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MINIMAL RATES OF SUMMABILITY

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1. Introduction. During the early nineteenth century much effort was spent on attempts to find a "universal comparison test": i.e., a sequence in l^1 that dominates every other member of l^1. The nonexistence of such a series converging at a minimal rate was demonstrated by Abel, et al. [**1**; **4**; **7**; **9**, pp. 298-304]. In this paper, we consider analogous questions about the rate of convergence of the sequence Ax or the series $\sum |(Ax)_n|$, where x is a complex number sequence and A is a matrix summability transformation given by $(Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k$. We shall view A as a sequence-to-sequence mapping into either l^1 or c. Throughout the paper, c will denote the set of convergent sequences; m denotes the bounded sequences; and σ denotes those x's such that $\sum x_k$ is convergent. Also,

$$c_A = A^{-1}[c]; \quad l_A = A^{-1}[l^1]; \quad d_A = \text{domain of } A.$$

(See [**12**, p. 289].) The matrix A is said to be *regular* provided that $\lim_k x_k = L$ implies $\lim_n (Ax)_n = L$, and A is called an $l - l$ *matrix* in case $l^1 \subseteq l_A$ [**5**; **10**].

The principal objective of our study will be to answer the following questions. Can l_A or c_A contain a sequence that is summed by A at a "minimal rate"? Does there exist a sequence x such that $\sum |(Ax)_n|$ diverges at a "minimal rate"? We must first define a natural way of comparing rates of summability by A. Then we shall answer these questions negatively, thus providing summability analogues of the Abel-Dini theorems.

2. Definitions. In order to obtain a comparison test for convergence, it was highly desirable, if not absolutely necessary, to use a term-by-term ratio for comparing two series. Thus the relationships $x_k = o(y_k)$ and $x_k = O(y_k)$ were used to define the statement "$\sum x_k$ is dominated by $\sum y_k$". Under a summability transformation, the frequent occurrence of zero terms makes a term-by-term ratio inapplicable in many cases. Therefore, we shall use ratios of the following three quantities for our comparison of convergence/divergence rates. If x is in l^1, let Rx denote the "remainder sequence" given by

$$R_m x = \sum_{n>m} |x_n|.$$

For any sequence w, let Sw denote the "sequence of partial sums of moduli" given by

$$S_m w = \sum_{n\leq m} |w_n|.$$

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If $\lim_k t_k = L$, let $\rho_m t$ denote the "maximum remaining difference" given by

$$\rho_m t = \max_{n>m} |t_n - L|.$$

It is obvious that $R_m x$, $S_m w$, and $\rho_m t$ tend monotonically to 0, ∞, and 0, respectively, if and only if x is in l^1, w is in $\sim l^1$, and t is in c.

Definition. The statement A *sums* x *faster than* y means that x and y are in l_A and $R_m Ax = o(R_m Ay)$.

Definition. The statement $\sum |w_n|$ *diverges faster than* $\sum |t_n|$ means that w and t are in $\sim l^1$ and $S_m t = o(S_m w)$.

Definition. The statement z *converges faster than* t means that z and t are in c and $\rho_m z = o(\rho_m t)$.

It is of interest to compare these definitions with the familiar ones based on the term-by-term ratios. In the following propositions, we observe that each of the above properties is implied by the corresponding term-by-term comparison property.

PROPOSITION 1. *If x and y are in l_A and $(Ax)_n = o((Ay)_n)$, then $R_m Ax = o(R_m Ay)$; i.e., A sums x faster than y.*

Proof. If $|(Ax)_n| = \epsilon_n |(Ay)_n|$, where $\lim_n \epsilon_n = 0$, then

$$R_m Ax = \sum_{n>m} \epsilon_n |(Ay)_n| \leqq (\rho_m \epsilon)(R_m Ay),$$

and the conclusion follows immediately.

