Canad. Math. Bull. Vol. **40** (4), 1997 pp. 395–401

D-SPACES AND RESOLUTION

ZINEDDINE BOUDHRAA

ABSTRACT. A space X is a D-space if, for every neighborhood assignment f there is a closed discrete set D such that $\bigcup f(D) = X$. In this paper we give some necessary conditions and some sufficient conditions for a resolution of a topological space to be a D-space. In particular, if a space X is resolved at each $x \in X$ into a D-space Y_x by continuous mappings $f_x : X - \{x\} \longrightarrow Y_x$, then the resolution is a D-space if and only if $\bigcup \{x\} \times \mathrm{Bd}(Y_x)$ is a D-space.

1**. Introduction.** Unless explicitly stated, no separation axioms are assumed. A neighborhood assignment for a space $(X, \mathcal{T})$ is a function $f: X \longrightarrow \mathcal{T}$ such that $x \in f(x)$. A space X is a D-space if, for every neighborhood assignment f there is a closed discrete set D such that $\bigcup f(D) = X$, [3]. As noted in [1], the property of being a D-space is a delicate covering property; for instance, it is not known whether each Lindelöf T_1 space is a D-space.

A fundamental operation in the construction of topological spaces is resolution. It will be shown that the resolution of a D-space X at each $x \in X$ into a compact space Y_x is always a D-space.

The main result of this article is Theorem 2.8 where we establish a necessary and sufficient condition for a resolution of an arbitrary topological space X to be a D-space.

2**. Resolutions of *D*-spaces.** All spaces considered in this section are T_1. Suppose that X is a topological space and $\{Y_x : x \in X\}$ are topological spaces and, for each $x \in X$, $f_x : X - \{x\} \longrightarrow Y_x$ is a continuous mapping. For each open set $U_x \subseteq X$ such that $x \in U_x$ and each open set $W \subseteq Y_x$ we let

$$U_x \otimes W = (\{x\} \times W_x) \cup \bigcup \Big\{ \{x'\} \times Y_{x'} : x' \in U_x \cap f_x^{-1}(W) \Big\}.$$

The collection $\{U_x \otimes W : x \in X\}$ is a basis for some topology on $Z = \bigcup \Big\{ \{x\} \times Y_x : x \in X \Big\}$. We call Z the resolution of X at each $x \in X$ into Y_x by the mapping f_x.

LEMMA 2.1. *Let Z be a resolution of X and $\mathcal{V}$ be an open cover of Z. Let $x_0 \in X$ and suppose that Y_{x_0} is compact. Then there is an open set U_{x_0} such that $x_0 \in U_{x_0}$ and $U_{x_0} \otimes Y_{x_0}$ is covered by finitely many elements of $\mathcal{V}$.*

For a proof, see the fundamental theorem of resolutions [3].

THEOREM 2.2. *If X is a D-space and each Y_x is compact, then the resolution Z of X is a D-space.*

Received by the editors February 21, 1996.
AMS subject classification: Primary: 54D20, 54B99; secondary: 54D10, 54D30.
Key words and phrases: D-space, neighborhood assignment, resolution, boundary.

https://doi.org/10.4153/CMB-1997-047-5 Published online by Cambridge University Press

PROOF. Let $F\colon Z \longrightarrow \mathcal{T}_Z$ be a neighborhood assignment for Z. For each (x, y_x) in $\{x\} \times Y_x$ choose a basic neighborhood $U_x \otimes W_{y_x}$ of (x, y_x) such that $U_x \otimes W_{y_x} \subseteq F(x, y_x)$. By compactness

$$U_x^1 \otimes W_{y_x^1}, \ldots, U^{n_x} \otimes W_{y_x^{n_x}} \text{ cover } \{x\} \times Y_x.$$

Let $V_x = \bigcap_{i=1}^{n} U_x^i$. By an argument similar to the proof of Lemma 2.1 we have

$$V_x \otimes Y_x = \bigcup \big\{ \{x'\} \times Y_{x'} : x' \in V_x \big\} \subseteq \bigcup_{i=1}^{n_x} U_x^i \otimes W_{y_x^i} \subseteq \bigcup_{i=1}^{n_x} F(x, y_x^i).$$

Define $G\colon X \longrightarrow \mathcal{T}_X$ by $G(x) = V_x$ and let D_X be a closed discrete set such that $X = \bigcup G(D_X)$, that is $X = \bigcup \{ V_x : x \in D_X \}$.

