is a slice of $\mathbf{Z}(Q, C')$. Since f is an isomorphism of quivers, we have that $S := f^{-1}(S')$ is a slice of $\mathbf{Z}(Q, C)$. Clearly, we have $C_S = C'$.

(iii) We shall show that the map $S \mapsto C_S$ is injective.

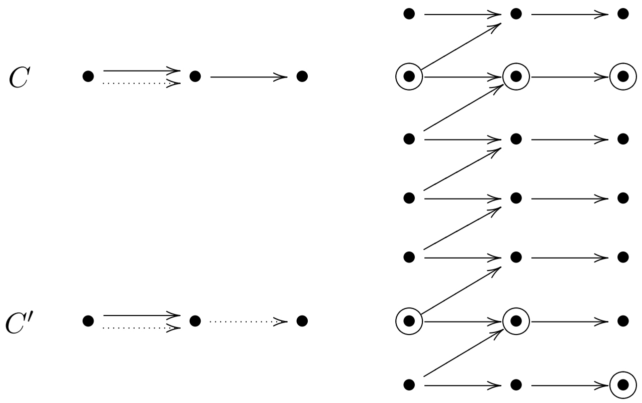
Let S and S' be slices in $\mathbf{Z}(Q, C)$ such that $C_S = C_{S'}$. Without loss of generality, we can assume that $S_0 \cap S'_0 \neq \emptyset$ by replacing S by $\tau^\ell S$ for some $\ell \in \mathbf{Z}$.

Let $X := \pi(S_0 \cap S'_0)$. Then X is a non-empty subset of Q_0. For any arrow $a : x \to y$ in Q, we will show that x belongs to X if and only if y belongs to X. Then we have $X = Q_0$ and $S = S'$ since Q is connected.

Assume that $x \in X$. Let $\widetilde{x} \in \pi^{-1}(x)$ be a vertex in $S_0 \cap S'_0$ and $\widetilde{a}$ be a lift of a starting at $\widetilde{x}$. If $a \notin C_S = C_{S'}$, then $\widetilde{a}$ belongs to $S_1 \cap S'_1$. Thus, $e(\widetilde{a})$ belongs to both S_0 and S'_0, and so $y = e(a) = \pi(e(\widetilde{a}))$ belongs to X. If $a \in C_S = C_{S'}$, then $\widetilde{a}$ belongs to neither S_1 nor S'_1. Since S and S' are slices, we have that $e(\widetilde{a})$ belongs to both $\tau^{-1}S_0$ and $\tau^{-1}S'_0$, and so $y = e(a) = \pi(e(\widetilde{a}))$ belongs to X. Thus, we have $y \in X$ in both cases.

Similarly, one can check that $y \in X$ implies that $x \in X$, so we are done. $\square$

Example 6.9. Consider the truncated quiver (Q, C) and its covering $\mathbf{Z}(Q, C)$ in Example 6.2. Then C has two compatible sets and $\mathbf{Z}(Q, C)$ has two slices up to $\tau^{\mathbf{Z}}$.



6.2 Transitivity of cut-mutation and slice-mutation

For a given subset C of Q_1, we have an operation to construct subsets of Q_1 which are compatible with C.

Definition 6.10. Let C be a subset of Q_1.

(a) We say that a vertex x of Q is a *strict source* of (Q, C) if all arrows ending at x belong to C and all arrows starting at x do not belong to C.

(b) For a strict source x of (Q, C), we define the subset $\mu_x^+(C)$ of Q_1 by removing all arrows in Q ending at x from C and adding all arrows in Q starting at x to C.

(c) Dually, we define a *strict sink* and $\mu_x^-(C)$.

We call these operations *cut-mutation*.

We have the following easy observation.

Lemma 6.11. *We have $\mu_x^+(C) \sim C$ (respectively, $\mu_x^-(C) \sim C$). Moreover, x is a strict sink (respectively, strict source) of $(Q, \mu_x^+(C))$ (respectively, $(Q, \mu_x^-(C))$) and we have $\mu_x^-(\mu_x^+(C)) = C$ (respectively, $\mu_x^+(\mu_x^-(C)) = C$).*

Proof. Clearly, we have $g_C(p) = g_{\mu_x^\pm(C)}(p)$ for any cyclic walk p in Q. Thus, we have $\mu_x^\pm(C) \sim C$. The other assertions are clear from the definition. □

It is natural to ask whether iterated cut-mutation acts transitively on the set of compatible subsets.

We need the following operation for slices.

Definition 6.12. Let S be a slice in $\mathbf{Z}(Q, C)$.

(a) We say that a vertex x of S is a *strict source* if all arrows in $\mathbf{Z}(Q, C)$ ending at x do not belong to S and all arrows in $\mathbf{Z}(Q, C)$ starting at x belong to S.

(b) For a strict source x of S, we define the full subquiver $\mu_x^+(S)$ of $\mathbf{Z}(Q, C)$ by $\mu_x^+(S)_0 = (S_0 \backslash \{x\}) \cup \{\tau^{-1}x\}$.

(c) Dually, we define a *strict sink* and $\mu_x^-(S)$.

We call these operations *slice-mutation*.

The following observation is clear.

Lemma 6.13. *We have that $\mu_x^+(S)$ (respectively, $\mu_x^-(S)$) is again a slice of $\mathbf{Z}(Q, C)$. Moreover, $\tau^{-1}x$ is a strict sink (respectively, τx is a strict source) of $\mu_x^+(S)$ (respectively, $\mu_x^-(S)$) and we have $\mu_{\tau^{-1}x}^-(\mu_x^+(S)) = S$ (respectively, $\mu_{\tau x}^+(\mu_x^-(S)) = S$).*

Proof. Let $a : y \to z$ be an arrow in $\mathbf{Z}(Q, C)$ such that $y \in \mu_x^+(S)_0$.

Assume that $y = \tau^{-1}x$. Then we have an arrow $\tau a : x \to \tau z$ in $\mathbf{Z}(Q, C)$. Since x is a strict source of S, we have $\tau z \in S_0 \backslash \{x\} \subset \mu_x^+(S)_0$. Thus, we have $z \in \tau^{-1}\mu_x^+(S)_0$.

Assume that $y \neq \tau^{-1}x$. Then $y \in S_0$, and we have $z \in S_0 \cup \tau^{-1}S_0$ since S is a slice. Since x is a strict source of S, we have $z \neq x$. Thus, we have $z \in (S_0 \backslash \{x\}) \cup \tau^{-1}S_0 \subset \mu_x^+(S)_0 \cup \tau^{-1}\mu_x^+(S)_0$.

Consequently, we have that $\mu_x^+(S)$ is a slice. The other assertions can be checked easily. □

Under the bijection in Theorem 6.8, slice-mutation corresponds to cut-mutation by the following observation.

Proposition 6.14. *Let S and Q_{C_S} be as in Theorem 6.8. Let $x \in S_0$.*

(a) *x is a strict source (respectively, strict sink) of S if and only if so is $\pi(x)$ in (Q, C_S).*

(b) *In this case, we have $\mu_{\pi(x)}^+(C_S) = C_{\mu_x^+(S)}$ (respectively, $\mu_{\pi(x)}^-(C_S) = C_{\mu_x^-(S)}$).*

Proof. (a) Since $\pi : S \to Q_{C_S}$ is an isomorphism of quivers, this is clear from Lemma 6.3.

