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RADON TRANSFORM ON AFFINE BUILDINGS OF RANK THREE

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Abstract

We define the Radon transform for functions on the set of chambers of affine, locally finite, rank three buildings. We investigate the problem of the inversion of this transform. Explicit inversion formulas are exhibited for functions which fulfill required summability conditions.

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0. Introduction

An affine building of rank three is a building **X** whose diagram is one of the following:

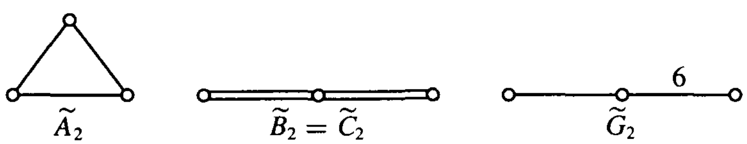

FIGURE 1.

We denote by $\mathscr{C}$ the collection of 2-simplices (or *chambers*) in **X** and by $\mathscr{A}$ the collection of apartments in **X**. Given a function f on $\mathscr{C}$ its Radon transform Rf is the function on $\mathscr{A}$ given by

$$Rf\,(A) = \sum_{c \in A} f\,(c) \quad \text{for each} \quad A \in \mathscr{A},$$

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provided that the series converges absolutely.

In analogy with the setting of $\mathbb{R}^n$ or of an affine building of rank two ([1,4]) we study the problem of recovering f from Rf or in other words of inverting R. It will be shown that an inversion formula can be obtained by applying to R an integral operator which is obtained by radializing, with respect to a measure m, defined on $\mathscr{A}$, a discrete inversion formula obtained case by case with geometric-combinatoric methods. The difficulties are related to the complicated geometry of buildings but two known facts simplify the matter. The first is that each apartment is a *retract* of the building, hence much work can be done by means of the combinatorics of the affine Coxeter group of the building. The second is related to the local geometry of affine buildings : for each simplex there is a spherical finite building to which we can *project* the whole affine building.

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1. Preliminaries

The basic notation is standard and we refer to the books [6, 9] and [11] for more details. We shall assume familiarity with elementary facts about Coxeter groups and Coxeter complexes (see also [8] and [7]).

In a *rank-three* affine building $\mathbf{X}$ a typical apartment arises from a tiling of the Euclidean plane. Each chamber can be identified with a triangle having angles π/k, π/l, π/m where $(k, l, m) = (3, 3, 3)$, $(2, 4, 4)$, $(2, 3, 6)$ in the respective cases $\widetilde{A}_2$, $\widetilde{B}_2$, $\widetilde{G}_2$. (Notice that other authors would call this kind of building 'rank-two'; by rank we mean the number of vertices of any chamber.) Each vertex of $\mathbf{X}$ is labelled with a type which is an integer in $I = \{0, 1, 2\}$, and each chamber has exactly one vertex of each type. A 1-simplex (or *edge*) σ has *cotype* i, (that is type $I \setminus \{i\}$), if the set of labels associated with the vertices of σ is $I \setminus \{i\}$). We always suppose $\mathbf{X}$ to be *thick* (that is each edge is contained in at least three chambers), in which case all edges of a given cotype i are contained in the same number of chambers $q_i + 1$.

Any two simplices σ, σ' in the building are contained in a common apartment. Furthermore if A and A' are apartments with a common chamber then there exists a unique isomorphism between A and A' fixing $A \cap A'$ pointwise, [6, p. 77]. Given two chambers c and c', a *gallery* of *extremes* c and c' and *length* n is a finite sequence $(c = c_0, c_1, \dots, c_n = c')$ where c_i and c_{i+1}, for $i = 0, \dots, n-1$, are *adjacent* (that is they have a common edge); if $c_i \neq c_{i+1}$ for all i, we say that the gallery is *simple*.