Let $D_Z = \{ (x, y_x^i) : i = 1, \ldots, n_x, x \in D_X \}$. If (x, y) is in Z, then there is $x_0 \in D_X$ such that $x \in V_{x_0}$ and hence

$$(x, y) \in V_{x_0} \otimes Y_{x_0} \subseteq \bigcup_{i=1}^{n_{x_0}} F(x_0, y_{x_0}^i) \subseteq \bigcup F(D_Z)$$

that is, $Z = \bigcup F(D_Z)$ and it remains to prove that D_Z is a closed discrete set. For this purpose, we show that D_Z has no cluster points.

Let (x, y) be a point in Z. First we assume $(x, y) = (x, y_x^i) \in D_Z$. Then x belongs to the closed discrete set D_X. Let H be an open set in X such that $H \cap D_X = \{x\}$ and choose a neighborhood W of y_x^i so that $y_x^j \notin W$ for $i \neq j$. Then $\big(H \cap f_x^{-1}(W)\big) \cap D_X = \{x\} \cap f_x^{-1}(W) = \emptyset$. It follows that $\bigcup \big\{ \{x'\} \times Y_{x'} : x' \in H \cap f_x^{-1}(W) \big\} \cap D_Z = \emptyset$ and

$$(H \otimes W) \cap D_Z = (\{x\} \times W) \cap D_Z = \{ (x, y_x^i) \}.$$

That is $(x, y) = (x, y_x^i)$ is not a cluster point of D_Z.

If $(x, y) \in Z - D_Z$, then either $x \notin D_X$ and hence

$$(x, y) \in (X - D_X) \otimes Y_x = \bigcup \big\{ \{x'\} \times Y_{x'} : x' \in X - D_X \big\} \subseteq Z - D_Z,$$

or $x \in D_X$ and $y \neq y_x^i$ for $i = 1, \ldots, n_x$. In this case we choose a neighborhood $\tilde{W}$ of y so that $\tilde{W} \cap \{ y_x^1, \ldots, y_n^{n_x} \} = \emptyset$ and a neighborhood H of x so that $D_X \cap H = \{x\}$. Then $(H \otimes \tilde{W}) \cap D_Z = \emptyset$ and again (x, y) is not a cluster point of D_Z.

DEFINITION 2.3. Let Z be a resolution of X at each point $x \in X$ into Y_x by the mapping f_x. Let $\mathrm{Bd}(\{x\} \times Y_x)$ be the boundary of $\{x\} \times Y_x$ in Z, and let $\pi_x\colon \{x\} \times Y_x \longrightarrow Y_x$ be the projection map. The subset $\mathrm{Bd}(Y_x) = \pi_x\big(\mathrm{Bd}(\{x\} \times Y_x)\big)$ is called the *boundary* of Y_x.

Therefore, $y \in \mathrm{Bd}(Y_x)$ if and only if for every neighborhood U_x of x and for every neighborhood W_y of y, we have $U_x \cap f_x^{-1}(W_y) \neq \emptyset$ [3].

LEMMA 2.4. *Let Z be a resolution for X. Suppose that for each $x \in X$ either* $\mathrm{Bd}(Y_x) = \{b_x\}$ *or* $\mathrm{Bd}(Y_x) = \emptyset$. *Let Ω be the set of $x \in X$ for which the boundary is not empty. Suppose that for every neighborhood U_x of x and every neighborhood W_{b_x} of b_x, $\big(U_x \cap f_x^{-1}(W_{b_x})\big) \cup \{x\}$ is an open set. Then Ω is homeomorphic to a closed subspace of Z.*

https://doi.org/10.4153/CMB-1997-047-5 Published online by Cambridge University Press

PROOF. Let $\Omega_Z = \{(x, b_x) : x \in \Omega\}$. Let $(x, y_x) \in Z - \Omega_Z$. Either $x \notin \Omega$, hence $\mathrm{Bd}(Y_x) = \emptyset$, or $x \in \Omega$ and $y_x \neq b_x$. In either case there are neighborhoods U_x and W_{y_x} such that $U_x \cap f_x^{-1}(W_{y_x}) = \emptyset$. Thus $(x, y) \in U_x \otimes W_{y_x} = \{x\} \times W_{y_x} \subseteq Z - \Omega_Z$.