(b) The difference of $C_{\mu_x^+(S)}$ and C_S are arrows starting or ending at $\pi(x)$. Since $\tau^{-1}x$ is a strict sink of $\mu_x^+(S)$ by Lemma 6.13, so is $\pi(x)$ in $C_{\mu_x^+(S)}$ by (a). Thus, we have $C_{\mu_x^+(S)} = \mu_{\pi(x)}^+(C_S)$. □

In the rest of this subsection, we study transitivity of iterated cut-mutations (respectively, slice-mutations).

DEFINITION 6.15. Let (Q,C) be a truncated quiver.

(a) (Q,C) has *enough compatibles* if $Q_1=\bigcup_{C'\sim C}C'$.

(b) (Q,C) is *sufficiently cyclic* if for each $a\in Q_1$ there is a cycle p containing a satisfying $g_C(p)\leqslant 1$.

If (Q,C) has either of these properties, then so does (Q,C') for any $C'\sim C$.

LEMMA 6.16. *Assume that (Q,C) is sufficiently cyclic. Then each source (respectively, sink) in Q_C is strict.*

Proof. Let x be a source (respectively, sink) in Q_C. We only have to show that any arrow $a\in Q_1$ starting (respectively, ending) at x does not belong to C. Since (Q,C) is sufficiently cyclic, there is a cycle p in Q containing a such that $g_C(p)\leqslant 1$. Since p is a cycle, it must contain an arrow b ending (respectively, starting) at x. Thus, $b\in C$ and $g_C(p)\geqslant g_C(a)+g_C(b)=g_C(a)+1$. Hence, $g_C(a)=0$ and $a\notin C$. □

We have a useful criterion for enough compatibility.

We call a numbering $x_1,\ldots,x_N$ of vertices of Q a *strict C-source sequence* if x_{i+1} is a strict source in $Q_{\mu^+_{x_i}\circ\cdots\circ\mu^+_{x_1}(C)}$ for any $0\leqslant i<N$. Dually, we define a *strict C-sink sequence.*

PROPOSITION 6.17. *For a truncated quiver (Q,C) which is sufficiently cyclic, the following conditions are equivalent.*

(a) *C has enough compatibles.*

(a′) *For any cycle p in Q, at least one arrow in p is contained in some $C'\sim C$.*

(b) *$Q_{C'}$ is an acyclic quiver for any $C'\sim C$.*

(b′) *Q_C is an acyclic quiver.*

(c) *There exists a strict C'-source sequence for any $C'\sim C$.*

(c′) *There exists a strict C-source sequence.*

(d) *There exists a strict C'-sink sequence for any $C'\sim C$.*

(d′) *There exists a strict C-sink sequence.*

Proof. (a) ⇒ (a′), (b) ⇒ (b′), and (c) ⇒ (c′) are clear.

(a′) ⇒ (b) Assume that $Q_{C'}$ has a cycle p. By (a′), there exists a cut C'' of Q such that some arrow in p is contained in C''. Since $C'\sim C\sim C''$, we have $0=g_{C'}(p)=g_{C''}(p)>0$, a contradiction.

(b′) ⇒ (c′) Assume that $x_1,\ldots,x_{i-1}$ are defined. Since Q_C is acyclic, so is the quiver $Q_C\backslash\{x_1,\ldots,x_{i-1}\}$. We define x_i as a source of this new quiver. It is easily checked that $x_1,\ldots,x_N$ is a strict C-source sequence. We can show (b) ⇒ (c) similarly.

(c′) ⇒ (a) Assume that there is an arrow $a:x\rightarrow y$ that does not belong to any $C'\sim C$. Then x is not a source of $Q_{C'}$ for any $C'\sim C$ since $a:x\rightarrow y$ belongs to $Q_{C'}$. Take a strict C-source sequence $x_1,\ldots,x_N$. Since x is not a source of $Q_{\mu^+_{x_i}\circ\cdots\circ\mu^+_{x_1}(C)}$ for any i, we have that x is none of $x_1,\ldots,x_N$, a contradiction. □

Now we are ready to prove the following main result in this subsection.

THEOREM 6.18 (Transitivity). *Let (Q,C) be a truncated quiver which has enough compatibles and is sufficiently cyclic.*

(a) *The set* $\{C' \subset Q_1 \mid C' \sim C\}$ *is transitive under iterated cut-mutations.*

(b) *The set of all slices in* $\mathbf{Z}(Q,C)$ *is transitive under iterated slice-mutations.*

Proof. By Proposition 6.14, it suffices to show (b).

Let S be a slice in $\mathbf{Z}(Q,C)$. Our aim is to show that S is slice-mutation equivalent to $S^0 := Q_C \times \{0\}$, which completes the proof. Set $C' := Q_1 \backslash \pi(S_1)$. Since (Q,C) has enough compatibles, $Q_{C'}$ is acyclic and has both a strict C'-source sequence and a strict C'-sink sequence by Proposition 6.17. Thus, by Proposition 6.14, S is acyclic and slice-mutation equivalent to $\tau^l S$ for any $l \in \mathbf{Z}$. In particular, we may assume that $S_0 \subset Q_0 \times \mathbf{Z}_{\geqslant 0}$. Observe also that each sink and source in S is strict by Proposition 6.14 and Lemma 6.16.

Let $x \in Q_0$ and define the height at x by $h_x(S) = l$, where $(x,l) \in S_0$. Moreover, define the volume under S to be $V(S) = \sum_{x \in Q_0} h_x(S)$. Let $a : (x,l) \to (y,m)$ be an arrow in S. Since $m \in \{l, l-1\}$, we have $h_x(S) = l \geqslant m = h_y(S)$. Because S is acyclic, it follows that $h_x(S)$ takes its maximal value at some x such that $(x, h_x(S))$ is a source in S. In particular, if $V(S) > 0$, there is a source (x,l) in S with $l > 0$. Let $S' = \mu_x^+(S)$. Then $h_y(S') = h_y(S)$ for all $y \neq x$ and $h_x(S') = h_x(S) - 1$. Thus, $h_y(S') \geqslant 0$ for all $y \in Q_0$ and $V(S') = V(S) - 1 < V(S)$. Hence, we may assume that $V(S) = 0$. But, this implies that $S_0 = S_0^0$ and thus $S = S^0$. □

7. Derived equivalence of truncated Jacobian algebras

In this section, we study the relationship between truncated Jacobian algebras of a fixed QP by algebraic cuts. Let us start with general observations on cut-mutation.

The following observations are easy.

LEMMA 7.1. *Let* (Q,W) *be a QP with a cut* C.

(a) *If* C' *is a subset of* Q_1 *such that* $C \sim C'$, *then* C' *is also a cut of* (Q,W).

(b) *If* x *is a strict source (respectively, strict sink) of* (Q,C), *then* $\mu_x^+(C)$ *(respectively,* $\mu_x^-(C)$*) is again a cut of* (Q,W).

Proof. (a) This is clear.

(b) By Lemma 6.11, we have $\mu_x^{\pm}(C) \sim C$. Thus, they are cuts of (Q,W) by (a). □

The following result is an immediate consequence of Theorem 6.18.

THEOREM 7.2. *Let* (Q,W) *be a QP with a cut* C *such that* (Q,C) *is sufficiently cyclic and has enough compatibles. Then the set of all cuts of* (Q,W) *compatible with* C *is transitive under iterated cut-mutations.*

Thus, it is natural to ask when a cut of a QP is sufficiently cyclic and has enough compatibles.