There always exists a non-unique shortest such gallery, called *minimal*; the *distance* between two chambers c and c', denoted by $\mathbf{d}(c, c')$, is the length of a minimal gallery connecting them. More generally the distance between a chamber c and a simplex σ in the building is the minimal length of a gallery $(c_0, c_1, \dots, c_n)$ with c_0 containing σ and $c_n = c$. A gallery which achieves this minimum is called a *stretched* gallery from σ to c.

Two chambers c and c' which have a common edge of cotype i, $i \in I$, are called i*-adjacent*, and denoted by $c \underset{i}{\sim} c'$. As two distinct adjacent chambers are i-adjacent for a unique i, a simple gallery $(c_0, c_1, \dots, c_n)$ has a well defined type $(i_0, \dots, i_{n-1})$ such that $c_j \underset{i_j}{\sim} c_{j+1}$ for $j = 0, \dots, n-1$.

The *diagram* of the building is the diagram of its (affine) Coxeter group W. Fixing an apartment A in the building, and a chamber $c \in A$, W is generated by the set S of reflections with respect to the edges of c (or *simple reflections*). The group W acts (on the left) on A by simplicial type-preserving automorphism (with c as a fundamental domain for this action) and its action is simply transitive on the set of chambers (indeed the stabilizer of c is the trivial subgroup). If for each subset $\mathbf{X}'$ of $\mathbf{X}$ we denote by $\mathscr{C}(\mathbf{X}')$ the set of chambers of $\mathbf{X}'$, then it follows that the set $\mathscr{C}(A)$ can be identified with W, via $wc \to w$. Moreover, if we label by s_i, $i = 0, 1, 2$, the reflection with respect to the edge of c of cotype i, two distinct elements $w, w' \in W$ correspond to i-adjacent chambers if and only if $w' = ws_i$. Let $l(w)$, or $l_S(w)$ if necessary, denote the length of w with respect to S. In view of the previous paragraphs, simple galleries starting at c are in 1-1 correspondence with S-words. Under this correspondence minimal galleries correspond to reduced words. Then it follows that the metric spaces $\mathscr{C}(A)$ and W are isometric, where the latter is endowed with the word distance.

A *reflection* r in W is a conjugate of some $s \in S$. The set M_r of all simplices in A fixed by r is a subcomplex of dimension one and is called the *wall* (in A) determined by r. Each wall partitions $\mathscr{C}(A)$ into two parts Φ_r^+ and Φ_r^- (with $c \in \Phi_r^+$) interchanged by r. These two parts of $\mathscr{C}(A)$ are called *half-apartments* determined by the wall M_r. They form complementary subsets of $\mathscr{C}(A)$ and are said to be opposite to each other. As subsets of W, $\Phi_r^+ = \{w \in W : l(rw) > l(w)\}$ and $\Phi_r^- = W - \Phi_r^+ = \{w \in W : l(rw) < l(w)\}$. The *boundary* $\partial\Phi_r = \Phi_r^+ \cap \Phi_r^-$ of Φ_r^+ (or of Φ_r^-) is the wall M_r. By the *interior* of a half-apartment we mean its topological interior. There are bijective correspondences between the set of reflections, the set of walls and the set of pairs of opposite half-apartments [9, Proposition 2.6].

There is an obvious action of W on the set of walls of an apartment and then an action of W on the set of half-apartments. For each $w \in W$ the number of positive half-apartments that w maps to negative one is equal to the length of w [8, Proposition 5.6]. By definition two walls are *parallel* if their distance (that is the distance from any point of one to the nearest point of the other) is bounded. A necessary and sufficient condition for two walls $\partial\Phi_r$ and $\partial\Phi_t$ to be parallel is that the element rt has infinite order [6, p. 142]. As there exists a bijective correspondence between the set I of types of $\mathbf{X}$ and the set S of generators of W (via $i \to s_i$) sometimes in the sequel for each reflection $r = s_{i_1}s_{i_2}\cdots s_{i_k}$ we will write $\Phi^{\pm}_{i_1 i_2\cdots i_k}$ for $\Phi^{\pm}_r$. A wall or a half-apartment in $\mathbf{X}$ is by definition a wall or a half-apartment in some apartment of $\mathbf{X}$.