The restriction f of the projection $\pi: Z \longrightarrow X$ to Ω_Z is a continuous bijective map onto Ω with inverse

$$f^{-1}(x): \Omega \longrightarrow \Omega_Z, \quad x \longmapsto (x, b_x).$$

Let $G = (U_x \otimes W_y) \cap \Omega_Z$ be a basic open set. Then

$$f(G) = \begin{cases} (U_x \cap f_x^{-1}(W_y) \cap \Omega) \cup \{x\} & \text{if } b_x \in W_y \text{ (hence } x \in \Omega) \\ U_x \cap f_x^{-1}(W_y) \cap \Omega & \text{if } b_x \notin W_y \end{cases}$$

in either case $f(G)$ is open in Ω.

COROLLARY 2.5. *If for each resolved point x, $\mathrm{Bd}(Y_x) = \{b_x\}$ and for every neighborhoods U_x and W_{b_x} of x and b_x the set $\big(U_x \cap f_x^{-1}(W_{b_x})\big) \cup \{x\}$ is open, then X is homeomorphic to a closed subspace of Z. In particular, if Z is a D-space then X is a D-space.*

COROLLARY 2.6. *Assume that we resolve only isolated points of X. If Z is a D-space then X is a D-space.*

PROOF. Let I_X be the set of isolated points of X; then $\mathrm{Bd}(Y_x)$ is empty whenever x is in I_X. Thus $X - I_X = \Omega$ is a *D*-space. The result follows from the following proposition.

PROPOSITION 2.7. *If $X = X_1 \cup X_2$, with X_1 and X_2 D-spaces and X_1 closed, then X is a D-space.*

With less effort, one can show that the resolution of a Lindelöf space X into compact spaces is always Lindelöf; the proof is a simple application of Lemma 2.1. However, there is no analogue to the following result for Lindelöf spaces.

THEOREM 2.8. *The resolution Z of a space X at each point x into a space Y_x is a D-space if and only if $\bigcup\big\{\{x\} \times \mathrm{Bd}(Y_x)\big\}$ is a D-space and for each $x \in X$, Y_x is a D-space.*

PROOF. Let $\Omega = \{x \in X : \mathrm{Bd}(Y_x) \neq \emptyset\}$ and let $F: Z \longrightarrow \mathcal{T}_Z$ be a neighborhood assignment for Z. For each $x \in \Omega$ and $b_x \in \mathrm{Bd}(Y_x)$ we choose a basic neighborhood $U_x \otimes W_{b_x}$ such that $(x, b_x) \in U_x \otimes W_{b_x} \subseteq F(x, b_x)$. Let $A = \bigcup\big\{\{x\} \times \mathrm{Bd}(Y_x)\big\}$. Define $\Gamma_A: A \longrightarrow \mathcal{T}_A$ by $\Gamma_A(x, b_x) = (U_x \otimes W_{b_x}) \cap A$. Since A is a *D*-space, there is a closed discrete set $\bar{D}_A \subseteq A$ such that $A = \bigcup\{\Gamma_A(\bar{d}) : \bar{d} \in \bar{D}_A\}$. We note that $\bar{D}_A$ is indeed closed in Z since A is a closed subset. To simplify our notation, we let

$\theta = \bigcup\{f_d^{-1}(W_{b_d}) \cap U_d : (d, b_d) \in \bar{D}_A\}$,

$\tilde{W}_d = \bigcup\{W_{b_d} : (d, b_d) \in \bar{D}_A\}$, and

$\pi: Z \longrightarrow X$ be the projection map.

Thus $Z - \bigcup\{U_d \otimes W_{b_d} : (d, b_d) \in \bar{D}_A\}$ is equal to

$$\cup\big\{\{d\} \times (Y_d - \tilde{W}) : d \in \pi(\bar{D}_A) - \theta\big\} \cup \bigcup\big\{\{x'\} \times Y_{x'} : x' \in X - \big(\theta \cup \pi(\bar{D}_A)\big)\big\}.$$

https://doi.org/10.4153/CMB-1997-047-5 Published online by Cambridge University Press

For each $d \in \pi(\bar{D}_A) - \theta$ we let $B_d = \{d\} \times (Y_d - \tilde{W}_d)$. Define

$$\Gamma_d\colon B_d \longrightarrow T_{B_d}, \quad (d, y) \longmapsto B_d \cap F(d, y)$$

and let $S_d \subseteq B_d$ be a closed discrete set such that $B_d = \bigcup\{\Gamma_d(s) : s \in S_d\}$. Finally, we let $\tilde{S} = \bigcup\{S_d : d \in \pi(\bar{D}_A) - \theta\}$. Clearly,

$$B = \bigcup\{B_d : d \in \pi(\bar{D}_A) - \theta\} \subseteq \bigcup\{F(s) : s \in \tilde{S}\}.$$