We have the following sufficient conditions for sufficient cyclicity.

PROPOSITION 7.3. *Let* (Q,W) *be a QP with a cut* C. *Then:*

(a) *if any arrow in* Q *appears in cycles in* W, *then* (Q,C) *is sufficiently cyclic;*

(b) *if* (Q,W) *is selfinjective, then* (Q,C) *is sufficiently cyclic.*

Proof. (a) For any $a \in Q_1$, take a cycle p in W. Then $g_C(p) = 1$.

(b) This is immediate from (a) and Proposition 3.8. ☐

We separate enough compatibility into the following two conditions.

Definition 7.4. Let (Q, W) be a QP.

(a) (Q, W) is *fully compatible* if all cuts are compatible with each other.

(b) (Q, W) has *enough cuts* if each arrow in Q is contained in a cut.

If (Q, W) is fully compatible and has enough cuts, then for each cut C in (Q, W) the truncated quiver (Q, C) has enough compatibles. Thus, we obtain the following result.

Corollary 7.5. *Let (Q, W) be a selfinjective, fully compatible QP with enough cuts. Then the set of all cuts of (Q, W) is transitive under iterated cut-mutations.*

Proof. Since (Q, W) is selfinjective, Proposition 7.3 implies that (Q, C) is sufficiently cyclic for any cut C. Thus, the assertion follow from Theorem 7.2. ☐

In § 8, we give a sufficient condition for QPs to be fully compatible.

7.1 Cut-mutation and 2-APR tilting

We say that two algebras A and A' of global dimension at most two are *cluster equivalent* if the corresponding generalized cluster categories $\mathcal{C}_A$ and $\mathcal{C}_{A'}$ are triangle equivalent. This is the case if A and A' are derived equivalent.

Proposition 7.6. *Let (Q, W) be a QP. Then all truncated Jacobian algebras $\mathcal{P}(Q, W)_C$ given by algebraic cuts C are cluster equivalent.*

Proof. By Propositions 2.4 and 3.3, the derived 3-preprojective DG algebra of $\mathcal{P}(Q, W)_C$ is quasi-isomorphic to $\Gamma(Q, W)$. Thus, the assertion follows. ☐

In [AO10], Amiot–Oppermann studied when two cluster-equivalent algebras are derived equivalent (see also [Ami09b]). Thus, it is natural to ask when the algebras $\mathcal{P}(Q, W)_C$ appearing above are derived equivalent to each other. In this section, we give a sufficient condition on (Q, W).

The main point of cut-mutation is its relation to 2-APR tilting modules [IO11] defined as follows: let x be a source and S the corresponding simple projective A-module. If $\mathrm{Ext}^1_A(DA, S) = 0$, then we have a tilting A-module

$$T := \tau^- \Omega^- S \oplus (A/S),$$

which we call a 2-*APR tilting* A-module. Dually, a 2-*APR cotilting module* T is defined. In both cases, we call $\mathrm{End}_A(T)$ a 2-*APR tilt* of A.

Theorem 7.7. *Let (Q, W) be a QP and C an algebraic cut. Let T be a 2-APR tilting (respectively, cotilting) $\mathcal{P}(Q, W)_C$-module associated with a source (respectively, sink) x of the quiver Q_C. Then x is a strict source (respectively, sink) of (Q, C), and $\mathrm{End}_{\mathcal{P}(Q,W)_C}(T)$ is isomorphic to $\mathcal{P}(Q, W)_{\mu_x^+(C)}$ (respectively, $\mathcal{P}(Q, W)_{\mu_x^-(C)}$).*

Proof. Let $A := \mathcal{P}(Q, W)_C$ and $B := \mathrm{End}_A(T)$. Since C is an algebraic cut, we have that x is a strict source of (Q, C). The changes of quivers with relations via 2-APR tilt was explicitly given in [IO11, Theorem 3.11]. In particular, we have that the QPs (Q_A, W_A) and (Q_B, W_B) are isomorphic, and the isomorphism $Q_A \simeq Q_B$ induces a bijection between $\mu_x^+(C_A)$ and C_B. Consequently, we have $B \simeq \mathcal{P}(Q_B, W_B)_{C_B} \simeq \mathcal{P}(Q_A, W_A)_{\mu_x^+(C_A)}$. □

Combining Corollary 7.5 with Theorem 7.7, we obtain the main result of this section.

THEOREM 7.8. *Let (Q, W) be a selfinjective, fully compatible QP with enough cuts. Then all truncated Jacobian algebras of (Q, W) are iterated 2-APR tilts of each other. In particular, they are derived equivalent.*

Proof. The first assertion is an immediate consequence of Corollary 7.5 and Theorem 7.7 since any source of 2-representation-finite algebras admits a 2-APR tilting module. The latter assertion is a direct consequence of the former one. □

To apply Theorem 7.8, it is important to have sufficient conditions for a QP to be fully compatible and have enough cuts. In § 8, we provide a sufficient condition for full compatibility.

Example 7.9. Let $Q = \bullet \leftarrow \bullet \rightarrow \bullet$. By Proposition 5.1 and Theorem 5.2, the QPs $(Q \tilde{\otimes} Q, W^{\tilde{\otimes}}_{Q,Q})$ and $(Q \,\square\, Q, W^{\square}_{Q,Q})$ are selfinjective. Their cut-mutations are displayed in Figure 1. For each of the two QPs, all cuts are connected by iterated cut-mutation. By Theorem 7.7, the corresponding 2-representation-finite algebras are derived equivalent.

8. The canvas of a QP

In this section, we provide a sufficient condition for a QP to be fully compatible.

In the theory of dimer models, QPs arise from finite bipartite tilings on compact Riemann surfaces (see [Bro11, Dav08, IU09] and references therein). Of main interest is the case when the surface is a torus. This is because the topological properties of the torus yield desirable properties of the corresponding QP.

The idea behind this section is to start with the QP instead of the surface. To each QP (Q, W), we assign a CW complex $X_{(Q,W)}$ called the canvas of (Q, W). One may think of (Q, W) as painted on $X_{(Q,W)}$; hence, the name canvas. In the dimer case we recover the torus in this way. Therefore, it is reasonable to hope that the topological properties of $X_{(Q,W)}$ will reflect interesting properties of (Q, W).

It turns out that for many examples of selfinjective QPs the canvas can be realized as a bounded simply connected region of the plane. Since a bipartite tiling on the torus can be thought of as an infinite periodic tiling on the plane, this is in stark contrast to dimer models.

8.1 Fundamental groups of QPs and simply connected QPs

In order to define CW complexes, we recall some basic notation. Denote by D^n the closed n-dimensional disc for each $n > 0$. Thus, $\partial D^n = S^{n-1}$, the $(n-1)$-dimensional sphere. In particular, $D^1 = [0, 1]$ and $\partial D^1 = S^0 = \{0, 1\}$. We use the inductive definition of CW complexes. Thus, a CW complex $X = \bigcup_{i \in \mathbf{N}} X^i$ with the weak topology, where X^0 is a discrete space and X^n is obtained from X^{n-1} by attaching n-dimensional discs D^n_a along continuous maps $\phi_a : S^{n-1} \to X^{n-1}$. Hence, there are continuous maps $\epsilon^n_a : D^n_a \to X$, embedding the interior of D^n_a in X, whose restriction to ∂D^n_a equals ϕ_a. The images of the interior under these embeddings will be referred to as n-cells. The elements in X^0 will be referred to as 0-cells.