Let us call a chamber subcomplex $\mathbf{X}'$ of $\mathbf{X}$ *convex* if any minimal gallery between two chambers of $\mathbf{X}'$ lies entirely in $\mathbf{X}'$. The *convex hull* of any subset $\mathbf{X}'$, which we denote by conv $\mathbf{X}'$, is by definition the intersection of all convex chamber subcomplexes containing $\mathbf{X}'$.

Let σ be any simplex in $\mathbf{X}$ and set $\mathscr{A}_\sigma = \{A \in \mathscr{A} : \sigma \in A\}$. Denote by $lk(\sigma)$ the set of all simplices in $\mathbf{X}$ containing σ (with the inclusion order relation induced by that of $\mathbf{X}$). It turns out that $lk(\sigma)$ is a spherical (finite) building whose set of apartments is equal to $\{A \cap lk(\sigma) : A \in \mathscr{A}_\sigma\}$. If σ has type $I \setminus J$ ($J \subseteq I$) and if W_J is the subgroup of W which is generated by s_j, $j \in J$ then the finite Coxeter group W_σ of $lk(\sigma)$ is equal to W_J. In terms of diagrams this has the following interpretation. Let us recall that the diagram M of $\mathbf{X}$ has a vertex for each $i \in I$. We pass from M to the diagram M_J of $lk(\sigma)$ by removing from M all vertices which belong to $I \setminus J$ [11, Proposition 3.12].

Since $lk(\sigma)$ is spherical, its *diameter* (that is $\sup\{\mathbf{d}(c, c') : c, c' \in \mathscr{C}(lk(\sigma))\}$), which we denote by diam $lk(\sigma)$, is finite. Let us call two chambers c, c' *opposite* if $\mathbf{d}(c, c') = \operatorname{diam} lk(\sigma)$. As the distance between two chambers in any building is equal to their distance in any apartment containing both, two chambers are opposite if and only if they are opposite in some apartment of $lk(\sigma)$. Moreover two opposite chambers are contained in one and only one apartment, namely conv$\{c, c'\}$.

The following facts are important and useful tools in the applications.

THEOREM 1.1. *Let A be an apartment and c a chamber in A. Then there exists a unique mapping $\rho_{c,A} : \mathbf{X} \to A$ such that for all apartments A' containing c, the map $\rho_{c,A}|_{A'}$ is the isomorphism fixing $A \cap A'$ pointwise.*

PROOF. See for example [6, p. 86]. □

The map $\rho_{c,A}$ is called the *retraction* of $\mathbf{X}$ onto A with center c. Let μ be the isometry from A to W with $\mu(c) = 1$ (the identity element) and define the mapping $\delta_c : \mathscr{C} \to W$ by $\delta_c(d) = \mu\left(\rho_{c,A}(d)\right)$. For each $d \in \mathscr{C}$ the group element $\delta_c(d)$ retains information about the relative position of d with respect to c: its length is equal to the distance between c and d, and the reduced decompositions of $\delta_c(d)$ are related to minimal paths from c to d. The following proposition shows the properties of the function δ_c and is proved in [12, Section 3].

PROPOSITION 1.2. *For any chamber c the following facts hold:*

(i) *if* $j \in I$, *if* d *and* d' *are two distinct,* j*-adjacent chambers and if* $\delta_c(d) = w$ *then* $\delta_c(d') \in \{w, ws_j\}$. *Moreover if* $l(ws_j) = l(w) + 1$ *then* $\delta_c(d') = ws_j$;

(ii) *if* $\delta_c(d) = w$ *then for any* $j \in I$ *there exists a chamber* d' *distinct from* d *and* j*-adjacent to it such that* $\delta_c(d') = ws_j$. *If* $l(ws_j) = l(w) - 1$ *then there is a unique such* d'.