Thus it remains to cover the subset of Z given by

$$T = \bigcup\Big\{\{x'\} \times Y_{x'} : x' \in X - \big(\theta \cup \pi(\bar{D}_A)\big)\Big\}.$$

For this purpose we note that

$$X - \big(\theta \cup \pi(\bar{D}_A)\big) \subseteq X - \Omega = \{x \in X : \mathrm{Bd}(Y_x) = \emptyset\}.$$

For each $a \in X - \big(\theta \cup \pi(\bar{D}_A)\big)$ we define

$$\Gamma_a\colon \{a\} \times Y_a \longrightarrow T_{\{a\}\times Y_a} \text{ by } (a, y) \longmapsto F(a, y) \cap \{a\} \times Y_a.$$

Let R_a be a closed discrete subset of $\{a\} \times Y_a$ such that

$$\{a\} \times Y_a = \bigcup\{\Gamma_a(r) : r \in R_a\}.$$

Put $\tilde{R} = \bigcup\Big\{R_a : a \in X - \big(\theta \cup \pi(\bar{D}_A)\big)\Big\}$. Clearly, $T \subset \cup\{F(r) : r \in \tilde{R}\}$.

Let $D = \bar{D}_A \cup \tilde{S} \cup \tilde{R}$. We shall prove that D has no cluster points, hence D is a closed discrete set with $Z = \bigcup F(D)$.

Let (x, y) be an arbitrary point in Z. We divide the proof into two cases.

CASE 1. $x \notin \theta \cup \pi(\bar{D}_A)$.

In this case $x \in X - \big(\theta \cup \pi(\bar{D}_A)\big)$ and hence $(x, y) \in T$. As we noted before, x must be in $X - \Omega$ and hence $\mathrm{Bd}(Y_x)$ is empty. Therefore there are open sets G_x and V_y containing x and y such that $G_x \cap f_x^{-1}(V_y) = \emptyset$. In other words, $(x, y) \in G_x \otimes V_y = \{x\} \times V_y$. Since each element of $\bar{D}_A \cup \tilde{S}$ is of the form (d, α) for some $d \in \Omega$, the open neighborhood $G_x \otimes V_y$ does not intersect $\bar{D}_A \cup \tilde{S}$. Thus $G_x \otimes V_y$ intersects at most $\tilde{R}$. Since $G_x \otimes V_y \subseteq \{x\} \times Y_x$, we have $(G_x \otimes V_y) \cap \tilde{R} = (G_x \otimes V_y) \cap R_x$. But R_x has no cluster points in Z, hence there is an open set H containing (x, y) such that $H \cap (G_x \otimes V_y) \cap R_x$ is at most $\{(x, y)\}$. It follows that (x, y) is not a cluster point of D.

CASE 2. $x \in \theta \cup \pi(\bar{D}_A)$.

If x is in θ then $x \in f_d^{-1}(W_{b_d}) \cap U_d$ for some $(d, b_d) \in \bar{D}_A$, thus

$$(x, y) \in U_d \otimes W_{b_d} = \{d\} \times W_{b_d} \cup \bigcup\{\{x'\} \times Y_{x'} : x' \in f_d^{-1}(W_{b_d}) \cap U_{b_d}\}.$$

From the definitions of B and T we obtain, $(U_d \otimes W_{b_d}) \cap (B \cup T) = \emptyset$. Since $\tilde{S} \subseteq B$ and $\tilde{R} \subseteq T$ we have $(U_d \otimes W_{b_d}) \cap (\tilde{S} \cup \tilde{R}) = \emptyset$. Therefore $(U_d \otimes W_{b_d}) \cap D = (U_d \otimes W_{b_d}) \cap \bar{D}_A$. Now the result follows from the fact that $\bar{D}_A$ is a closed discrete subset of Z.

https://doi.org/10.4153/CMB-1997-047-5 Published online by Cambridge University Press

Therefore we may assume $x = d \in \pi(\bar{D}_A) - \theta$. Let us divide the rest of the proof into two sub-cases.