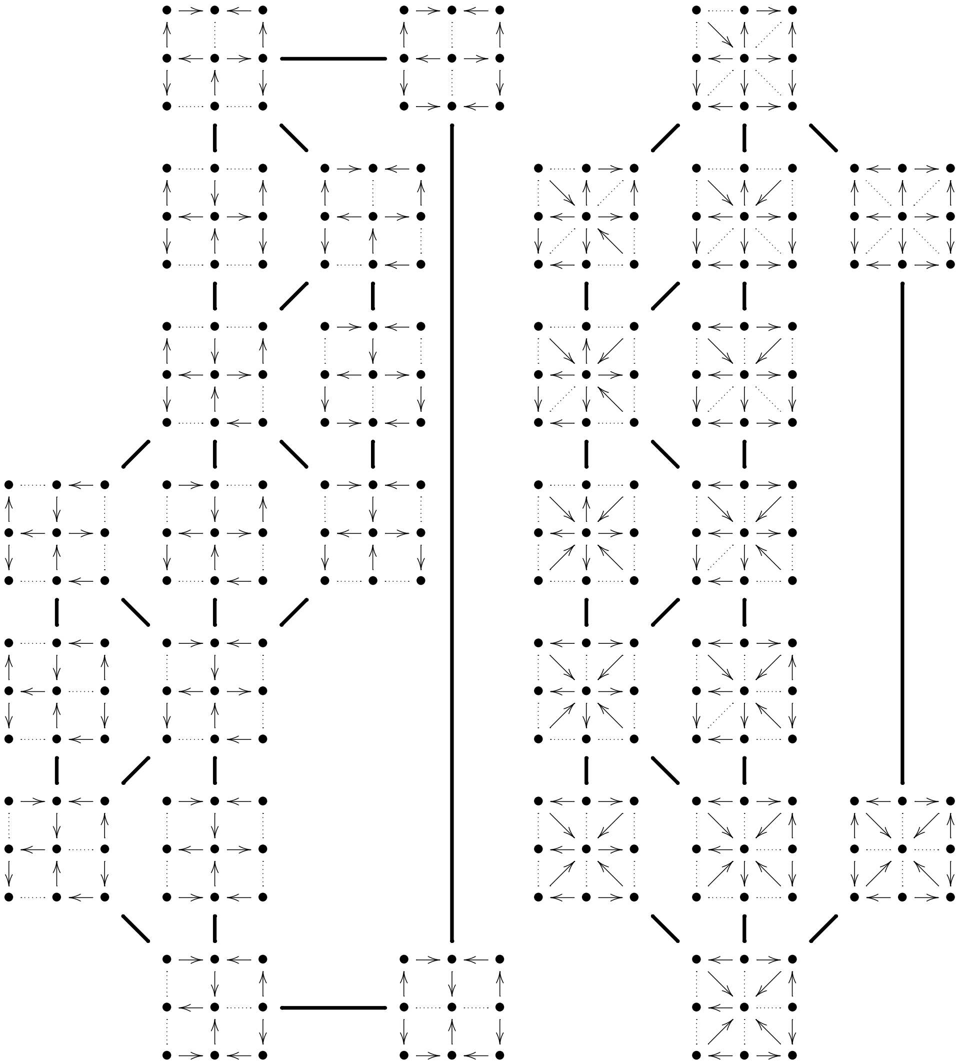


FIGURE 1. Cut-mutation lattices.

DEFINITION 8.1. (a) Given a quiver Q, define the CW complex X_Q as follows. The 0-cells are the vertices of Q, i.e. $X_Q^0 := Q_0$. The 1-cells are indexed by the arrows $a \in Q_1$ with attaching maps $\phi_a : S^0 \to Q_0$ defined by $\phi_a(0) = s(a)$ and $\phi_a(1) = e(a)$. These are all cells, i.e. $X_Q = X_Q^1$.

(b) Let (Q, W) be a QP. Write $W = \sum_c \lambda_c c$, where c runs through all cycles in Q and $\lambda_c \in K$. Let Q_2 be the set of cycles c such that $\lambda_c \neq 0$. Define the CW complex $X_{(Q,W)}$, which we call the *canvas* of (Q, W), by attaching to X_Q one 2-cell for each $c = a_0 \cdots a_{s-1} \in Q_2$ via the attaching map $\phi_c : S^1 \to X_Q$ defined by

$$\phi_c\left(\cos\left(\frac{2\pi}{s}(i+t)\right), \sin\left(\frac{2\pi}{s}(i+t)\right)\right) = \epsilon^1_{a_i}(t)$$

for each integer $0 \leqslant i < s$ and real number $0 \leqslant t < 1$.

Remark 8.2. The space $X_{(Q,W)}$ is not invariant under right equivalence of QPs.

We proceed to compute the fundamental groups of X_Q and $X_{(Q,W)}$. Fix a base point $x_0 \in Q_0 \subset X_Q$. Let $G_{x_0}(Q)$ be the group of equivalence classes of cyclic walks in Q (see § 6) starting and ending at x_0. For each arrow $a \in Q_1$, we have two paths $f_a, f_{a^{-1}} : [0, 1] \to X_Q$ defined by $f_a(t) = \epsilon^1_a(t)$ and $f_{a^{-1}}(t) = \epsilon^1_a(1-t)$.

For each cyclic walk $p = a_1^{s_1} \cdots a_\ell^{s_\ell}$ starting at x_0 ($s_i \in \{\pm 1\}$), we get a closed curve

$$f_p : [0, 1] \to X_Q$$

by composing the paths $f_{a_1^{s_1}}, \ldots, f_{a_\ell^{s_\ell}}$. This induces a group morphism $f : G_{x_0}(Q) \to \Pi^1_{x_0}(X_Q)$.

Proposition 8.3. *The group morphism*

$$f : G_{x_0}(Q) \to \Pi^1_{x_0}(X_Q)$$

is an isomorphism.

Proof. Choose a maximal subquiver $T \subset Q$ such that the underlying graph is a tree. Let $A = Q_1 \backslash T_1$. Since X_T is a contractible closed subcomplex of X_Q, the quotient map $X_Q \to X_Q/X_T$ is a homotopy equivalence. Moreover, X_Q/X_T is a wedge sum of loops, one for each element in A. Let F_A be the free group on A and $g : \Pi^1_{x_0}(X_Q) \to F_A$ the induced isomorphism.

Every element in $G_{x_0}(Q)$ is written uniquely in the form $p = t_1 a_1^{s_1} \cdots t_\ell a_\ell^{s_\ell} t_{\ell+1}$, where t_i is a reduced walk in T, $a_i \in A$, and $s_i \in \{\pm 1\}$ such that $a_i^{s_i} \neq a_{i+1}^{-s_{i+1}}$. Moreover, $g(f(p)) = a_1^{s_1} \cdots a_\ell^{s_\ell}$. Since every reduced word in F_A appears as $g(f(p))$ for some $p \in G_{x_0}(Q)$, the morphism $g \circ f$ is an isomorphism and, therefore, so is f. □

Now choose for each cycle $c \in Q_2$ a walk p_c from x_0 to the starting point of c. Let N be the normal subgroup of $G_{x_0}(Q)$ generated by $\{p_c c p_c^{-1} \mid c \in Q_2\}$.

