BULL. AUSTRAL. MATH. SOC. 20D10, 20F17
VOL. 53 (1996) [209–212]

A NOTE ON HALL CLOSURE OF METANILPOTENT FITTING CLASSES

MARTIN MENTH

Metanilpotent Lockett classes are Hall closed. There is an example of a supersoluble, not Hall closed Fitting class.

Let $\mathfrak{F}$ be a Fitting class, that is a class of finite soluble groups that is closed with respect to forming normal subgroups and normal products. If π is a set of primes, $\mathfrak{F}$ is said to be *Hall-π-closed* provided that whenever H is a Hall-π-subgroup of G and $G \in \mathfrak{F}$, then $H \in \mathfrak{F}$. The class $\mathfrak{F}$ is said to be *Hall-closed* if it is Hall-π-closed for all sets of primes. If G is a finite soluble group, we denote by $G_{\mathfrak{F}}$ the join of all normal $\mathfrak{F}$-subgroups of G. A Fitting class $\mathfrak{F}$ satisfying $(G \times H)_{\mathfrak{F}} = G_{\mathfrak{F}} \times H_{\mathfrak{F}}$ for all finite soluble groups G and H is called a *Lockett class*. For a Fitting class $\mathfrak{F}$ there is a uniquely determined smallest Lockett class $\mathfrak{F}^* \supseteq \mathfrak{F}$. The intersection of all Fitting classes $\mathfrak{X}$ with $\mathfrak{X}^* = \mathfrak{F}^*$ is denoted by $\mathfrak{F}_*$. A Fitting class $\mathfrak{F}$ is said to be a *Fischer class* if for every normal subgroup K of $G \in \mathfrak{F}$ and nilpotent subgroup H/K of G/K we have $H \in \mathfrak{F}$.

Imposing an additional condition for nilpotent length of a Fitting class $\mathfrak{F}$ sometimes yields further closure properties of $\mathfrak{F}$. For instance, every metanilpotent Fischer class is subgroup closed [**5**, 3.7]. By weakening the hypothesis 'Fischer class' to 'Lockett class' we cannot expect subgroup closure, but we have:

THEOREM. *Every metanilpotent Lockett class is Hall-closed.*

PROOF: Let $\mathfrak{F}$ be a metanilpotent Fitting class that is not Hall-closed. There exists a set π of primes and a group $G \in \mathfrak{F}$ such that G has a Hall-π-subgroup $H \notin \mathfrak{F}$. Set $F = Fit(G)$, and let $p_1, \dots, p_n$ be the prime divisors of $|F|$. Then F is the direct product of its Sylow-p_i-subgroups P_i, $1 \leqslant i \leqslant n$, and G/F is nilpotent. Having numbered the primes suitably, there is an integer k $(1 \leqslant k < n)$ such that $p_1, \dots, p_k$ are the p_i in π. Then $P = P_1 \cdot \ldots \cdot P_k = H \cap F$. The quotient H/P is isomorphic to a subgroup of G/F and therefore nilpotent. So H/P is a subnormal product of cyclic groups $\langle x_i P \rangle$. At least one of the subgroups $\langle P, x_i \rangle$ of H is not an $\mathfrak{F}$-group, let us say $H^* = \langle P, x \rangle$. Now we may replace G by $G^* = \langle F, x \rangle$, because $G^* \in \mathfrak{F}$ and

Received 27th April, 1995

H^* is a Hall-π-subgroup of G^*. Set $Q = P_{k+1} \cdot \ldots \cdot P_n$. We define a direct product $D = \langle P, x_1 \rangle \times \langle Q, x_2 \rangle$, where $\langle P, x_1 \rangle$ is a copy of H^* and $\langle Q, x_2 \rangle$ is a copy of $Q\langle x \rangle$. Then $K = PQ\langle x_1 x_2 \rangle$ is a normal subgroup of D and isomorphic to G^*, so $K \leqslant D_{\mathfrak{F}}$. On the other hand, $x_1 \notin \langle P, x_1 \rangle_{\mathfrak{F}}$. Hence $D_{\mathfrak{F}} \neq \langle P, x_1 \rangle_{\mathfrak{F}} \times \langle Q, x_2 \rangle_{\mathfrak{F}}$, and $\mathfrak{F}$ is not a Lockett class. ∎

Brison [1, 6.4] proved that for an arbitrary Fitting class $\mathfrak{F}$, the class $\mathfrak{F}_*$ is Hall closed if and only if $\mathfrak{F}^*$ is Hall closed. So we get:

Corollary 1. *$\mathfrak{F}^*$ and $\mathfrak{F}_*$ are Hall closed for every metanilpotent Fitting class $\mathfrak{F}$.*

For every supersoluble Fitting class $\mathfrak{F}$, the Lockett closure $\mathfrak{F}^*$ is supersoluble [4, 1.1]. Therefore we obtain from the preceeding theorem:

Corollary 2. *Every supersoluble Fitting class is a subclass of a supersoluble Hall-closed Fitting class.*

The following example shows that not all supersoluble Fitting classes are Hall closed. We need some notation.