(i) $y \in \mathrm{Bd}(Y_d)$: then $(x,y) \in A$ and $(x,y) = (d,y) \in U_{d'} \otimes W_{b_{d'}}$. But if $d \neq d'$ then $(d,y) \in \bigcup\big\{\{x'\} \times \underline{Y_{x'}} : x' \in U_{d'} \cap f_{d'}^{-1}(W_{b_{d'}})\big\}$ and hence $x \in U_{d'} \cap f_{d'}^{-1}(W_{b_{d'}}) \subseteq \theta$ which contradicts $x \in \pi(\bar{D}_A) - \theta$. Therefore $(x,y) = (d,y) \in U_d \otimes W_{b_d}$ and the rest of the argument is exactly the same as the one used in the previous paragraph.

(ii) $y \notin \mathrm{Bd}(Y_d)$: we choose a neighborhood of $(x,y) = (d,y)$ of the form $(G_d \otimes V_y) = \{d\} \times V_y \subseteq \{d\} \times Y_d$ *i.e.*, $G_d \cap f_d^{-1}(V_y) = \emptyset$.

The open neighborhood $(G_d \otimes V_y)$ does not intersect $\bar{D}_A$ for: if $(\alpha, \beta) \in \bar{D}_A \cap (G_d \otimes V_y)$, then $\alpha = d$ and $\beta = b_d$ for some $b_d \in \mathrm{Bd}(Y_d)$ and $G_d \cap f_d^{-1}(V_y)$ would be non empty since $b_d \in V_y$.

The neighborhood $(G_d \otimes V_y)$ does not intersect $\tilde{R}$ for: if $(d,\gamma) \in (G_d \otimes V_y) \cap \tilde{R}$, then $(d,\gamma) \in \tilde{R} \subseteq T$ would imply $d \in X - \big(\theta \cup \pi(\bar{D}_A)\big)$.

Therefore $(G_d \otimes V_y) \cap D = (G_d \otimes V_y) \cap \tilde{S} = (G_d \otimes V_y) \cap S_d$, and the final conclusion follows from the fact that S_d is a closed discrete subset of Z.

For the converse, we observe that both $\{x\} \times Y_x$ and $\bigcup\big\{\{x\} \times \mathrm{Bd}(Y_x)\big\}$ are closed in Z; hence they are D-spaces.

Resolutions of each point into an arbitrary space by constant mappings are important and they are the source of several famous spaces [3].

COROLLARY 2.9. *Let Z be a resolution for X by constant mappings. Then Z is a D-space if and only if X is a D-space and for each x, Y_x is a D-space.*

PROOF. Suppose that $f_x(y) = b_x$. If I_X is the set of isolated points of X, then by Lemma 2.4

$$X - I_X \simeq \{(x, b_x) : x \in X - I_X\} = \bigcup\{x\} \times \mathrm{Bd}(Y_x).$$

The result follows from Proposition 2.7 and Theorem 2.8.

COROLLARY 2.10. *Assume that we resolve only isolated points of X. Then Z is a D-space if and only if X is a D-space and for each x, Y_x is a D-space.*

PROOF. The resolution is independent of the mapping f_x since we resolve only isolated points.

EXAMPLE. The resolution of a Lindelöf space X at each point x into a Lindelöf space Y_x need not be Lindelöf even if $\bigcup\{x\} \times \mathrm{Bd}(Y_x)$ is Lindelöf.

Let $X = (0,1)$ regarded as a subspace of the Sorgenfrey line. We observe that a resolution Z is discrete if and only if each Y_x is discrete and has empty boundary. For each $x \in X$, choose an integer n_x large enough so that $(x - \frac{1}{n_x}, x + \frac{1}{n_x}) \subset X$. Let $Y_x = \{x + \frac{1}{n_x + i} : i = 0, 1, \ldots\}$.

Let $I_0 = (0, x - \frac{1}{n_x}) \cup [x + \frac{1}{n_x}, 1)$, and for $k \geq 1$ we let

$$I_k = \Big[x - \frac{1}{n_x + k - 1}, x - \frac{1}{n_x + k}\Big) \cup \Big[x + \frac{1}{n_x + k}, x + \frac{1}{n_x + k - 1}\Big).$$

https://doi.org/10.4153/CMB-1997-047-5 Published online by Cambridge University Press

Therefore we have a sequence $\{I_k\}$ of pair wise disjoint open sets with $\cup I_k = X - \{x\}$.