Proposition 8.4. *The group morphism f induces an isomorphism*

$$G_{x_0}(Q)/N \to \Pi^1_{x_0}(X_{(Q,W)}).$$

Proof. According to [Hat02, Proposition 1.26], the inclusion of X_Q in $X_{(Q,W)}$ induces an epimorphism $\Pi^1_{x_0}(X_Q) \to \Pi^1_{x_0}(X_{(Q,W)})$ with kernel generated by $\{f_{p_c c p_c^{-1}} \mid c \in Q_2\}$. Hence, the kernel is $f(N)$. □

Definition 8.5. We call a QP (Q, W) *simply connected* if $X_{(Q,W)}$ is simply connected.

We now obtain a sufficient condition for full compatibility.

Proposition 8.6. *Every simply connected QP is fully compatible.*

Proof. Let (Q, W) be a simply connected QP and p a cyclic walk in Q at $x \in Q_0$. Since $X_{(Q,W)}$ is simply connected, $N = G_x(Q)$ by Proposition 8.4 and thus $p \in N$. Hence, there are cycles $c_i \in Q_2$, walks p_i, and $s_i \in \{\pm 1\}$ such that

$$p \sim (p_1 c_1^{s_1} p_1^{-1})(p_2 c_2^{s_2} p_2^{-1}) \cdots (p_\ell c_\ell^{s_\ell} p_\ell^{-1}).$$

For any cut C in (Q, W), we have

$$g_C(p) = \sum_{i=1}^{\ell} g_C(c_i^{s_i}) = \sum_{i=1}^{\ell} s_i,$$

which does not depend on C. □

The following is an immediate consequence of Theorem 7.8.

THEOREM 8.7. *Let (Q, W) be a selfinjective, simply connected QP with enough cuts. Then all truncated Jacobian algebras of (Q, W) are iterated 2-APR tilts of each other. In particular, they are derived equivalent.*

9. Planar QPs

In this section, we introduce a class of QPs called planar. Many examples of selfinjective planar QPs are provided at the end of this section.

DEFINITION 9.1. A QP (Q, W) is called *planar* if $X_{(Q,W)}$ is simply connected and equipped with an embedding $\epsilon : X_{(Q,W)} \to \mathbf{R}^2$.

Observe that since planar QPs are simply connected, they are fully compatible by Proposition 8.6. In particular, we have the following result.

THEOREM 9.2. *Let (Q, W) be a selfinjective planar QP with enough cuts. Then all truncated Jacobian algebras of (Q, W) are iterated 2-APR tilts of each other. In particular, they are derived equivalent.*

PROPOSITION 9.3. *Let (Q, W) be a planar QP. Write $W = \sum_c \lambda_c c$ and define $W' = \sum_{\lambda_c \neq 0} c$. Then $X_{(Q,W)} = X_{(Q,W')}$ and (Q, W) is right equivalent to (Q, W').*

Proof. If $W = 0$, there is nothing to show. Assume that this is not the case. Let c be a cycle in W containing an arrow a on the boundary of $X_{(Q,W)}$. Consider the subquiver $\widehat{Q} = Q \backslash \{a\}$ and the potential $\widehat{W}$ on $\widehat{Q}$ satisfying $\widehat{W} + \lambda_c c = W$. The QP $(\widehat{Q}, \widehat{W})$ is planar but $\widehat{W}$ contains fewer cycles than W. By induction, we may assume that all coefficients in $\widehat{W}$ are 0 or 1. Now consider the automorphism $KQ \to KQ$ defined by $a \mapsto \lambda_c^{-1} a$ and $b \mapsto b$ for all $b \neq a$. It sends W to $\widehat{W} + c$. □

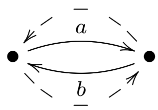
Remark 9.4. Motivated by Proposition 9.3, we assume in the following that all planar QPs have potentials where every coefficient is 0 or 1. In particular, the potential of a planar QP (Q, W) is determined by the restricted embedding $X_Q \to \mathbf{R}^2$. Therefore, we will omit the potential and simply write (Q, ϵ) for a planar QP (Q, W) with restricted embedding ϵ.

Let us consider 2-reduction of planar QPs.

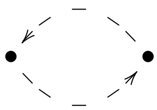
DEFINITION 9.5. Let (Q, ϵ) be a planar QP with a 2-cycle ab not lying entirely on the boundary and set $Q' := Q \backslash \{a, b\}$. Then we define a *planar 2-reduction* (Q', ϵ') of (Q, ϵ) as follows (dashed arrows denote paths).

(a) Both a and b are in the interior.

(Q, ϵ) :

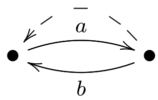


(Q', ϵ') :

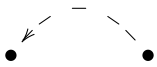


(b) The arrow a is in the interior and b is on the boundary (the case b is in the interior and a is on the boundary is similar).

(Q, ϵ) :

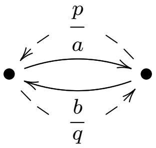


(Q', ϵ') :



Proposition 9.6. *(Q, ϵ) is right equivalent to a direct sum of (Q', ϵ') and a trivial QP. In particular, we have $\mathcal{P}(Q, \epsilon) \simeq \mathcal{P}(Q', \epsilon')$.*

Proof. We only show the case (a). We name the paths in Q as follows.



Then we can write $W = W' + ap + ab + qb$, where any cycle in W' does not contain a and b. Since $W = W' - qp + (a+q)(p+b)$, the automorphism $f : \widehat{KQ} \to \widehat{KQ}$ defined by $f(a) = a+q$, $f(b) = b+p$, and $f(c) = c$ for any $c \in Q_1 \backslash \{a, b\}$ satisfies $f(W' - qp + ab) = W$. Since the QP $(Q, W' - qp + ab)$ is right equivalent to a direct sum of (Q', ϵ') and a trivial QP, we have the assertion. □

We proceed to interpret mutation in the planar setting.

9.1 Planar mutation

Let (Q, ϵ) be a planar QP. We fix a vertex $k \in Q_0$ satisfying one of the following conditions:

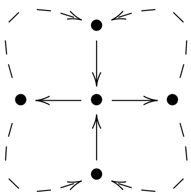
(i) k is in the interior and exactly four arrows start or end at k;

(ii) k is at the boundary and at most four arrows start or end at k.

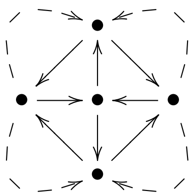
Definition 9.7. We define a planar QP $\mu_k(Q, \epsilon)$ as follows (dashed arrows denote paths).

(a) k is in the interior and exactly four arrows start or end at k.

(Q, ϵ) :

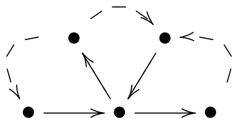


$\mu_k(Q, \epsilon)$:

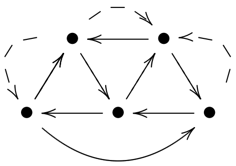


(b) k is at the boundary and exactly four arrows start or end at k.

(Q, ϵ) :

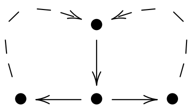


$\mu_k(Q, \epsilon)$:

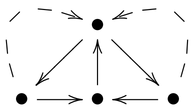


(c) k is at the boundary and exactly three arrows start or end at k.