PROPOSITION 2. *If w and t are sequences in $\sim l^1$ such that $t_n = o(w_n)$, then $S_m t = o(S_m w)$.*

Proof. If $|t_n| = \epsilon_n |w_n|$, where $\lim_n \epsilon_n = 0$, then

$$S_m t / S_m w = \sum_{n \leqq m} \epsilon_n \left(|w_n| \Big/ \sum_{n \leqq m} |w_n| \right) = (\bar{N}_{|w|} \epsilon)_m,$$

where $\bar{N}_{|w|}$ is the Nörlund-type mean determined by the sequence $|w|$. (See [**8**, p. 57] and [**11**, pp. 45–46].) Since $|w|$ is not in l^1, $\bar{N}_{|w|}$ is regular, whence $\lim_m (\bar{N}_{|w|}\, \epsilon)_m = 0$.

PROPOSITION 3. *If $\lim_n z_n = \zeta$, $\lim_n t_n = \tau$, and $z_n - \zeta = o(t_n - \tau)$, then $\rho_m z = o(\rho_m t)$; i.e., z converges faster than t.*

The proof is similar to that of Proposition 1. The author is indebted to Professor David Borwein for raising the question that led to this result.

3. Theorems for $l - l$ summability. The first of the main results deals with the possibility of l_A containing a sequence that is summed at a minimal rate. In answering this question negatively for a class of matrices that includes the $l - l$ matrices, we shall prove even more: viz., l_A contains sequences that are summed at arbitrarily slow rates. If l_A were mapped *onto* l^1 by A, then this assertion would follow trivially from Proposition 1. But in general, the range of A does not include l^1, so we must employ a sliding humps argument to construct the sequences that are summed at the desired rate. The sliding humps will be constructed not in A, as is usually done, but in the matrix AX whose n, k-th entry is $a_{nk}x_k$.

THEOREM 1. *Let A be a summability matrix such that $d_A \neq l_A$ and whose column sequences are in l^1. If t is in l^1, then l_A contains a sequence y satisfying $R_m t = o(R_m Ay)$.*

Proof. Let x be a sequence in $d_A \sim l_A$, and let AX be the matrix given by $AX_{nk} = a_{nk}x_k$. We begin the sliding humps construction by choosing $\nu(0) = \kappa(0) = -1$. After $\nu(j)$ and $\kappa(j)$ have been chosen for $j \leqq 2m$, the observation that each column of AX is in l^1 while the sequence of row sums is in $\sigma \sim l^1$ allows us to choose $\nu(2m + 1) > \nu(2m)$ so that

$$(1) \qquad \sum_{n=1+\nu(2m)}^{\nu(2m+1)} \left| \sum_{k=1+\kappa(2m)}^{\infty} a_{nk}x_k \right| > 1.$$

Then choose $\kappa(2m + 1) > \kappa(2m)$ so that

$$(2) \qquad \sum_{n=1+\nu(2m)}^{\nu(2m+1)} \left| \sum_{k=1+\kappa(2m)}^{\kappa(2m+1)} a_{nk}x_k \right| > 1.$$

Next choose $\nu(2m + 2) > \nu(2m + 1)$ satisfying

$$(3) \qquad \sum_{n>\nu(2m+2)} \sum_{k=0}^{\kappa(2m+1)} |a_{nk}x_k| < \epsilon_{m+1},$$

where ϵ is defined recursively by $\epsilon_0 \equiv R_0 t$ and $\epsilon_{m+1} \equiv \min \{\epsilon_m/2, R_m t/2\}$. (We assume that $R_m t$ never vanishes; for if $R_m t = 0$, then $t_{m+k} \equiv 0$, and the conclusion is trivial.) Note that ϵ is in l^1 and satisfies

$$(4) \qquad R_m\epsilon \leqq R_m t \quad \text{and} \quad R_m \epsilon \leqq \epsilon_m.$$

Finally, since each row of AX is in σ, we can select $\kappa(2m + 2) > \kappa(2m + 1)$ so that if $j > i > \kappa(2m + 2)$, then

$$(5) \qquad \sum_{n=0}^{\nu(2m+2)} \left| \sum_{k=i}^{j} a_{nk}x_k \right| < \epsilon_{m+1}.$$

Let us introduce two notational abbreviations:

$$(6) \qquad H_m = \sum_{n=1+\nu(2m)}^{\nu(2m+2)} \left| \sum_{k=1+\kappa(2m)}^{\kappa(2m+1)} a_{nk}x_k \right| ,$$

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and

(7) $\alpha_m = H_m^{-1} \sum_{n=1+\nu(2m)}^{\nu(2m+2)} |\tau_n|$, where τ is in l^1 and

(8) $R_m t = o(R_{\nu(2m)}\tau)$.

