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ON INDUCED OPERATORS

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ABSTRACT. We show that when a positive contraction of type (p, q) is equipped with a positive norming function having full support, then it is related in a natural way to operators on other L_p spaces.

1. **Introduction.** Operator ergodic theory was born of Von Neumann's observation that the action of composing functions with a measure preserving transformation defines a bounded linear operator on L_2. In fact, such an operator is an isometry, simultaneously defined on every Banach space L_p, $p \in [1, \infty]$. It is also a positive operator, by which we mean that $Tf \geq 0$ a.e. whenever $f \geq 0$ a.e.

More generally, a linear operator may be defined by composing functions with a measurable non-singular bijection on the underlying measure space. In this case, the composed function must be weighted by the p^{th} root of the Radon-Nikodym derivative of the induced measure in order to define a bounded operator on L_p. This is an important example of a class of operators which are not necessarily defined on more than one L_p space, but which have "cousins" in every L_r, $r \in [1, \infty)$.

It is natural to ask for sufficient conditions under which a positive L_p operator may give rise to operators on L_p spaces of different index. In [B] and [AB], such sufficient conditions were given, along with an explicit definition of the related operator: we say u is *semi-invariant* for a positive L_p contraction T if it has full support and $\|Tu\| = \|u\|$. If $p, r \in (1, \infty)$ and T is such an operator, then the equation

$$T_r f = (Tu)^{\frac{p}{r}-1} T\left(u^{1-\frac{p}{r}} f\right)$$

defines a positive contraction of L_r. Furthermore, this *induced operator* is independent of the choice of semi-invariant function u.

The utility of this construction has been demonstrated by its use in giving a common generalisation to a theorem of Rota [R] and a theorem of Akcoglu and Sucheston [AS4].

In Section 3 of this paper, we investigate further properties of induced operators. In Section 2, the construction is carried out in the case where T is an operator between L_p spaces of different index; that is, operators of strong type (p, q) where q bears no special relation to p. Specifically,

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THEOREM. *Suppose $T\colon L_p(\mathbf{X}) \longrightarrow L_q(\mathbf{Y})$ is a positive linear operator of type (p,q), with $p,q \in [1,\infty)$. Suppose that u is a norming function for T and that u and $v = Tu$ both have full support. If $1 \leq s \leq r < \infty$, then the equation*

$$T_{r,s}f = v^{\frac{q}{s}-1}T\left(u^{1-\frac{p}{r}}f\right)$$

defines a positive bounded linear operator of type (r,s) with $\|T_{r,s}\| = \|T\|^{\frac{q}{s}}\|u\|_p^{\frac{q}{s}-\frac{p}{r}}$.

In Section 3, we will see that if $p = q$, then the induced operator is independent of the choice of u if and only if $r = s$. There are open questions in the case $p \neq q$, although we show in Section 5 that there is an important special class of type (p,q) operators which have only one norming function, up to scalar multiplicity.

In Section 4 we prove a more general version of the alternating sequence theorem in [AB], and provide a survey of related results in this area of research.

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2. **Existence of induced operators.**

DEFINITIONS 2.1. Throughout this paper, $\mathbf{X} = (X, \mathcal{F}, \mu)$ and $\mathbf{Y} = (Y, \mathcal{G}, \nu)$ always denote σ-finite measure spaces. Let $\mathcal{M}(\mathbf{X})$ be the vector space of $\mathcal{F}$-measurable complex-valued functions defined on X. Let $\mathcal{M}^+(\mathbf{X})$ and $\overline{\mathcal{M}}^+(\mathbf{X})$ denote the subsets of $\mathcal{M}(\mathbf{X})$ consisting of functions whose ranges are subsets of $\mathbb{R}^+ = [0,\infty)$ and $\overline{\mathbb{R}}^+ = [0,\infty]$ respectively. When $p \in [1,\infty)$, $L_p(\mathbf{X})$ is the Banach space of functions in $\mathcal{M}(\mathbf{X})$ such that $\int \|f\|^p\,dx < \infty$. L_∞ is the space of essentially bounded functions. We use the usual L_p norms. $L_p^+(\mathbf{X}) = \mathcal{M}^+(\mathbf{X}) \cap L_p(\mathbf{X})$ for every $p \in [1,\infty]$. All of the relations between the functions in these classes are in the μ-a.e. sense, even when this is not made explicit.

With the convention $0\cdot\infty = 0$, functions in $\overline{\mathcal{M}}^+(\mathbf{X})$ may be multiplied pointwise. Note that this convention implies $\infty^0 = 1$. In particular, $f^0 = \mathbf{1} = \mathbf{1}_X$ for any $f \in \overline{\mathcal{M}}^+(\mathbf{X})$, where $\mathbf{1}_A$ is the characteristic function of the set A. The *support* of a function $f \in \mathcal{M}(\mathbf{X})$ is $\{x \in X \mid f(x) \neq 0\}$.

Whenever a real number $p \in (1,\infty)$ is understood, the symbol p' always stands for the real number $p(p-1)^{-1}$. Following the usual convention, when $p = 1$ we take $p' = \infty$ and vice versa.

Suppose $p, r \in [1,\infty)$. When $f \in L_p^+(\mathbf{X})$ then $f^{\frac{p}{r}} \in L_r^+(\mathbf{X})$. For a general $f \in L_p(\mathbf{X})$ and an $x \in X$, we use $f^{\frac{p}{r}}(x)$ as shorthand for $\operatorname{sgn}\big(f(x)\big)|f(x)|^{\frac{p}{r}}$, where $\operatorname{sgn}(z)$ is the complex number of unit modulus having the same argument as z. Observe that if $f \in L_p(\mathbf{X})$, then $f^{p-1} \in L_{p'}(\mathbf{X})$ and $\|f^{p-1}\|_{p'} = \|f\|_p^{p-1}$.

By a *monotone operator* from $\overline{\mathcal{M}}^+(\mathbf{X})$ to $\overline{\mathcal{M}}^+(\mathbf{Y})$, we mean a mapping

$$T\colon \overline{\mathcal{M}}^+(\mathbf{X}) \longrightarrow \overline{\mathcal{M}}^+(\mathbf{Y})$$

which is linear with respect to scalars in $\mathbb{R}^+$ and is *order continuous,* in the sense that $Tf_n \uparrow Tf$ ν-a.e. whenever $f_n \uparrow f$ μ-a.e. (the arrows indicate monotone non-decreasing pointwise convergence in $\mathbb{R}^+$).