(Q, ϵ) :

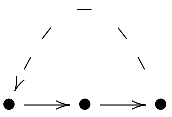


$\mu_k(Q, \epsilon)$:

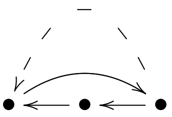


(d) k is at the boundary and exactly two arrows start or end at k.

(Q, ϵ) :



$\mu_k(Q, \epsilon)$:



One can easily check that the above planar mutation of QPs is compatible with the usual Derksen–Weyman–Zelevinsky mutation of QPs.

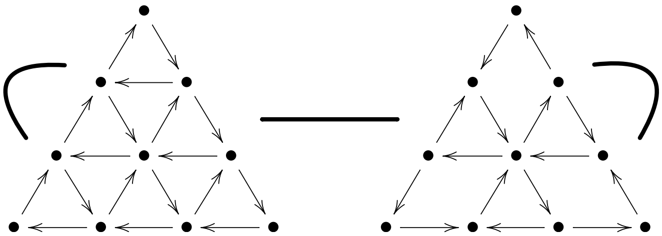


FIGURE 2. Planar mutation lattice for $(Q^{(4)}, \epsilon^{(4)})$.

THEOREM 9.8. *Assume that (Q, ϵ) is a selfinjective planar QP and $k \in Q_0$ satisfies the conditions in Definition 4.1. Then $\mu_{(k)}(Q, \epsilon) = \mu_{\sigma^{m-1}k} \circ \cdots \circ \mu_k(Q, \epsilon)$ is again a selfinjective planar QP.*

Proof. By Theorem 4.2, the planar QP $\mu_{(k)}(Q, \epsilon)$ is selfinjective. □

We say that two selfinjective planar QPs are related by *planar mutation* if one is obtained from the other by the process described in Theorem 9.8.

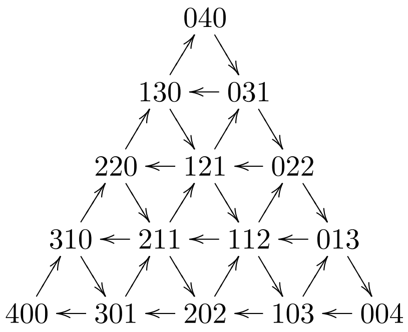
In the rest of this section, we exhibit explicit selfinjective planar QPs by considering already known examples and applying planar mutation.

9.2 Examples from triangles

The first family of selfinjective planar QPs that we consider are the 3-preprojective algebras of Auslander algebras of quivers of type $\mathbb{A}$ with linear orientation. We recall their description from [IO11]. Define $Q = Q^{(s)}$ as follows:

$$Q_0 := \{(x_1, x_2, x_3) \in \mathbf{Z}^3_{\geqslant 0} \mid x_1 + x_2 + x_3 = s - 1\},$$
$$Q_1 := \{x \xrightarrow{i} x + f_i \mid 1 \leqslant i \leqslant 3,\, x,\, x + f_i \in Q_0\},$$

where $f_1 := (-1, 1, 0)$, $f_2 := (0, -1, 1)$, and $f_3 := (1, 0, -1)$. The vertices Q_0 lie in the plane in $\mathbf{R}^3$ defined by the equation $x_1 + x_2 + x_3 = s - 1$. Identifying this with $\mathbf{R}^2$, we get an induced embedding $\epsilon^{(s)} : X_{Q^{(s)}} \to \mathbf{R}^2$. For instance, $(Q^{(5)}, \epsilon^{(5)})$ is the following planar QP.



For each $1 \leqslant i \leqslant 3$, the subset $\{x \xrightarrow{i} x + f_i \mid x, x + f_i \in Q_0\} \subset Q_1$ is a cut. Thus, all QPs $(Q^{(s)}, \epsilon^{(s)})$ have enough cuts.

In contrast to the selfinjective cluster tilted algebras, this family of planar QPs gives rise to many others by use of planar mutation. For instance, the planar mutation lattices of $(Q^{(4)}, \epsilon^{(4)})$ and $(Q^{(5)}, \epsilon^{(5)})$ are displayed in Figures 2 and 4, respectively.

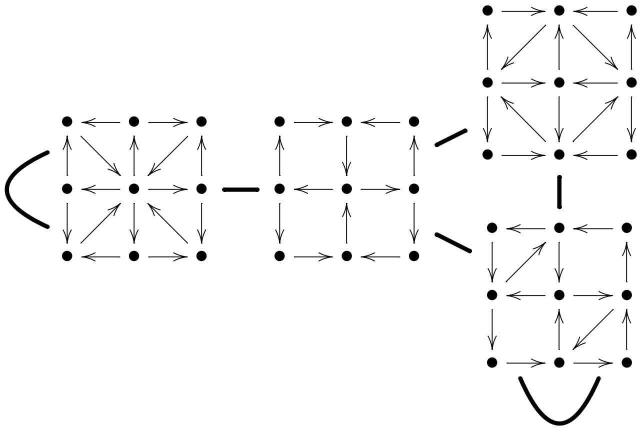


FIGURE 3. Planar mutation lattice for $(\mathbb{A}_3 \,\square\, \mathbb{A}_3, W^{\square}_{\mathbb{A}_3,\mathbb{A}_3})$.

9.3 Examples from squares

Let Q be a Dynkin quiver of type $\mathbb{A}_s$ with alternating orientation. By Theorem 5.2, $(Q \,\square\, Q, W^{\square}_{Q,Q})$ is selfinjective. Moreover, it is planar. The planar mutation lattice of $(Q \,\square\, Q, W^{\square}_{Q,Q})$ when $s = 3$ is found in Figure 3.

We proceed to define a class of planar QPs that we call square-shaped. Every planar QP appearing in Figure 3 lies in this class. Moreover, so does $(Q^1 \,\tilde{\otimes}\, Q^2, W^{\tilde{\otimes}}_{Q^1,Q^2})$ and $(Q^1 \,\square\, Q^2, W^{\square}_{Q^1,Q^2})$ for all Q^1, Q^2 Dynkin of type $\mathbb{A}_s$ for some $s \in \mathbf{N}$.

Let Q be a quiver without loops constructed in the following way.

- The vertex set is $Q_0 = \{1, \dots, s\}^2 \subset \mathbf{R}^2$.
- For each $1 \leqslant i, j < s$, let P be the full subquiver of Q with vertex set

  $$P_0 = \{a = (i, j), b = (i+1, j), c = (i, j+1), d = (i+1, j+1)\}.$$

  Then P or P^{op} appears in the list below.

$$\begin{tikzcd}
c \arrow[r] & d \arrow[d] \\
a \arrow[u] & b \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
c \arrow[dr] & d \arrow[l] \\
a \arrow[u] & b \arrow[l] \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
c \arrow[d] & d \arrow[l] \arrow[d] \\
a \arrow[ur] & b \arrow[l]
\end{tikzcd}$$

Let (Q, ϵ) be the planar QP arising from the pictures of P and P^{op} above. Furthermore, all arrows in Q lie in one of the subquivers P. We call such a QP *square-shaped*. Here is one example.

$$\begin{tikzcd}
14 \arrow[r] & 24 \arrow[dl] \arrow[dr] & 34 \arrow[l] \arrow[r] & 44 \arrow[d] \\
13 \arrow[u] \arrow[r] & 23 \arrow[u] \arrow[d] & 33 \arrow[u] \arrow[l] \arrow[dr] & 43 \arrow[l] \\
12 \arrow[u] \arrow[d] & 22 \arrow[l] \arrow[r] \arrow[d] & 32 \arrow[u] \arrow[l, phantom] \arrow[d] & 42 \arrow[l] \arrow[u] \\
11 \arrow[ur] & 21 \arrow[l] \arrow[r] & 31 \arrow[r] \arrow[ul] & 41 \arrow[u]
\end{tikzcd}$$

Define $\sigma : Q_0 \to Q_0$ by $\sigma(i, j) = (s - i + 1, s - j + 1)$. We call (Q, ϵ) *symmetric* if it has an automorphism that coincides with σ on Q_0.