From (2) we see that $H_m > 1$, so α is bounded.

The sequence y, whose existence is asserted in the theorem, can now be defined by

$$(9) \quad y_k = \begin{cases} x_k\alpha_m, & \text{if } \kappa(2m) < k \leqq \kappa(2m+1), \\ 0, & \text{if } \kappa(2m+1) < k \leqq \kappa(2m+2), \end{cases} \quad (m \geqq 0).$$

Thus, for each m,

$$\sum_{n=1+\nu(2m)}^{\nu(2m+2)} |(Ay)_n| = \sum_{n=1+\nu(2m)}^{\nu(2m+2)} \left| \sum_{j=0}^{\infty} \alpha_j \sum_{k=1+\kappa(2j)}^{\kappa(2j+1)} a_{nk}x_k \right|,$$

so

$$(10) \quad \left| \sum_{n=1+\nu(2m)}^{\nu(2m+2)} |(Ay)_n| - \alpha_m H_m \right|$$
$$= \sum_{n=1+\nu(2m)}^{\nu(2m+2)} \left\{ \left| \sum_{j=0}^{\infty} \alpha_j \sum_{k=1+\kappa(2j)}^{\kappa(2j+1)} a_{nk}x_k \right| - \alpha_m \left| \sum_{k=1+\kappa(2m)}^{\kappa(2m+1)} a_{nk}x_k \right| \right\}$$
$$\leqq \sum_{n=1+\nu(2m)}^{\nu(2m+2)} \left\{ \sum_{j<m} \alpha_j \left| \sum_{k=1+\kappa(2j)}^{\kappa(2j+1)} a_{nk}x_k \right| + \sum_{j>m} \alpha_j \left| \sum_{k=1+\kappa(2j)}^{\kappa(2j+1)} a_{nk}x_k \right| \right\}$$
$$\leqq \left(\max_{j<m} \alpha_j \right) \sum_{n=1+\nu(2m)}^{\nu(2m+2)} \sum_{k=0}^{\kappa(2m-1)} |a_{nk}x_k| + \sum_{j>m} \alpha_j \epsilon_j$$
$$\leqq \left(\max_{j<m} \alpha_j \right) \epsilon_m + \sum_{j>m} \alpha_j \epsilon_j$$
$$\leqq (\sup \alpha_j) R_{m-1}\epsilon_m$$
$$= O(\epsilon_m).$$

Now use (7) to replace $\alpha_m H_m$ in the left-hand member of (10), which gives

$$(11) \quad \sum_{n=1+\nu(2m)}^{\nu(2m+2)} \{ |\tau_n| - |(Ay)_n| \} = O(\epsilon_m).$$

Since τ and ϵ are in l^1, (11) guarantees that y is in l_A. Also, by summing (11) from m to ∞, we get

(12) $R_{\nu(2m)}Ay = R_{\nu(2m)}\tau + O(R_m\epsilon)$.

We now multiply through (12) by $(R_m t)^{-1}$, then use (4) and (8) to conclude that

(13) $R_m t = o(R_{\nu(2m)}Ay)$.

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Since $R_{\nu(2m)}Ay \leqq R_mAy$, (13) implies that $R_mt = o(R_mAy)$, and the proof is complete.

In the theorem just proved, t can be replaced by Ax, where x is an arbitrary sequence in l_A. Thus we conclude that l_A cannot contain a sequence that is summed by A at a minimal rate. This result is stated precisely as follows.