Given such a T, there exists a unique monotone operator

$$T^*: \overline{\mathcal{M}}^+(Y) \longrightarrow \overline{\mathcal{M}}^+(X),$$

called the *adjoint* of T, such that

$$\int_X f \cdot T^* g \, d\mu = \int_Y Tf \cdot g \, d\nu$$

for every $f \in \overline{\mathcal{M}}^+(X)$ and $g \in \overline{\mathcal{M}}^+(Y)$. The proof is an easy consequence of the Radon-Nikodym theorem; see [AB].

Suppose $p, q \in [1, \infty]$. A monotone operator T on $\overline{\mathcal{M}}^+(X)$ is said to be of *type* (p, q) (or, more properly, of strong type (p, q)) if there is an $M \in \mathbb{R}^+$ such that $\|Tf\|_q \leq M\|f\|_p$. The infimum of all values of M satisfying this inequality is the (p, q) *norm* of T and is denoted $\|T\|_{p,q}$. When $\|T\|_{p,q} \leq 1$, we say that T is a type (p, q) *contraction.* When $p = q$, the norm is denoted simply $\|T\|_p$, and T may be called a monotone L_p operator or L_p contraction.

We note that T is a monotone operator of type (p, q) if and only if T^* is a monotone operator of type (q', p') and that, whenever either of these conditions hold, then $\|T\|_{p,q} = \|T^*\|_{q',p'}$. This follows the defining property of T^* and the fact that

$$\|T\|_{p,q} = \sup \int Tf \cdot g \, d\nu,$$

the supremum being taken relative to functions of unit norm in $\overline{\mathcal{M}}^+(X)$ and $\overline{\mathcal{M}}^+(Y)$ (see [E], p. 143).

If $T: L_p(X) \longrightarrow L_q(Y)$ is a bounded linear operator in the usual sense, we say that T is *positive* if $TL_p^+(X) \subseteq L_q^+(Y)$. It is easy to see that its restriction of such an operator to $L_p^+(X)$ can be extended uniquely to a monotone operator on $\overline{\mathcal{M}}^+(X)$. The next lemma is something of a converse.

LEMMA 2.2. *If T is a monotone operator on $\overline{\mathcal{M}}^+(X)$ of type (p, q), then its restriction to $L_p^+(X)$ may be extended uniquely to a bounded linear operator of type (p, q).*

PROOF. The result follows from Lemma 2.2 of [AS4], although if one is concerned only with real-valued L_p spaces, then the result follows by considering the usual decomposition of functions into their positive and negative parts. In either case, we note that the norm of T when viewed as an L_p operator has the same value as when it is thought of as a monotone operator on $\overline{\mathcal{M}}^+(X)$. ∎

THEOREM 2.3. *Let T be a monotone operator from $\overline{\mathcal{M}}^+(X)$ to $\overline{\mathcal{M}}^+(Y)$ such that $T\mathbf{1} \leq \mathbf{1}$ and $T^*\mathbf{1} \leq \mathbf{1}$. Suppose either that $p = q \in [1, \infty]$ or that $1 \leq q < p < \infty$ and X is a finite measure space. Then T is a monotone operator of type (p, q) with*

$$\|T\|_{p,q} \leq \big(\mu(X)\big)^{\frac{p-q}{pq}}.$$

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PROOF. (This result is well-known for the case $p = q$.) If $f \in L^+_\infty(X)$ and α is its essential supremum, then $Tf \leq \alpha\mathbf{1}$, since T maps the positive function $\alpha\mathbf{1} - f$ to a positive function. Thus T is a type (∞, ∞) contraction.

Let $t = p/q$ and let $f \in L^+_t(X)$. Then

$$\begin{aligned}\int Tf\,d\nu &= \int fT^*\mathbf{1}\,d\mu \\ &\leq \int f\mathbf{1}\,d\mu.\end{aligned}$$

If $p = q$, observe that this means that $\|Tf\|_1 \leq \|f\|_1$. Therefore, T is a type $(1, 1)$ contraction. By the Riesz-Thorin theorem (also called the Riesz interpolation theorem, see, e.g., [E], p. 151), T is a contraction of L_p.

If $p > q$, observe that $\mathbf{1} \in L_r(X)$ for every $r \in [1, \infty]$, and that $t' = p(p - q)^{-1}$. By the Hölder inequality,

$$\int f\mathbf{1}\,d\mu \leq \|f\|_t\|\mathbf{1}\|_{t'}.$$

In other words,

$$\|Tf\|_1 \leq \big(\mu(X)\big)^{\frac{p-q}{p}}\|f\|_t.$$

Thus, T has type $(t, 1)$, and we may apply the Riesz-Thorin theorem to conclude that T is a type (p, q) operator with the desired norm. ∎

DEFINITION 2.4. Suppose T is a monotone operator from $\overline{\mathcal{M}}^+(X)$ to $\overline{\mathcal{M}}^+(Y)$ of type (p, q). We say that $u \neq 0$ is a (p, q) *norming function* for T if

$$\|Tu\|_q = \|T\|_{p,q} \cdot \|u\|_p.$$

When the indices (p, q) are understood, we may refer to u simply as a norming function for T.

LEMMA 2.5. *If $p, q \in [1, \infty)$, then u is a norming function for a monotone operator T from $\overline{\mathcal{M}}^+(X)$ to $\overline{\mathcal{M}}^+(Y)$ of type (p, q), if and only if*

$$T^*(Tu)^{q-1} = (\|T\|^q_{p,q} \cdot \|u\|^{q-p}_p)u^{p-1}. \tag{2.6}$$

If either of these conditions holds, then $(Tu)^{q-1}$ is a norming function for T^*.

REMARK. The case $p = q$ appears in [AS2] as Lemma 2.2. The reflexive case, i.e., $p, q > 1$, is in [K2] as Lemma 2.10.

PROOF. The converse follows by a straightforward computation. For the implication, we first suppose that $\|T\|_{p,q} = 1$.

$$\begin{aligned}\|Tu\|^q_q &= \int u\big(T^*(Tu)^{q-1}\big)\,d\mu \\ &\leq \|u\|_p\|T^*(Tu)^{q-1}\|_{p'} \\ &\leq \|u\|_p\|Tu\|^{q-1}_q \\ &= \|u\|^q_q \\ &= \|Tu\|^q_q,\end{aligned}$$

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where the third line follows by Hölder's Inequality. Thus, we have equality in Hölder's Inequality, so $T^*(Tu)^{q-1}$ is equal to αu^{p-1}, for some real number α. It follows that $\alpha = \|u\|_p^{q-p}$. (2.6) now follows by considering $T/\|T\|_{p,q}$.