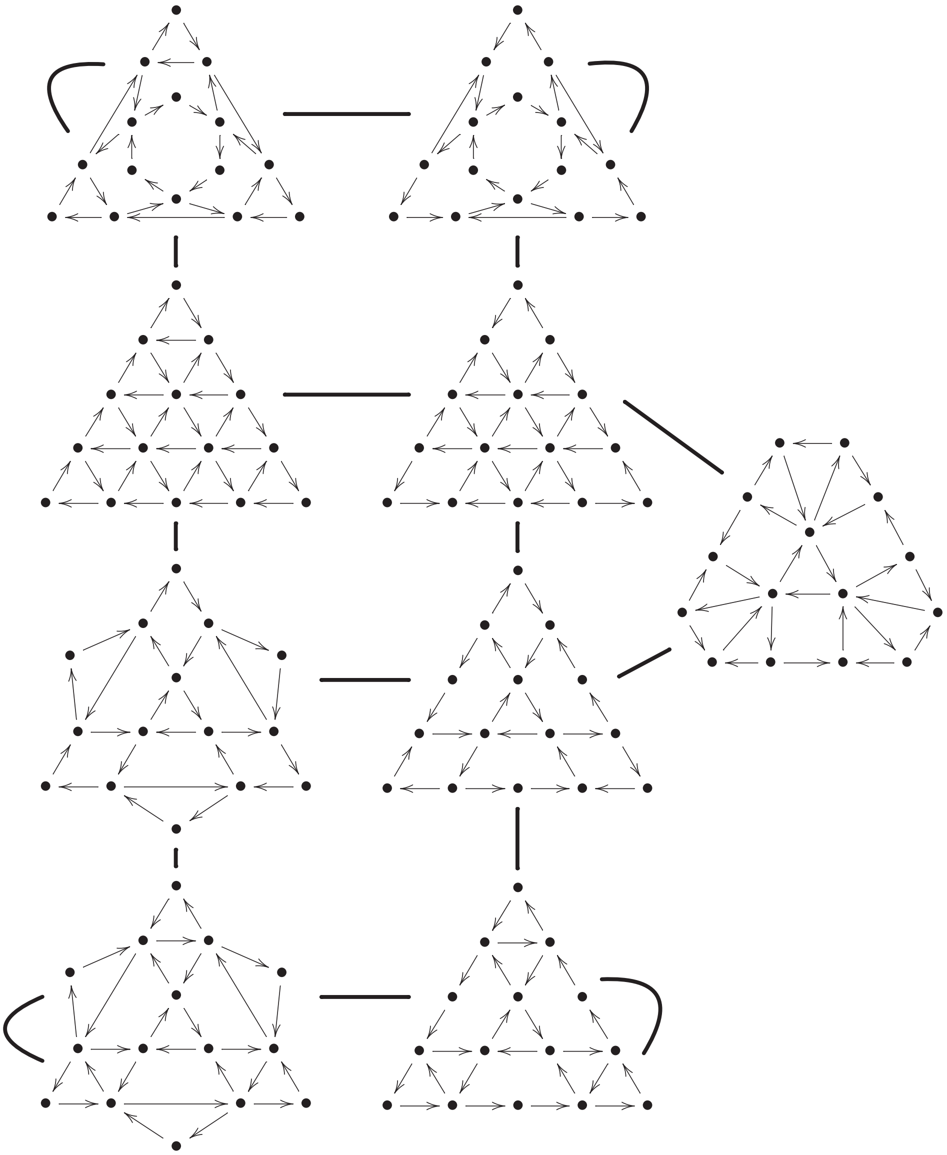


Figure 4. Planar mutation lattice for $(Q^{(5)}, \epsilon^{(5)})$.

Theorem 9.9. *Every symmetric square-shaped QP is selfinjective with Nakayama permutation σ.*

Proof. Let (Q, ϵ) be a symmetric square-shaped QP. We regard $X = X_{(Q,\epsilon)}$ as a filled square in the plane. For each subquiver P of Q, we also regard X_P as a subset of X.

Consider the partition

$$Q_0 = Q_0^0 \coprod Q_0^1,$$

where Q_0^k consists of all vertices (i, j) such that $i + j \equiv k \bmod 2$. Define the diagonal arrows of Q to be $D_1 = D_1^0 \coprod D_1^1$, where D_1^k is the set of arrows a with $s(a), e(a) \in Q_0^k$. For each $k \in \{0, 1\}$, set $D^k_0 = s(D^k_1) \cup e(D^k_1)$ and let D^k be the subquiver of Q with vertices D^k_0 and arrows D^k_1. Denote the union of D^0 and D^1 by D.

We call the vertices in $Q_0 \backslash D_0$ mutable. Mutation at a mutable vertex is planar and produces a new square-shaped QP. Let us make this observation more precise. Let x be mutable and $y, z \in Q_0$ be of the form $x \pm (1, 0)$ and $x \pm (0, 1)$, respectively. Moreover, let $d^Q_{yz} \in \{0, 1\}$ be the number of diagonal arrows between y and z in Q. Then $d^{\mu_x Q}_{yz} = 1 - d^Q_{yz}$. Notice that if $x \in Q^k_0$, then the diagonal arrows affected by mutation at x lie in D^{k+1}_1.

If D is empty, then (Q, ϵ) is the square product of two Dynkin quivers of type $\mathbb{A}_s$. By Theorem 5.2, (Q, ϵ) is selfinjective with Nakayama permutation σ. Now assume that D is not empty. We proceed to show that there is a set of mutable vertices in Q invariant under σ such that after mutation at these vertices the square-shaped QP obtained will have fewer diagonal arrows than Q. This will complete the proof.

A subquiver B of D is called a border if it is either of type $\tilde{\mathbb{A}}$ with at most one vertex on the boundary ∂X of X or of type $\mathbb{A}$ with exactly two vertices on ∂X. Starting with a diagonal arrow, we may extend it in both directions until a border is achieved. Thus, D is the union of all borders.

Let B be a border. By possibly attaching a connected part of ∂X to X_B, we get a simple closed curve c. By the Jordan curve theorem, $\mathbf{R}^2 \backslash c$ decomposes into two connected components A and U, where A is bounded and U is unbounded. Since U intersects $\mathbf{R}^2 \backslash X$, we have $\mathbf{R}^2 \backslash X \subset U$ and so $A \subset X$. Moreover, for every point x in $U \cap X$ there is a path in U connecting x with some point in $\mathbf{R}^2 \backslash X$. This path intersects $U \cap \partial X$, which is connected. Hence, $U \cap X$ is connected. It follows that $X \backslash X_B$ has exactly two connected components. We denote these components by Y^1_B and Y^2_B, where $x_0 \in Y^2_B$ for some fixed point $x_0 \in X \backslash X_Q$ not depending on B. Since $c = \partial A = \partial U$, we may recover B from either Y^1_B or Y^2_B.