THEOREM 2. *Let A be a summability matrix such that $d_A \neq l_A$ and whose columns are in l^1. If x is in l_A, then there is a sequence y such that A sums x faster than y.*

Since A is an $l - l$ matrix if and only if $\sup_k \sum_{n=1}^{\infty} |a_{nk}| < \infty$, Theorems 1 and 2 apply to $l - l$ matrices that satisfy $d_A \neq l_A$. The hypothesis that $d_A \neq l_A$ was needed to achieve equation (1) in the above construction. But more to the point is the fact that the conclusion is not valid without this hypothesis, as is shown by the following example.

Let A be the matrix given by $a_{nk} = 2^{-n}$. Then $(Ax)_n = 2^{-n} \sum_{k=0}^{\infty} x_k$, for every x in σ, the set of (conditionally) convergent series. Clearly $l_A = d_A = \sigma$. Also, for every x in σ, $R_mAx = 2^{-m} |\sum_{k=0}^{\infty} x_k|$. Hence, there is no sequence satisfying $\sum_{k=0}^{\infty} x_k \neq 0$ that is summed by A faster than any other sequence.

Next we develop a summability analogue of the Abel theorem that asserts that no series diverges at a minimal rate. As above, we first prove a stronger result which shows that for many matrices A, there exists a sequence y such that $\sum |(Ay)_n|$ diverges at an arbitrarily slow rate.

THEOREM 3. *Let A be a summability matrix such that $d_A \neq l_A$ and whose columns are in l^1. If t is a sequence in $\sim l^1$, then $d_A \sim l_A$ contains a sequence y such that $\sum |t_n|$ diverges faster than $\sum |(Ay)_n|$.*

Proof. The sliding humps are constructed exactly as in the proof of Theorem 1 until equation (7) is reached. This time we use a sequence τ that is *not* in l^1 and satisfies $S_{\nu(2m)}\tau = o(S_mt)$. It is clear that we can choose τ so that again α is bounded. Then the definition (9) of y and the calculations (10) and (11) are the same as before. Since τ is in $\sim l^1$, it follows that y is in $d_A \sim l_A$, and

(14) $\quad S_{\nu(2m)}Ay = S_{\nu(2m)}\tau + O(S_m\epsilon).$

From the choice of τ and the fact the ϵ is in l^1, (14) implies that $S_{\nu(2m)}Ay = o(S_mt)$, whence $S_mAy = o(S_mt)$. Hence, $\sum |t_n|$ diverges faster than $\sum |(Ay)_n|$.

The necessity of the hypothesis that $d_A \neq l_A$ is again shown by the example in which $a_{nk} = 2^{-n}$. Also, this theorem applies to $l - l$ matrices, a priori. We can replace t with Ax, where x is an arbitrary sequence in $d_A \sim l_A$, and thus conclude that for any such x, $\sum |(Ax)_n|$ cannot diverge at a minimal rate. This is stated precisely in the next result.

THEOREM 4. *Let A be a summability matrix such that $d_A \neq l_A$ and whose columns are in l^1. If x is in $d_A \sim l_A$, then there is a sequence y in $d_A \sim l_A$ such that $\sum |(Ax)_n|$ diverges faster than $\sum |(Ay)_n|$.*

Since the assumption that $d_A \neq l_A$ played a crucial role in the above proofs, it would be very desirable to have an explicit characterization of this property that is given by row/column conditions on A. Unfortunately, such a result seems to be deceptively elusive. The following proposition gives a necessary condition on A in order that $d_A \neq l_A$. But we shall see that it is not a sufficient condition.