To see that $(Tu)^{q-1}$ is a norming function for T^*, observe that

$$\begin{aligned}\|T^*(Tu)^{q-1}\|_{p'} &= \|T\|_{p,q}^q \|u\|_p^{q-p} \|u\|_p^{p-1} \\ &= \|T\|_{p,q}(\|T\|_{p,q}\|u\|_p)^{q-1} \\ &= \|T^*\|_{q',p'}\|(Tu)^{q-1}\|_{q'}. \end{aligned}$$ ■

THEOREM 2.7. *Let T be a monotone operator from* $\overrightarrow{\mathcal{M}}^+(X)$ *to* $\overrightarrow{\mathcal{M}}^+(Y)$. *Suppose* $u \in \mathcal{M}^+(X)$ *and* $v = Tu$ *have full support. Suppose that* $1 \le q \le p < \infty$ *and, if* $p \neq q$, *that* $u \in L_p^+(X)$. *If there is a* $\lambda \in \mathbb{R}^+$ *such that*

$$T^*(Tu)^{q-1} \le \lambda^q u^{p-1}, \tag{2.8}$$

then T is a monotone operator of type (p, q) *with* $\|T\|_{p,q} \le \lambda \|u\|_p^{\frac{p}{q}-1}$.

REMARK. This was observed by Kan ([K1], Remark 4). We include a short proof to emphasize the role played by u in re-distributing the measure μ and to make this paper more self-contained. The case $p = q$ is a well-known.

PROOF. If $\lambda = 0$, then $T = 0$, since for any $f \in \mathcal{M}^+(X)$,

$$\begin{aligned}\int Tf(Tu)^{q-1}\,d\nu &= \int fT^*(Tu)^{q-1}\,d\mu \\ &\lambda^q \int fu^{p-1}\,d\mu = 0.\end{aligned}$$

If $\lambda > 0$ and $v = Tu$ then one may show that v is finite ν-a.e., for example by the techniques in [AS1], p. 391.

Let $d\mu' = u^p\,d\mu$ and $d\nu' = (v/\lambda)^q\,d\nu$. Let $X' = (X, \mathcal{F}, \mu')$ and $Y' = (Y, \mathcal{G}, \nu')$. Define an operator

$$R\colon \overrightarrow{\mathcal{M}}^+(X) \to \overrightarrow{\mathcal{M}}^+(Y')$$

by

$$Rf = \frac{1}{v}T(uf),$$

for $f \in \overrightarrow{\mathcal{M}}^+(X')$. The adjoint of R is easily computed. When $g \in \overrightarrow{\mathcal{M}}^+(Y')$, we have

$$R^*g = \frac{1}{\lambda^q u^{p-1}}T^*(v^{q-1}g).$$

$R\mathbf{1} = \mathbf{1}$ and $R^*\mathbf{1} = \mathbf{1}$, so by Theorem (2.3), R is a type (p, q) operator with norm $\zeta = \|u\|_p^{\frac{p}{q}-1}$. This means that if $f \in \overrightarrow{\mathcal{M}}^+(X')$, then

$$\int (Rf)^q\,d\nu' \le \zeta^q\Big(\int f^p\,d\mu'\Big)^{\frac{q}{p}}.$$

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If $f \in \overline{\mathcal{M}}^+(X)$, let $f_1 = \frac{f}{u}$, where $f_1 \in \overline{\mathcal{M}}^+(X')$. Hence

$$\begin{aligned}\int (Tf)^q \, d\nu &= \int \big(T(uf_1)\big)^q \, d\nu \\ &= \lambda^q \int (Rf_1)^q \, d\nu' \\ &\leq (\lambda\zeta)^q \Big(\int f_1^p \, d\mu'\Big)^{\frac{q}{p}} \\ &= (\lambda\zeta)^q \Big(\int f^p \, d\mu\Big)^{\frac{q}{p}} \\ &= (\lambda\zeta \, \|f\|_p)^q.\end{aligned}$$

Thus $\|Tf\|_q \leq (\lambda\zeta)\|f\|_p$, as desired. ∎

LEMMA 2.9. *Suppose that T is a monotone operator on $\overline{\mathcal{M}}^+(X)$ and that $u \in \mathcal{M}^+(X)$ and $v = Tu$ both have full support. Suppose that there is a $\lambda \in \mathbb{R}^+$ such that inequality (2.8) is satisfied, where $p, q \in [1, \infty)$. Suppose $1 \leq s \leq r < \infty$. If $r \neq s$, then suppose also that $u \in L_p^+(X)$. Then the formula*

$$Sf = v^{\frac{q}{s}-1} T\big(u^{1-\frac{p}{r}} f\big),$$

for $f \in \overline{\mathcal{M}}^+(d\mu)$, defines a monotone operator of type (r, s) from $\overline{\mathcal{M}}^+(d\mu)$ to $\overline{\mathcal{M}}^+(d\nu)$ with $\|S\|_{r,s} \leq \lambda^{\frac{q}{s}} \|u\|_p^{p(\frac{1}{s}-\frac{1}{r})}$.

PROOF. $S^*: \overline{\mathcal{M}}^+(Y) \longrightarrow \overline{\mathcal{M}}^+(X)$ is easily computed: if $g \in \overline{\mathcal{M}}^+(Y)$, then

$$S^* g = u^{1-\frac{p}{r}} T^*\big(v^{\frac{q}{s}-1} g\big).$$

Let $\tilde{u} = u^{\frac{p}{r}}$, then $S\tilde{u} = v^{\frac{q}{s}}$. Thus

$$\begin{aligned}S^*(S\tilde{u})^{s-1} &= u^{1-\frac{p}{r}} T^*(v^{q-1}) \\ &\leq \lambda^q u^{p-\frac{p}{r}} \\ &= \lambda^q \tilde{u}^{r-1}.\end{aligned}$$

The hypotheses of Theorem 2.7 are satisfied for the function $\tilde{u}$, the indices r and s and the constant $\lambda^{\frac{q}{s}}$. Thus, S is a monotone operator of type (r, s) and because

$$\|\tilde{u}\|_r^{\frac{r-s}{s}} = \|u\|_p^{\frac{p(r-s)}{rs}},$$

the norm of S has the stated upper bound. ∎

THEOREM 2.10. *Suppose $T: L_p^{\mathbb{C}}(X) \longrightarrow L_q^{\mathbb{C}}(Y)$ is a positive bounded linear operator of type (p, q), with $p, q \in [1, \infty)$. Suppose that u is a norming function for T and that u and $v = Tu$ both have full support. Suppose $1 \leq s \leq r < \infty$. Then the equation*

$$T_{u,r,s} f = v^{\frac{q}{s}-1} T\big(u^{1-\frac{p}{r}} f\big)$$

defines a positive bounded linear operator of type (r, s)*. Furthermore,* $u^{\frac{p}{r}}$ *is a norming function for* $T_{u,r,s}$*, and* $\|T_{u,r,s}\|_{r,s} = \|T\|_{p,q}^{\frac{q}{s}} \|u\|_p^{\frac{q}{s}-\frac{p}{r}}$.