Define $f_x\colon X - \{x\} \longrightarrow Y_x$ by $f_x(I_k) = x + \frac{1}{n_x+k}$. Clearly f_x is continuous. Let $y = x + \frac{1}{n_x+k} \in Y_x$. Choose ϵ_k small enough so that $I_k \cap [x-\epsilon_k, x+\epsilon_k) = \emptyset$. Thus $f_x^{-1}(\{y\}) \cap [x-\epsilon_k, x+\epsilon_k) = \emptyset$ and $y \notin \mathrm{Bd}(Y_x)$. By the above remark Z is discrete.

We note that, if Z is Lindelöf then the set of $x \in X$ for which the boundary is empty need not be countable:

Let X be the set of ordinals $\leq \omega_1$, the first uncountable ordinal, and let Y_{ω_1} be a one point space. For each $x \neq \omega_1$ we let $f_x\colon X - x \longrightarrow X - x$ be the identity map. Clearly Z is Lindelöf and the boundary is not empty only if $x = \omega_1$.

We say that a subset Y of a space X is countably located in X if every subset F of Y that is closed in X is countable [2].

PROPOSITION 2.11. *Let Ω be the set of x in X for which the boundary is not empty, and assume that for each $x \in \Omega$ the space Y_x is compact. The resolution Z of X is Lindelöf if and only if*

1. *Each Y_x is Lindelöf*
2. *$\bigcup\{x\} \times \mathrm{Bd}(Y_x)$ is Lindelöf*
3. *$(X - \Omega)$ is countably located in X.*

PROOF. If Z is Lindelöf, then certainly $\bigcup\{x\} \times \mathrm{Bd}(Y_x)$ and $\{x\} \times Y_x$ are Lindelöf. Suppose that $(X - \Omega)$ is not countably located in X. There is an uncountable closed set F in X such that $\Omega \subset X - F$. For each $x \in \Omega$, let U_x be a neighborhood of x such that $U_x \subset X - F$. Let $\mathcal{U}_1 = \{U_x \otimes Y_x : x \in \Omega\}$. The set $T = (X - \bigcup U_x)$ is uncountable and

$$Z - \bigcup\{U_x \otimes Y_x : x \in \Omega\} = \bigcup\Big\{\{x'\} \times Y_{x'} : x' \in T\Big\}.$$

Since $T \subseteq (X - \Omega)$, we can choose for each $x \in T$ and each $y_x \in Y_x$ neighborhoods $V_x^{y_x}$ and W_{y_x} such that $V_x^{y_x} \cap f_x^{-1}(W_{y_x}) = \emptyset$. Let $\mathcal{U}_x = \{V_x^{y_x} \otimes W_{y_x}\}$. Clearly $\mathcal{U}_1 \cup \{\mathcal{U}_x : x \in T\}$ is an open cover of Z which has no countable subcover.

Let $\mathcal{U}$ be an cover for Z. For each $x \in \Omega$, there is a neighborhood U_x of x such that $U_x \otimes Y_x$ is covered by finitely many elements of $\mathcal{U}$. Since $\bigcup\{x\} \times \mathrm{Bd}(Y_x)$ is Lindelöf, the open cover $\{U_x \otimes Y_x : x \in \Omega\}$ has a countable subcover $\{U_{x_i} \otimes Y_{x_i} : i = 1, \ldots\}$. Therefore, $\bigcup\{x\} \times \mathrm{Bd}(Y_x)$ is covered by countably many elements of $\mathcal{U}$ and, at most, it remains to cover

$$\bigcup\Big\{\{x'\} \times Y_{x'} : x' \in X - \bigcup U_{x_i}\Big\}.$$

Since $X - \Omega$ is countably located, $(X - \bigcup U_{x_i})$ is countable. Thus, for each $x \in (X - \bigcup U_{x_i})$ we cover $\{x\} \times Y_x$ by countably many elements of $\mathcal{U}$. It follows that Z is Lindelöf.

REFERENCES

1. E. K. van Douwen and W. F. Pfeffer, *Some Properties of the Sorgenfrey Line and Related Spaces*. Pacific J. Math. (2) **81**(1979).

https://doi.org/10.4153/CMB-1997-047-5 Published online by Cambridge University Press

2. R. Engelking, *General Topology*. Polish Scientific Publishers, 1977.
3. S. Watson, *The Construction of Topological Spaces.* Recent Progress in General Topology, North-Holland, 1992.

Bowie State University
Department of Mathematics
Bowie, Maryland 20715
U.S.A.
e-mail: boudhraa@mcs.kent.edu