PROPOSITION 4. *If A is a summability matrix whose columns are in l^1 and $d_A \neq l_A$, then the rows of A form an infinite dimensional family of sequences.*

Proof. Each row of A belongs to a maximal family of linearly dependent rows. Let the row $\{a_{n(1),k}\}$ belong to the family $\{t_j^{(1)} a_{n(1),k}: j \in J_1\}$, which consists of all the rows that are multiples of $\{a_{n(1),k}\}$. Since every column of A is in l^1, for each k we have, $\sum_{j\in J} |t_j^{(1)} a_{n(1),k}| < \infty$. Hence, $t^{(1)}$ is in l^1. Suppose there are only m such maximal linearly dependent families, say

$$\{t_j^{(i)}a_{n(i),k}: j \in J_i\}, i = 1, \ldots, m.$$

Then for each n there is an i in $\{1, \ldots, m\}$ and a j in J_i such that for each x in d_A,

$$(Ax)_n = \sum_{k=0}^{\infty} t_j^{(i)}a_{n(i),k}x_k = t_j^{(i)}(Ax)_{n(i)}.$$

Therefore,

$$\sum_{n=0}^{\infty} |(Ax)_n| = \sum_{i=1}^{m} \sum_{j\in J_i} |t_j^{(i)}||(Ax)_{n(i)}| < \infty,$$

because each $t^{(i)}$ is in l^1. Hence, x is in l_A, and we conclude that $d_A = l_A$.

In order to see that the condition given in Proposition 4 is not sufficient to imply that $d_A \neq l_A$, consider the following example:

$$a_{nk} = \begin{cases} 2^{-n}, & \text{if } k \geqq n, \\ 0, & \text{if } k < n. \end{cases}$$

The rows of A are obviously linearly independent. But

$$(Ax)_n = 2^{-n+1} \sum_{k=n}^{\infty} x_k,$$

so $d_A = l_A = \sigma$.

4. Theorems for ordinary summability. In this section we shall prove analogues of Theorems 1 and 2 in the setting of ordinary summability. The question that is answered in the negative is the following: if A is a regular matrix, can c_A contain a sequence such that Ax converges at a minimal rate? As above, we shall show that the summability field contains sequences that converge arbitrarily slowly.

THEOREM 5. *If A is a regular matrix and t is a nonincreasing null sequence (of real numbers), then there is a null sequence y such that t converges faster than Ay.*

Proof. Using the well-known Silverman-Toeplitz conditions for the regularity of A, we construct sliding humps in A by choosing increasing index sequences ν and κ satisfying

$$\sum_{k=0}^{\kappa(m-1)} |a_{\nu(m),k}| < (t_m/t_0)^{1/2}4^{-1},$$

$$\sum_{k=1+\kappa(m)}^{\infty} |a_{\nu(m),k}| < \sqrt{t_m}/4, \quad \text{and}$$

$$\left|\sum_{k=1+\kappa(m-1)}^{\kappa(m)} a_{\nu(m),k}\right| > 3/4.$$

Then define $y_k = \sqrt{t_m}$, if $\kappa(m-1) < k \leqq \kappa(m)$. Since A is regular, we have $\lim_n (Ay)_n = 0$. Also, for each m,

$$|(Ay)_{\nu(m)}| \geqq - \sum_{k=0}^{\kappa(m-1)} |a_{\nu(m),k}|\sqrt{t_0} + \left|\sum_{k=1+\kappa(m-1)}^{\kappa(m)} a_{\nu(m),k}\sqrt{t_m}\right| - \sqrt{t_m}\sum_{k>\kappa(m)} |a_{\nu(m),k}| > \sqrt{t_m}/4.$$

Therefore,

$$\begin{aligned}\rho_m t/\rho_m Ay &= t_m/\rho_m Ay \\ &\leqq t_m/|(Ay)_{\nu(m)}| \\ &= 4\sqrt{t_m}.\end{aligned}$$

Hence, $\rho_m t = o(\rho_m Ay)$.

For any sequence x in c_A, we can substitute $\rho_m Ax$ for t in the preceding theorem, which yields the following assertion.