PROOF. The restriction of T to $L_p^+(X)$ may be extended to a monotone operator on $\overline{\mathcal{M}}^+(X)$. From this it easily follows that the equation for $T_{u,r,s}$ also defines a monotone operator on $\overline{\mathcal{M}}^+(X)$. By Lemma 2.2 it suffices to show that $T_{u,r,s}$ has type (r, s) and the above norm when viewed as a monotone operator on $\overline{\mathcal{M}}^+(X)$.

Because of Lemma 2.5, we may apply Lemma 2.9 with $\lambda = \|T\|_{p,q}\|u\|_p^{\frac{1-p}{q}}$ to conclude that $T_{u,r,s}$ has type (r, s) with norm no greater than the stated one. To finish the proof, one makes a straightforward computation to show that this norm is actually achieved by $u^{\frac{p}{r}}$. This requires equality in (2.8). ∎

REMARK. We have assumed that both u and v have full support. This is not essential in the case of v, as long as we define the value of $T_{u,r,s}f$ to be zero outside of the support of v when $q < s$. On the other hand, we need the condition on u in order to re-distribute the measure μ. If T does not use the entire space X, then we may relax the condition on the support of u accordingly. To be precise: if u is a norming function such that $Tf = 0$ whenever $u \cdot f = 0$, then all of the results in this section are valid with minor modifications. See [B].

3. **Uniqueness in the case** $p = q$**.** For the remainder of this paper, all operators under consideration are the usual bounded linear operators on L_p spaces.

DEFINITIONS 3.1. We say $\tau : X \longrightarrow X$ is *non-singular* if $\mu(\tau^{-1}E) = 0$ whenever $\mu E = 0$. An *automorphism* of a measure space X is an invertible mapping τ such that both τ and τ^{-1} are measurable and non-singluar. Thus, the Radon-Nikodym derivative

$$\rho = \frac{d(\mu \circ \tau^{-1})}{d\mu}$$

exists and is positive almost everywhere. When $\rho = \mathbf{1}$, we say that τ is a *measure-preserving* transformation (this term is also widely used for maps which need not be invertible). If $p \in [1, \infty)$, we define the L_p *isometry induced by* τ by the equation

$$Q_p f = \rho^{\frac{1}{p}}(f \circ \tau^{-1}),$$

for $f \in L_p$.

Classical ergodic theory is concerned with the study of measure-preserving transformations and the operators they induce. $\mathbf{1}$ is an invariant function for each Q_p in this case, and every function in $L_p(X)$ is a norming function. Furthermore, $Q_r = Q_p$ for every $r \in [1, \infty)$. The set of positive L_p contractions with positive norming functions broadly generalises this extensively-studied class of operators.

The class of isometries induced by automorphisms is also properly included in the class of positive contractions with positive norming functions. It is a natural example of a class of operators which do not necessarily contract more than one L_p space each, but where there is an L_r operator associated with every Q_p in a natural way: multiplication by $\rho^{\frac{1}{r}-\frac{1}{p}}$. An operator in this class does not necessarily have invariant functions, but once again every function is norming.

DEFINITION 3.2. If T is a positive L_p contraction we say $u \in L_p$ is *semi-invariant for* T if u and $v = Tu$ both have full support and $\|u\| = \|v\|$.

In the case $p = q$ and $r = s$ of Theorem 2.10, the operator $T_{u,r,s}$ is independent of the choice of norming function u with full support. Thus, we may suppress two of the subscripts. This gives the following; for a proof see [AB] or [B].

THEOREM 3.3. *Suppose $p, r \in [1, \infty)$, $p > 1$, and T is a positive L_p contraction with a semi-invariant function u. Let $v = Tu$. Then the mapping T_r defined by*

$$T_r f = v^{\frac{p}{r}-1} T\left(u^{1-\frac{p}{r}} f\right)$$

is a positive contraction of L_r with semi-invariant function $u^{\frac{p}{r}}$. Furthermore, T_r is independent of the choice of u. T_r is called the L_r operator induced by T.

The proof of uniqueness hinges on the fact that if u_1 and u_2 are both norming functions for T with full support, then for any $\alpha \in \mathbb{R}^+$ the set

$$E_\alpha = \left\{ x \in X \;\middle|\; \frac{u_2(x)}{u_1(x)} > \alpha \right\}$$

is a *reducing* set for T. That is, if the support of f is in E_α and the support of g is disjoint from E_α, then $Tf \cdot Tg = 0$. We note that $p > 1$ is needed for the proof to work.

THEOREM 3.4. *Suppose T is a positive contraction of L_p with $p \in (1, \infty)$ and that u is semi-invariant for T. If $r \in (1, \infty)$ and $s \in [1, \infty)$ then*

(a) $T_p = T$,

(b) $(T_r)^* = (T^*)_{r'}$, *and*

(c) $(T_r)_s = T_s$.

PROOF. Part (a) is obvious and part (c) follows by a routine computation, using the fact that $u^{\frac{p}{r}}$ is semi-invariant for T_r. For part (b), observe that v^{p-1} is semi-invariant for T^* and that $T^*(v^{p-1}) = u^{p-1}$, where $v = Tu$. This follows from (2.5). Thus when $g \in L_{r'}$,

$$(T^*)_{r'} = u^{1-\frac{p}{r}} T^*\left(v^{\frac{p}{r}-1} g\right).$$

On the other hand, if $f \in L_r$ and $g \in L_{r'}$ then

$$\begin{aligned}\int f\left[(T_r)^* g\right] d\mu &= \int [T_r f] g \, d\nu \\ &= \int \left[v^{\frac{p}{r}-1} T\left(u^{1-\frac{p}{r}} f\right)\right] g \, d\nu \\ &= \int f\left[u^{1-\frac{p}{r}} T^*\left(v^{\frac{f}{r}-1} g\right)\right] d\mu. \end{aligned}$$

■

THEOREM 3.5. *Suppose τ is an automorphism of X and $p, r \in [1, \infty)$. Then $(Q_p)_r = Q_r$.*

PROOF. First suppose X is a probability space. Any function with full support is semi-invariant for Q_p and, in view of (3.3), any of them for the construction. $u = \mathbf{1}$ is the most convenient. In this case

$$u^{1-\frac{p}{r}} = \mathbf{1} \text{ and } v^{\frac{p}{r}-1} = \rho^{\frac{1}{r}-\frac{1}{p}}.$$

Thus,

$$(Q_p)_r f = \rho^{\frac{1}{r}}(f \circ \tau^{-1}),$$

as desired. The argument in the infinite measure case is not much more difficult, in view of the fact that every subset of X is a reducing set for Q_p. ∎

REMARKS. 1) Something better may be proved: if $T = EQ_pE$ for a conditional expectation operator E, then $T_r = EQ_rE$, see [AB]. This is of independent interest, as operators of the form EQE are central in reducing the proof of the pointwise ergodic theorem (PET) for positive L_p contractions (see [A]) to the PET for positive isometries (see [I]).