Let B and B' be borders. We claim that $X_B \subset Y^1_{B'} \cup X_{B'}$ holds if and only if $Y^2_{B'} \subset Y^2_B$. If $X_B \subset Y^1_{B'} \cup X_{B'}$, then $Y^2_{B'} = X \backslash (Y^1_{B'} \cup X_{B'}) \subset X \backslash X_B$. Since Y^2_B is a connected component of $X \backslash X_B$ intersecting $Y^2_{B'}$ at x_0, we have $Y^2_{B'} \subset Y^2_B$ as $Y^2_{B'}$ is connected. On the other hand, if $Y^2_{B'} \subset Y^2_B$, then $X_B \subset X \backslash Y^2_{B'} = Y^1_{B'} \cup X_{B'}$. Hence, we get a partial order $B \leqslant B'$ defined by $X_B \subset Y^1_{B'} \cup X_{B'}$.

Now let B be a minimal border with respect to $\leqslant$. By symmetry, we may assume that B is a subquiver of D^0. Set $V := Q^1_0 \cap Y^1_B$. We claim that all vertices in V are mutable. Assume otherwise. Then there is an arrow $a \in D^1_1$ such that $s(a), e(a) \in V$. As explained earlier, we can extend a to a border B' in D^1. Since $X_{B'}$ is connected and does not intersect X_B, we have that $X_{B'} \subset Y^1_B$, which contradicts the minimality of B.

Mutating at V will remove the arrows in B and will not create or remove any other diagonal arrows. Similarly, mutating at σV will only affect the arrows in σB. If $B = \sigma B$, then we mutate at $\sigma V = V$ to reduce the number of diagonal arrows. Otherwise, we mutate at $(V \cup \sigma V) \backslash (V \cap \sigma V)$ to remove the arrows in $(B_1 \cup \sigma B_1) \backslash (B_1 \cap \sigma B_1)$. □

In the above proof, we saw that symmetric square-shaped QPs of fixed size s are obtained from each other via planar mutation. However, for $s \geqslant 4$, many other planar QPs can be constructed by planar mutation as well. For instance, the planar mutation lattice for the case $s = 4$ is displayed in Figure 5.

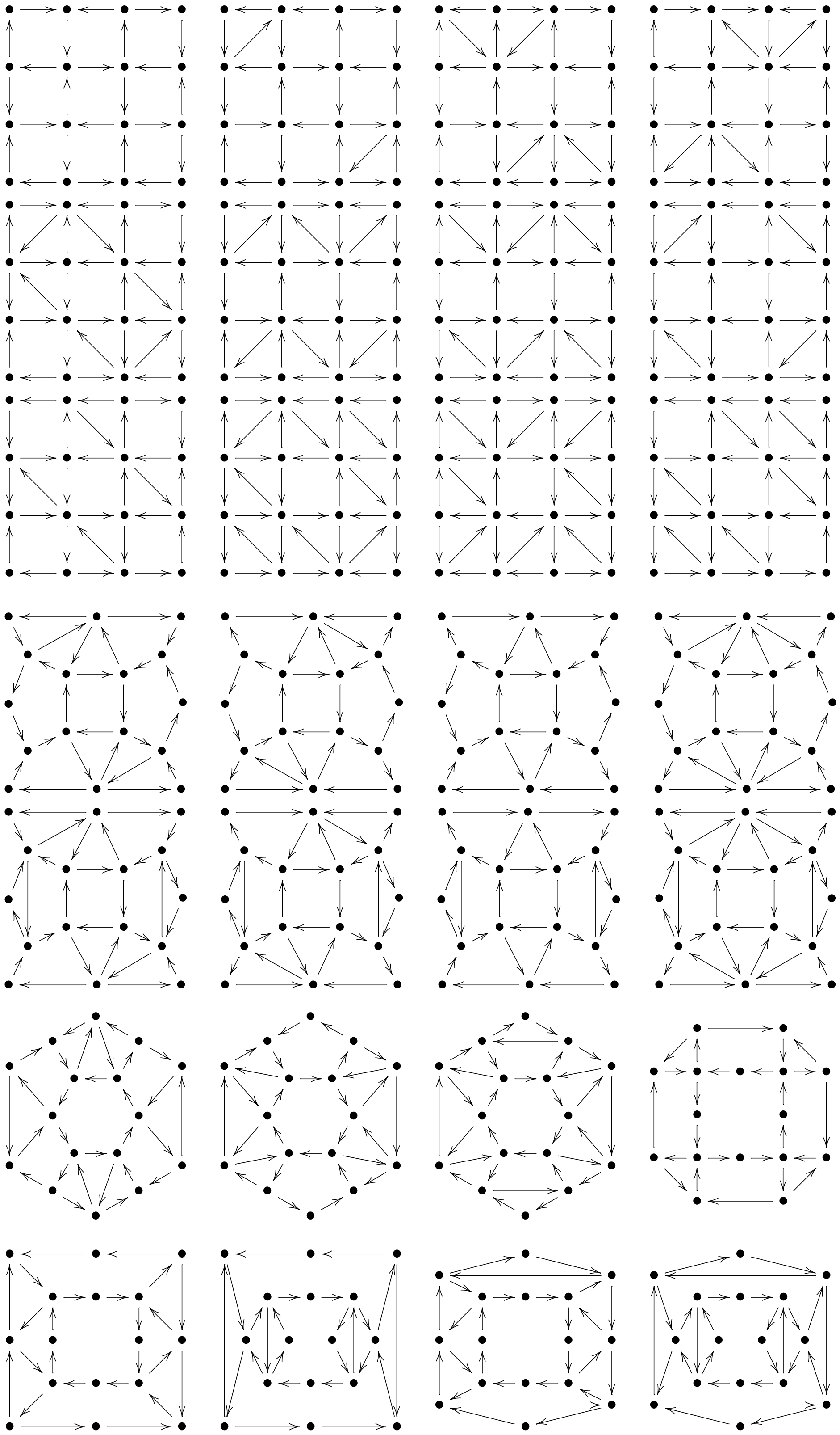
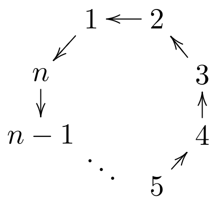


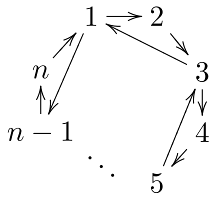
Figure 5. Planar mutation lattice for $s = 4$.

9.4 Examples from n-gons

The two families of selfinjective QPs given in § 5.1 are planar. For each $n \geqslant 4$, define the following planar QP.



If n is even, then also define the following planar QP.



For each even $n \geqslant 4$, the above pair forms a single planar mutation class.

10. Concluding remarks

We started out by showing that the study of 2-representation-finite algebras can be reduced to selfinjective QPs and their cuts. In § 9, we saw that even in the quite restrictive setting of planar QPs many selfinjective ones appear. At present, all examples of selfinjective planar QPs that we are aware of can be obtained from triangles, squares, and n-gons. Also, as far as we know they all have enough cuts. Thus, we pose the following questions.

Question 10.1. (1) Is every selfinjective planar QP related by iterated planar mutation to one of the triangles in § 9.2, squares in § 9.3, or n-gons in § 9.4?

(2) Does every selfinjective planar QP have enough cuts?

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