Let p be a prime, $p \equiv 1 \bmod 3$ and n a primitive third root of unity in $GF(p)$. Set $T_p = \langle a, b \mid a^p = b^p = [a,b,a,a] = [a,b,a,b] = [a,b,b,b] = 1 \rangle$ and $U_p = \langle T, s \mid s^3 = 1,\ a^s = a^n,\ b^s = b^n \rangle$. The Fitting class $\mathfrak{U}_p$ generated by U_p can be described in the following way: Let $\mathfrak{U}_p^0$ be the class of all finite groups $G = XY$, where $X = O_p(G)$ and $Y \in Syl_3(G)$, such that

(i) X is a central product of copies P_i of T_p;

(ii) $Y/C_Y(P_i) \cong C_3$ and $P_iY/C_Y(P_i) \cong U_p$ for all indices i.

Then $\mathfrak{U}_p$ is the class of all groups $G \in \mathfrak{S}_p\mathfrak{S}_3$ such that $O^p(G) \in \mathfrak{U}_p^0$. So $\mathfrak{U}_p$ is a Fitting class of 'Dark type', and a supersoluble Lockett class (see [3] and [4]).

Example 1. Set $A = T_7$ and $B = T_{13}$, and construct a semidirect product $K = (A \times B) \rtimes \langle x \rangle$ in the following way: x raises all elements of A modulo A' to the power of 2 and all elements of B modulo B' to the power of 3. That means $\langle A, x \rangle \cong U_7$ and $\langle B, x \rangle \cong U_{13}$. Then the Fitting class $\mathfrak{K}$ generated by K has the following properties:

(a) $\mathfrak{K}$ is a supersoluble Fitting class;

(b) the Hall-$\{7,3\}$-subgroups and Hall-$\{13,3\}$-subgroups of K are not in $\mathfrak{K}$.

Proof: The class $\mathfrak{K}$ can be described in the same way as $\mathfrak{U}_p$, replacing T by $P = A \times B$ and U by K. Denote $X = O_{\{7,13\}}(G)$ and define $\mathfrak{K}^0$ in the same way as $\mathfrak{U}_p^0$. Then $\mathfrak{K}$ is the class of all groups $G \in \mathfrak{N}_{\{7,13\}}\mathfrak{S}_3$ such that $O^{\{7,13\}}(G) \in \mathfrak{K}^0$. To prove this one can use the same steps as in [3], replacing T by P. In particular $\mathfrak{K}$ is supersoluble. If G is a $\{7,13\}$-perfect group in $\mathfrak{K}$ then $O_{\{7,13\}}(G)$ is a central product of m copies of A and of the same number of copies of B. The subgroup $A\langle x \rangle$ of K is a Hall-$\{7,3\}$-subgroup of K, and $B\langle x\rangle$ is a Hall-$\{13,3\}$-subgroup of K. Both of them are $\{7,13\}$-perfect, so they are not $\mathfrak{K}$-groups. $\square$

Since Fischer classes are Lockett classes, an example of Brison (proof of [**1**, Proposition 3.3.a]) shows that the theorem cannot be generalized by dropping the metanilpotency hypothesis. For every set π of primes, the class $\mathfrak{H}_\pi = (G \in \mathfrak{S} \mid O_\pi(G) \leqslant Z_\infty(G))$ is a Fischer class [**2**, IX.3.7b]. The intersection $\mathfrak{H}_3 \cap \mathfrak{N}^3$ is a Fischer class of nilpotent length three that is not $\{2,3\}$-Hall closed.