2) Taken together, Theorems 3.3–3.5 argue eloquently that induced operators are natural and intrinsic objects in the case $p = q$ and $r = s$. On the other hand, if $p = q$ and $s < r$, then operator $T_{u,r,s}$ depends on the choice of u. Indeed, $\|u\|_p$ affects the operator norm in this case, but uniqueness fails even if we limit ourselves to unit vectors. Consider, for example, the case where T is the operator Q_p induced by an automorphism τ on a probability space, and u is a non-constant function with full support and unit norm. Then for $f \in L_r$

$$\begin{aligned} T_{u,r,s}f &= v^{\frac{p}{s}-1}T\big(u^{1-\frac{p}{r}}f\big) \\ &= v^{p(\frac{1}{s}-\frac{1}{r})}T_r f, \end{aligned}$$

whereas

$$T_{\mathbf{1},r,s} = \rho^{(\frac{1}{s}-\frac{1}{r})}T_r.$$

We conclude this section with an observation about the possibility of defining induced operators using a function u with full support satisfying the weaker condition

$$T^*(Tu)^{p-1} \leq \lambda^p u^{p-1} \tag{3.6}$$

rather than being a norming function. Of course, this is only of interest in the case where the left-hand side is not a scalar multiple of the right, for otherwise we would simply adjust the constant. Equality in (3.6) is used in this paper only in (2.10) to show that $u^{\frac{p}{r}}$ is norming and hence that $T_{u,r,s}$ actually achieves the stated upper bound on its norm. Thus, existence of induced operator follows from this weaker assumption on u. Equality in (3.6) is also used in the uniqueness portion of (3.3), and we will demonstrate by means of a counterexample that this use is essential.

In doing so, we also show that a *bistochastic* operator (i.e., one satisfying $T\mathbf{1} = T^*\mathbf{1} = \mathbf{1}$) need not have a semi-invariant function in the infinite measure case. Clearly, $\mathbf{1}$ is semi-invariant in the finite measure case.

DEFINITION 3.7. Let X be two-sided sequence space over the complex numbers, with counting measure. If $x \in X$, then x is a two-sided infinite tuple of complex numbers where x_i and $(x)_i$ both denote the i^{th} co-ordinate of x, for each $i \in \mathbb{Z}$. The integral of x is simply $\sum_{i=-\infty}^{\infty} x_i$.

Let $\ell_p = L_p(X)$ for $p \in [1, \infty]$ and let H be the "half-shift" operator from $\overline{\mathcal{M}}^+(X)$ to $\overline{\mathcal{M}}^+(X)$ given by

$$(Hx)_i = \frac{x_i + x_{i+1}}{2},$$

for $x \in X$ and $i \in \mathbb{Z}$. This is the average of the identity and the shift operator. It follows that

$$(H^*x)_i = \frac{x_i + x_{i-1}}{2},$$

for $x \in X$ and $i \in \mathbb{Z}$. Furthermore, $H\mathbf{1} = H^*\mathbf{1} = \mathbf{1}$. Thus H is an ℓ_p contraction for every $p \in [1, \infty]$.

PROPOSITION 3.8. *If $p \in (1, \infty)$, then there is no norming function for H in ℓ_p^+. Thus, a bistochastic operator need not have a norming function.*

PROOF. Suppose $\|Hx\|_p^p = \|x\|_p^p$. Then

$$\sum_{i=-\infty}^{\infty} \left(\frac{x_i + x_{i+1}}{2}\right)^p = \sum_{i=-\infty}^{\infty} \frac{x_i^p + x_{i+1}^p}{2}.$$

If $a, b \in \mathbb{R}^+$ and $p > 1$, then $(a+b)^p \leq 2^{p-1}(a^p + b^p)$, with equality if and only if $a = b$. Thus $x_i = x_j$ for every $i, j \in \mathbb{Z}$. But then $\|x\|_p = 0$ or ∞, which is a contradiction. ∎

PROPOSITION 3.9. *Suppose $p \in (1, \infty)$, and let $(u_1)_i = 2^{-|i|}$ and $(u_2)_i = 2^{-|i-1|}$. Then $u_j \in \ell_p$ for $j = 1, 2$, and*

$$H^*(Hu_j)^{p-1} \leq \left(\frac{3}{2}\right)^{p-1} u_j^{p-1}.$$

The proof is an easy computation.

PROPOSITION 3.10. *Suppose $p, r \in (1, \infty)$, and $p \neq r$. Let H, u_1 and u_2 be as defined above, and let $v_j = Hu_j$ for $j = 1, 2$. If*

$$S_j x = v_j^{\frac{p}{r}-1} H\left(u_j^{1-\frac{p}{r}} x\right)$$

for $j = 1, 2$ and $x \in \overline{\mathcal{M}}^+(d\mu)$, then $S_1 \neq S_2$.

PROOF. One may verify that for $j = 1, 2$,

$$(S_j x)_i = \begin{cases} 3^{\frac{p}{r}-1} 2^{-\frac{p}{r}} \left[2^{1-\frac{p}{r}} x_i + x_{i+1}\right] & \text{if } i > 0, \\ 3^{\frac{p}{r}-1} 2^{-\frac{p}{r}} \left[x_i + 2^{1-\frac{p}{r}} x_{i+1}\right] & \text{if } i < 0. \end{cases}$$

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However

$$(S_1x)_0 = \frac{1}{3}\left(\frac{3}{2}\right)^{\frac{p}{r}}\left(2^{1-\frac{p}{r}}x_0 + x_1\right)$$

whereas

$$(S_2x)_0 = \frac{1}{3}\left(\frac{3}{2}\right)^{\frac{p}{r}}\left(x_0 + 2^{1-\frac{p}{r}}x_1\right). \qquad \blacksquare$$

4. **An application.**

THEOREM 4.1. *Suppose* $1 < p < \infty$ *and* $1 < r < \infty$. *Suppose that* $\langle T_n \rangle_{n=1}^{\infty}$ *is a sequence of positive contractions where*

$$T_n: L_p(\boldsymbol{X}_{n-1}) \longrightarrow L_p(\boldsymbol{X}_n)$$

for each $n \geq 1$, *where* $\langle \boldsymbol{X}_n \rangle_{n=0}^{\infty}$ *is a sequence of* σ*-finite Lebesgue spaces. If every* T_n *has a semi-invariant function then*

$$\lim_{n \to \infty} (T_1^*) \cdots (T_n^*)_r (T_n \cdots T_1 f)^{\frac{p}{r}} \tag{4.2}$$

exists almost everywhere, for every $f \in L_p(\boldsymbol{X}_0)$.