THEOREM 6. *If A is regular and x is in c_A, then there is a sequence y in c_A such that Ax converges faster than Ay.*

In Theorems 5 and 6 we do not need the full regularity hypothesis. Although we used the assumption that A preserves zero limits, the choice of ν would require only $\limsup_n |\sum_{k=0}^{\infty} a_{nk}| > 0$, rather than the Silverman-Toeplitz condition: $\lim_n \sum_{k=0}^{\infty} a_{nk} = 1$. Therefore, Theorems 5 and 6 could be stated with the hypothesis "A is regular" replaced by "A is coregular [**12**, p. 93] and preserves zero limits." It is easy to see that the conclusion is not valid if one assumes only that A is conservative: e.g., suppose $a_{nk} = 0$ if $k > 1$.

In comparing Theorems 5 and 6 with those of § 3, we see that in the present case it is not necessary to assume that $d_A \neq c_A$. This is guaranteed by the regularity of A, since $m \subseteq d_A$, but $m \nsubseteq c_A$.

We conclude the remarks on ordinary summability by noting that no analogue is given for Theorems 3 and 4. This omission seems inevitable due to the fact that there is no convenient device such as R_m, S_m, or ρ_m for measuring the rate of nonconvergence of a sequence.

5. Questions on the relation of rate of summability to inclusion. It is natural to seek some relationship between the strength of the matrix A and the rate at which A sums sequences. For example, the statement "A is stronger than B" (i.e., $l_B \subseteq l_A$) seems to be related to the statement "A sums x faster than B". The fact is that there is no simple relationship of this type. This is illustrated most simply by taking one of the two matrices, say B, to be the identity matrix, then constructing A such that $l_A = l_I = l^1$ and either (i) A "slows" the rate of summability of x in l^1, or (ii) A "speeds up" the rate of summability.

In case (i), we want $R_m x = o(R_m Ax)$. Let A' be the matrix given by

$$a_{nk}' = \begin{cases} 1, & \text{if } n = k = 0 \\ 1, & \text{if } n = 2^k,\ k = 0, 1, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

This gives

$$\sum_{n=0}^{2^m} |(A'x)_n| = \sum_{k=0}^{m} |x_k|;$$

so if $x_k = 2^{-k}$, then $R_m x = 2^{-m}$ and $R_m A'x = 2^{-\mu}$, where $2^\mu \leqq m < 2^{\mu+1}$. Therefore, x is summed more slowly by A' than by I, yet $l_{A'} = l_I = l^1$. Similarly, if $x_k = k$, then $\sum_n |(Ax)_n|$ diverges slower than $\sum_k |x_k|$. The idea that is employed here is to construct A' by inserting rows that are identically zero between the rows of I. In this way, we could construct an A that sums x as slowly as we wished and still have $l_A = l^1$.

In case (ii), we want $R_m Ax = o(R_m x)$. Consider the matrix A'' given by

$$a_{nk}'' = \begin{cases} 2, & \text{if } n = k, \\ -1, & \text{if } n = k + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $l_A = l^1$ by [**6**, Th. 6]. If $x_k = (k + 1)/2^{k+1}$, then $(A''x)_n = 2^{-n}$ when $n > 0$; so $R_m A''x = o(R_m x)$.

The question of preserving the rate of (ordinary) convergence under a regular matrix transformation was investigated by D. F. Dawson, in [**2**; **3**] where the term-by-term ratio was used to compare rates. (Cf. [**9**, p. 279].) By virtue of Proposition 3, this provides an example in which A is equivalent to I, yet $\rho_m Ax = o(\rho_m x)$. This is also true for the example A'' in the preceding paragraph. In addition, we can modify the previous example A' to give a regular matrix A^* that is equivalent to x, and A^* slows the rate convergence of some sequence. This is achieved by inserting repeated rows between the rows of I:

$$a_{nk}^* = \begin{cases} 1, & \text{if } n = k = 0, \\ 1, & \text{if } 2^k \leqq n < 2^{k+1} (k = 0, 1, \ldots), \\ 0, & \text{otherwise.} \end{cases}$$

Then $(A^*x)_n = x_m$ whenever $2^m \leqq n < 2^{m+1}$. Therefore, A^* is equivalent to I, and it is easy to find an x such that $\rho_m x = o(\rho_m Ax)$.

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