For Fitting classes $\mathfrak{F}$ and $\mathfrak{G}$ the product

$$\mathfrak{F}\mathfrak{G} = (G \in \mathfrak{S} : \text{ there exists } N \trianglelefteq G \text{ such that } N \in \mathfrak{F} \text{ and } G/N \in \mathfrak{G})$$

is again a Fitting class (note that this is different from the Fitting class product $\mathfrak{F}\diamond\mathfrak{G}$). If ρ is a set of primes and $\mathfrak{F}_r$ a Fitting class for every $r \in \rho$, then the class $\bigcap_{r\in\rho} \mathfrak{F}(r)\mathfrak{S}_r\mathfrak{S}_{r'}$ is called *locally defined* by the family $\{\mathfrak{F}(r) \mid r \in \rho\}$. By [**2**, IX.3.7c], every locally defined Fitting class is a Fischer class. It is easy to see that for Hall closed Fitting classes $\mathfrak{F}$ and $\mathfrak{G}$ the product $\mathfrak{F}\mathfrak{G}$ is also Hall closed, and that a Fitting class which is locally defined by a family of Hall closed Fitting classes is again Hall closed. But in general locally defined Fitting classes are not Hall closed:

EXAMPLE 2. Let $\mathfrak{K}$ be the supersoluble Fitting class of Example 1. Then $\mathfrak{K}\mathfrak{S}_3\mathfrak{S}_{3'}$ is Hall closed, while $\mathfrak{K}\mathfrak{S}_7\mathfrak{S}_{7'}$ and $\mathfrak{K}\mathfrak{S}_7\mathfrak{S}_{7'} \cap \mathfrak{N}^3$ are not Hall closed.

PROOF: Let G be a group in $\mathfrak{K}\mathfrak{S}_3\mathfrak{S}_{3'}$. By quotient group closure of $\mathfrak{S}_3\mathfrak{S}_{3'}$ we see $G/G_{\mathfrak{K}} \in \mathfrak{S}_3\mathfrak{S}_{3'}$. If H is a Hall-π-subgroup of G, then $H \cap G_{\mathfrak{K}}$ is a Hall-π-subgroup of $G_{\mathfrak{K}}$, and therefore $H \cap G_{\mathfrak{K}} \in \mathfrak{K}\mathfrak{S}_3$. Moreover $H/(H \cap G_{\mathfrak{K}}) \cong HG_{\mathfrak{K}}/G_{\mathfrak{K}} \leqslant G/G_{\mathfrak{K}} \in \mathfrak{S}_3\mathfrak{S}_{3'}$. This shows $H/(H \cap G_{\mathfrak{K}}) \in \mathfrak{S}_3\mathfrak{S}_{3'}$ and finally $H \in \mathfrak{K}\mathfrak{S}_3\mathfrak{S}_{3'}$. Set $G = K \wr \langle s\rangle$, where $\langle s\rangle$ is cyclic of order 7, and $N = K_1 \times \ldots \times K_7$ is the base group of G. Set $P_i = A_i \times B_i = O_{\{7,13\}}(K_i)$ and let $\langle t_i\rangle$ be a Sylow-3-subgroup of K_i. Then $U_i = \langle A_i, t_i\rangle$ is a Hall-$\{3,7\}$-subgroup of K_i, and $U = U_1 \times \ldots \times U_7$ is a Hall-$\{3,7\}$-subgroup of N. Now $H = \langle U, s\rangle$ is a Hall-$\{3,7\}$-subgroup of G. Obviously $O_7(U) \leqslant H_{\mathfrak{K}}$. If $s \in H_{\mathfrak{K}}$, so $t_1t_2^{-1} \in H_{\mathfrak{K}}$, and $L = \langle A_1 \times A_2, t_1t_2^{-1}\rangle$ is a subnormal subgroup of $H_{\mathfrak{K}}$. Therefore $L \in \mathfrak{K}$. On the other hand L is $\{7,13\}$-perfect, a contradiction. This implies $H_{\mathfrak{K}} = O_7(U)$. Finally we have $H/H_{\mathfrak{K}} \cong C_3 \wr C_7 \notin \mathfrak{S}_7\mathfrak{S}_{7'}$. $\square$

REFERENCES

[1] O. Brison, 'Hall operators for Fitting classes', *Arch. Math. (Basel)* **33** (1979), 1–9.

[2] K. Doerk and T. Hawkes, *Finite soluble groups* (deGruyter, Berlin, New York, 1992).

[3] M. Menth, 'A family of Fitting classes of supersoluble groups', *Math. Proc. Cambridge Philos. Soc.* **118** (1995), 49–57.

[4] M. Menth, 'Examples of supersoluble Lockett sections', *Bull. Austral. Math. Soc.* **49** (1994), 325–332.

[5] K. Müller, 'Fittingklassen mit zusätzlichen Abschlußeigenschaften,', *Arch. Math. (Basel)* **50** (1988), 19–24.

Mathematisches Institut
Universität Würzburg
Am Hubland
D-97074 Würzburg
Germany