The proof is deferred until the end of this section. By σ-finite Lebesgue space, we mean a space $\boldsymbol{X}$ where X is a complete metric space, $\mathcal{F}$ is the Borel σ-algebra, and μ is σ-finite. We note that the case $\boldsymbol{X}_n = \boldsymbol{X}_0$ for every $n \geq 1$ appears in [AB]. What follows is a survey of related results. The proofs of the following theorems appear in [BC], [R], [St] and [AS4] respectively.

THEOREM 4.3 (BURKHOLDER AND CHOW 1961). *If* E_1 *and* E_2 *are conditional expectation operators over a probability space and* $T_{2n-1} = E_1$, $T_{2n} = E_2$ *for every* $n \geq 1$. *Then*

$$\lim_{n \to \infty} T_n \cdots T_1 f$$

exists almost everywhere for every $f \in L_2$.

THEOREM 4.4 (ROTA 1962). *If* $\langle T_n \rangle_{n=1}^{\infty}$ *is a sequence of positive bistochastic operators over a probability space, then*

$$\lim_{n \to \infty} T_1^* \cdots T_n^* T_n \cdots T_1 f$$

exists almost everywhere for every $f \in L_p$, *when* $1 < p < \infty$.

THEOREM 4.5 (STEIN 1961). *If* T *is a self-adjoint positive contraction on* L_2, *then*

$$\lim_{n \to \infty} T^{2n} f$$

exists almost everywhere for every $f \in L_2$.

THEOREM 4.6 (AKCOGLU AND SUCHESTON 1988). *If* $\langle T_n \rangle_{n=1}^{\infty}$ *is a sequence of positive contractions of* L_p *of a* σ*-finite measure space, where* $1 < p < \infty$, *then*

$$\lim_{n\to\infty} T_1^* \cdots T_n^*(T_n \cdots T_1 f)^{p-1}$$

exists almost everywhere for every $f \in L_p$.

The following implications hold:

$$\begin{array}{ccccccc} (4.1) & & \longrightarrow & (4.4) & \longrightarrow & & (4.3) \\ & \searrow & & & & \nearrow & \\ & & (4.6) & \longrightarrow & (4.5) & & \end{array}$$

The theorem of Burkholder and Chow appears to assert the convergence of T^n for the operator $T = E_2E_1$; it is more useful to think of T as $E_1E_2E_1$. As a case in point, we note that the convergence of $(E_3E_2E_1)^n$, where all E_is are conditional expection operators, remains open almost 30 years later, although the method of proof of Burkholder and Chow implies the convergence of $(E_1E_2E_3E_2E_1)^n$.

The theorems of Rota and of Stein provide deeper but different insights into the phenomenon described in Burkholder and Chow's theorem, both relying on the fact that conditional expectation operators are positive, self-adjoint and idempotent. (4.3) follows immediately from Rota's theorem with the T_ns defined in the same way for both. Stein's theorem, as originally stated, asserts the convergence of T^n when the L_2 contraction T is positive, self-adjoint and non-negative definite, properties enjoyed by $E_1E_2E_1$ and by the square of any positive self-adjoint contraction of L_2.

The theorem of Akcoglu and Sucheston clearly implies Stein's theorem as stated. It also implies the conclusion of Rota's theorem when $p = 2$, but apparently not for all p, and so offers only a partial resolution of the Rota/Stein dichotomy.

It is easy to see that Theorem (4.1) implies Rota's theorem: **1** is semi-invariant for each T_n^* and $(T_n^*)_r = T_n^*$ for each r, so Rota's theorem is the case $r = p$ in (4.1). It is not immediately apparent that (4.1) also implies the theorem of Akcoglu and Sucheston, since its hypothesis requires the existence of semi-invariant functions. However, $(T_n^*)_{p'} = T_n^*$ by Theorem 3.4(a), so semi-invariant functions are not really needed for the statement of (4.1) in the case $r = p'$. The only place where existence of semi-invariant functions is used in the proof of (4.1), other than for the definition of $(T_n^*)_r$, is to scale the infinite measure spaces X_n to probability spaces without destroying the existence of semi-invariant functions (see the proof of (4.1) below). This may be done instead by the standard rescaling of a σ-finite measure when one wishes only to capture the theorem of Akcoglu and Sucheston.

We state another theorem relevant to this line of research, see [S]. The proof incorporates techniques of Doob [D]. It appears to be the last major contribution to the alternating sequence problem until 1987, when the norm convergence version of the Akcoglu and Sucheston theorem was published [AS3].

THEOREM 4.7 (STARR 1966). *Suppose $\langle T_n\rangle_{n=1}^{\infty}$ is a sequence of positive contractions where*

$$T_n\colon L_p(X_{n-1}) \longrightarrow L_p(X_n)$$

for each $n \geq 1$, where $\langle X_n\rangle_{n=0}^{\infty}$ is a sequence of σ-finite measure spaces. If $T_n\mathbf{1} \leq \mathbf{1}$ and $T_n^*\mathbf{1} \leq \mathbf{1}$ for every $n \geq 1$, then

$$\lim_{n\to\infty} T_1^* \cdots T_n^* T_n \cdots T_1 f$$

exists almost everywhere, for every $f \in L_p$, $1 < p < \infty$.

Starr's theorem generalizes Rota's theorem in three ways: the operators map between L_p spaces over different measure spaces, those measure spaces may be σ-finite, and bistochastic operators have been replaced by $L_1 - L_\infty$ operators. The first of these improvements is incorporated in (4.1), but we have seen that even a bistochastic operator need not have a semi-invariant function. On the other hand, the hypothesis $T\mathbf{1} \leq \mathbf{1}$ and $T^*\mathbf{1} \leq \mathbf{1}$ implies $T^*(T\mathbf{1})^{p-1} \leq \mathbf{1}^{p-1}$. Thus, we may construct induced operators for operators of the type considered by Starr, although they may depend on the choice of function satisfying (3.6). As long as we agree only to use $\mathbf{1}$ in the construction $(T_n^*)_p$, then, we may deduce Starr's theorem from (4.1) with $r = p$.

PROOF OF THEOREM 4.1. We reduce the general case to the case $X_n = X_0$ for $n \geq 1$, the proof of which appears in [AB].

In the proof in [AB], the pointwise convergence of the sequence (4.2) is shown to follow from two maximal estimates: given a sequence $\langle T_n\rangle_{n=1}^{\infty}$ as in the statement of this theorem and a function $f \in L_p(X_0)$, let $g_0 = f^{\frac{p}{r}}$ and, for $n \geq 1$,

$$g_n = (T_1^*)_r \cdots (T_n^*)_r (T_n \cdots T_1 f)^{\frac{p}{r}}.$$

We say that *Estimate A* holds for such a sequence of operators if

$$\|\sup_{n\geq 0} |g_n|\,\|_r \leq (p'\|f\|_p)^{\frac{p}{r}}$$

for every $f \in L_p(X_0)$. We say that *Estimate B* holds for $\langle T_n\rangle_{n=1}^{\infty}$ if for every $\varepsilon > 0$ there is a $\delta > 0$, depending only on ε, p and r, such that

$$\|\sup_{n\geq 0} |g_n - g_0|\,\|_r < \varepsilon \|f\|_p^{\frac{p}{r}}$$

whenever $f \in L_p(X_0)$ is such that

$$\|f\|_p - \lim_{n\geq 0} \|V_n f\|_p < \delta\,\|f\|_p.$$

It is then demonstrated that if Estimates A and B hold for every such sequence of operators, then for every $f \in L_p(X_0)$, g_n converges a.e. That is, the sequence (4.2) converges a.e.

It is further shown that if the Estimates A or B fail, then they fail at some finite stage where the operators can be assumed to be of a very simple sort. Finally, it is shown that these estimates always hold for finite sequences of such operators, using the martingale inequality and a dilation argument similar to the one in [A]. To deduce the present theorem, we need only show that if Estimates A or B fail for a finite string of g_ns with different measure spaces $\boldsymbol{X}_n$, then A or B fails for a sequence $\langle T'_n\rangle_{n=1}^{\infty}$ where the $\boldsymbol{X}_n$s all coincide. This is done in two steps, with the intermediate stage being the case where the measure spaces $\boldsymbol{X}_n$ are all probability spaces. We give the argument for Estimate A; the argument for Estimate B is similar, and uses the same operators T'_k and spaces $\boldsymbol{X}'_k$.

Suppose, then, that $\langle T_n\rangle_{n=1}^{\infty}$ is a sequence for which Estimate A fails. Then there is an $f \in L_p(\boldsymbol{X}_0)$ and an $n \geq 1$ such that $\| \max_{0\leq k\leq n} |g_k| \|_r > (p'\|f\|_p)^{\frac{p}{r}}$, with g_k as defined above.

Let $\boldsymbol{X}_k = (X_k, \mathcal{F}_k, \mu_k)$ for each $0 \leq k \leq n$. Let $u_{k-1} \in L_p(\boldsymbol{X}_{k-1})$ be semi-invariant for T_k and suppose, without loss of generality, that $\|u_k\|_p = 1$ for each k, $0 \leq k \leq n$. For each such k, let $d\mu'_k = u_k^p\, d\mu_k$. Let $\boldsymbol{X}'_k = (X_k, \mathcal{F}_k, \mu'_k)$. Let

$$T'_k\colon L_p(\boldsymbol{X}'_{k-1}) \longrightarrow L_p(\boldsymbol{X}'_k)$$

be given by

$$T'_k f' = \frac{1}{u_k} T(u_{k-1} f')$$

for $f' \in L_p(\boldsymbol{X}'_{k-1})$. Observe that $f' \in L_p(\boldsymbol{X}'_{k-1})$ if and only if $u_{k-1}f' \in L_p(\boldsymbol{X}_{k-1})$ and that $\mathbf{1}$ is semi-invariant for each T'_k.

For each $g \in L_{p'}(\boldsymbol{X}_k)$,

$$(T'_k)^* g = u_{k-1}^{1-p}\, T_k^*\left(u_k^{p-1} g\right).$$

Furthermore, $(T_k u_{k_1})^{p-1} u_k^{1-p}$ is a semi-invariant function for $(T'_k)^*$, whose image is $\mathbf{1}$. Thus, for every $g \in L_r(\boldsymbol{X}'_k)$, we have

$$\left((T'_k)^*\right)_r g = u_{k-1}^{-\frac{p}{r}} (T_k^*)_r \left(u_k^{\frac{p}{r}}\, g\right),$$

as may be seen by an awkward but entirely routine computation.

Let $f' = \frac{f}{u_0} \in L_p(\boldsymbol{X}'_0)$ and define g'_k from f and $\langle T_n\rangle_{n=1}^{\infty}$ in a manner entirely analogous to the definition of g_k from f and $\langle T_n\rangle_{n=1}^{\infty}$. It follows that

$$g'_k = \frac{g_k}{u_0^{\frac{p}{r}}},$$

and so

$$\max_{o\leq k\leq n} |g'_k| = \frac{1}{u_0^{\frac{p}{r}}} \max_{0\leq k\leq n} |g_k|.$$

Hence

$$\| \max_{0\leq k\leq n} |g'_k| \|_{r,\boldsymbol{X}'_0} = \| \max_{0\leq k\leq n} |g_k| \|_{r,\boldsymbol{X}_0}.$$

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Furthermore,

$$\|f'\|_{p,X_0'} = \|f\|_{p,X_0},$$

so

$$\|\max_{0\le k\le n}|g_k'|\,\|_r > (p'\|f'\|_p)^{\frac{p}{r}}.$$

Thus $\langle T_1',\dots,T_n'\rangle$ forms the initial portion of a sequence $\langle T_n'\rangle_{n=1}^{\infty}$ defined over probability spaces for which Estimate A fails.

Now suppose that $\langle T_n\rangle_{n=1}^{\infty}$ is a sequence defined over probability spaces for which Estimate A fails. Again, there is an $f\in L_p(X_0)$ and an $n\ge 1$ such that $\|\max_{0\le k\le n}|g_k|\,\|_r > (p'\|f\|_p)^{\frac{p}{r}}$.

Let $X = X_0\times\cdots\times X_n$. For each k, $0\le k\le n$, we identify each function $f\in L_p(X_k)$ with a function $f'\in L_p(X)$ which depends only on the k^{th} coordinate. We identify T_k with an operator on $L_p(X)$ which maps f' (where $f\in L_p(X_{k-1})$) to $(T_kf)'$.

More formally, for each i,j, where $0\le i,j\le n$, define

$$\mathcal{D}_{ij} = \begin{cases} \mathcal{F}_i & \text{if } i=j,\\ \{\emptyset, X\} & \text{otherwise.}\end{cases}$$

Let k range through $\{0,\dots,n\}$. Let $\mathcal{D}_k = \mathcal{D}_{k0}\times\cdots\times\mathcal{D}_{kn}$ and $E_k = E(\cdot\mid\mathcal{D}_k)$ be the associated conditional expectation operator. Let

$$\mathcal{S}_{p,k} = \{f\in L_p(X)\mid f(x_0,\dots,x_n) = f(y_0,\dots,y_n) \text{ whenever } x_k = y_k\}.$$

Then E_k is a mapping of $L_p(X)$ onto $\mathcal{S}_{p,k}$ for each p. Let

$$P_{p,k}\colon L_p(X_k)\longrightarrow \mathcal{S}_{p,k}$$

be given by $(P_{p,k}f)(x_0,\dots,,x_n) = f(x_k)$. We observe that the finiteness of μ_k is needed in order to assert that $P_{p,k}f\in L_p(X)$. Clearly $P_{p,k}$ is a bijection; in fact, it is an invertible isometry. Furthermore, $P_{p,k}^* = P_{p',k}^{-1}$ for every p and k. When $f\in L_p(X_k)$, f' also denotes $P_{p,k}f$. If $g\in\mathcal{S}_{p,k}$, then $g=f'$ for some $f\in L_p(X_k)$.

Now let k range through $\{1,\dots,n\}$. Define

$$T_k'\colon L_p(X)\longrightarrow L_p(X)$$

by

$$T_k' = E_kP_{p,k}T_kP_{p,k-1}^{-1}E_{k-1}.$$

A simple induction shows that if $f\in\mathcal{S}_{p,0}$, then

$$T_k'\cdots T_1'f' = P_{p,k}T_k\cdots T_1f.$$

We have

$$(T_k')^* = E_{k-1}P_{p',k-1}T_k^*P_{p',k}^{-1}E_k,$$

a well-defined $L_{p'}(X)$-operator. Since u'_{k-1} is semi-invariant for T'_k, $(T'_k u'_{k-1})^{p-1}$ is semi-invariant for $(T'_k)^*$, with image $(u'_{k-1})^{p-1}$. Thus for $g' \in \mathcal{S}_{r,k}$,

$$\big((T'_k)^*\big)_r g' = P_{r,k-1}[(T^*_k)_r g].$$

A simple induction using this fact shows that when $g \in L_r(X_k)$, then

$$\big((T'_1)^*\big)_r \cdots \big((T'_k)^*\big)_r g' = P_{r,0}(T^*_1)_r \cdots (T^*_k)_r g.$$

If f is the function for which Estimate A fails, then $\|f\|_{p,X_0} = \|f'\|_{p,X}$. We also have

$$\max_{0\leq k\leq n} |g'_k| = \max_{0\leq k\leq n} |g_k|$$

and consequently

$$\|\max_{0\leq k\leq n} |g'_k|\,\|_{r,X} = \|\max_{0\leq k\leq n} |g_k|\,\|_{r,X_0}.$$

So $\langle T'_1, \ldots, T'_n\rangle$ forms the initial part of a sequence $\langle T'_n\rangle_{n=1}^{\infty}$ defined over the same measure space X for which Estimate A fails.

This completes the proof. ∎

5. **An open question.** When $p \neq q$, the induced operator in Theorem 2.10 depends on $\|u\|_p$, the norm of the norming function, unless $\frac{p}{r} = \frac{q}{s}$. It would be interesting to know if $T_{u,r,s}$ is independent of the choice of u when $\frac{p}{r} = \frac{q}{s}$. The method of proof in [AB] for the case $p = q$ does not seem to apply in this case.

If independence does hold, then the argument used in Section 3 would imply that $T_{u,r,s}$ depends on the choice of u when $\frac{p}{r} \neq \frac{q}{s}$, even when only norming functions of unit norm are considered. This would entirely solve the question of which indices p, q, r, s give rise to type (r, s) operators which are dependent only on the operator and not on the shape of the norming function.

There is a special class of type (p, q) operators which have only one norming function, up to scalar multiplicity; we close with a brief study of them.

DEFINITION 5.1. Let $1 \leq q < p < \infty$, and suppose τ is an automorphism of a σ-finite measure space X. Let ρ and Q_p be as defined in (3.1). Suppose that $h \in L_1^+(X)$ has full support. Let $w = h^{\frac{p-q}{pq}}$. Define $W_{h,p,q}\colon L_p(X) \longrightarrow L_q(X)$ by

$$W_{h,q}f = wQ_pf$$

for $f \in L_p$.

PROPOSITION 5.2. *$W_{h,p,q}$ is an operator of type (p, q) with*

$$\|W_{h,p,q}\|_{p,q} = \|h\|_1^{\frac{p-q}{pq}}.$$

Up to scalar multiple, the only norming function for $W_{h,p,q}$ is

$$Q_p^{-1}h^{\frac{1}{p}} = (\rho\circ\tau)^{-\frac{1}{p}}(h\circ\tau)^{\frac{1}{p}}.$$

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PROOF.

$$\begin{aligned}\| W_{h,p,q}f\|_q^q &= \int h^{\frac{p-q}{p}} (Q_p f)^q \, d\mu \\ &\leq \|h^{\frac{p-q}{p}}\|_{\frac{p}{p-q}} \, \|(Q_p f)^q\|_{\frac{p}{q}} \\ &= \|h\|_1^{\frac{p-q}{p}} \, \|f\|_p^q,\end{aligned}$$

as desired. The second line follows from Hölder's inequality, and so we have equality if and only if h is a scalar multiple of $(Q_p f)^p$. ∎

REMARKS. 1) This is the type (p, q) analog to L_p isometries induced by automorphisms, a rich example for studying the case $p = q$. When $p \neq q$, there are no non-trivial isometries of type (p, q) induced by point transformations; see [B]. Briefly put, the requirements of linearity and isometry are incompatible, even in the 2 point space. Even when τ is the identity, the mapping Q_q fails to map L_p functions to L_q functions in the infinite measure case. Thus, a weighted isometry is needed. Furthermore, the weight function must be in $L_{\frac{p}{p-q}}$ for a useful application of Hölder's inequality, since we may not asssume any power of an L_p function f other than the p^{th} is integrable.

2) It is easy to verify that for any $1 \leq r < s < \infty$, $W_{h,r,s} = (W_{h,p,q})_{u,r,s}$ where $u = Q_p^{-1} h^{\frac{1}{p}}$.

3) If $\mu \circ \tau^{-1}$ is finite, then one may take $h = \rho$. $W_{\rho,p,q}f = \rho^{\frac{1}{q}}(f \circ \tau^{-1})$, as expected. In particular, the identity is a type (p, q) operator if and only if either $q = p$ or $q < p$ and X is a finite measure space.

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