2. Measure on the space of apartments

Let A_0 be the Coxeter complex of W (that is the representative of the isomorphism class of the set of apartments of $\mathbf{X}$). Given any subset of chambers $C \subseteq A_0$ we define a map $\lambda : C \to \mathbf{X}$ to be an isometry if it preserves the labelling and the distance between any two chambers. Denote by Λ the set of isometries $\lambda : A_0 \to \mathbf{X}$. Note that each apartment in $\mathbf{X}$ is an isometric image $\lambda(A_0)$ of A_0 into $\mathbf{X}$. Moreover each isometry λ is uniquely determined by its image $\lambda(A_0)$ together with its value on a fixed chamber c_0, because if λ' is another such isometry, then $\lambda^{-1}\lambda'$ is an isometry of A_0 fixing c_0 and is therefore the identity of W. We endow Λ with the open compact topology: a fundamental neighborhood for $\lambda \in \Lambda$ consists of the set of maps λ' which agree with λ on a fixed finite set of chambers. This topology makes Λ a locally compact, totally disconnected, Hausdorff and separable space. We remark that the group W acts on Λ via $\lambda \mapsto \lambda \circ w$, and that the map $\Lambda \to \mathscr{A}$ defined by $\lambda \mapsto \lambda(A_0)$, gives a bijective correspondence between the set of W-orbits $W\backslash\Lambda$ and $\mathscr{A}$.

Let c_0 be a fixed chamber in A_0 and C_n, for $n \geq 0$, be the compact convex set consisting of the convex hull of c_0 and the set of chambers which are at distance n from c_0, ($C_0 = c_0$). Let us denote by Λ_n the set of isometries of C_n into $\mathbf{X}$. Let $\psi \in \Lambda_n$ and $K_{n,\psi} = \{\lambda \in \Lambda : \lambda|_{C_n} = \psi\}$. We remark that the family $\{K_{n,\psi}\}_{n\in\mathbb{N},\psi\in\Lambda_n}$ forms a basis for the open-compact topology on Λ. Moreover for each n we get

$$\Lambda = \bigcup_{\psi\in\Lambda_n} K_{n,\psi}$$

where the union is disjoint.

We denote by $\mathscr{M}$ the σ-algebra generated by the sets $K_{n,\psi}$, $n \in \mathbb{N}$ and $\psi \in \Lambda_n$ and by $C_c(\Lambda)$ the space of continuous, compactly supported functions on Λ.

PROPOSITION 2.1. *If a measure* m *exists, which is positive, finite on compact sets and regular on* $\mathscr{M}$ *and such that the following conditions are satisfied*

(i) $m(K_{0,\psi}) = 1$,

(ii) *for each* n, $m(K_{n,\psi})$ *does not depends on* ψ,

then it is unique.

PROOF. Let m_1 and m_2 be positive measures satisfying the hypothesis, and let us define on $C_c(\Lambda)$ the functional F_i, $i = 1, 2$, by $F_i(f) = \int_\Lambda f(\lambda) dm_i(\lambda)$. We will prove that $F_1 \equiv F_2$. We start by proving that $F_1 f = F_2 f$ when f is a finite linear combination of characteristic functions of sets $K_{n,\psi}$. To this end we note that for each ψ and for each n the set $K_{0,\psi}$ can be covered by a finite number of subsets $K_{n,\psi_1}, \dots, K_{n,\psi_{p_n}}$. Moreover this covering forms a partition of $K_{0,\psi}$. Since (ii) holds, p_n does not depend on ψ and from (i) it follows that $m_i(K_{n,\psi_j}) = 1/p_n$ for $j = 1, \dots, p_n$. As the sets $K_{0,\psi}$ cover Λ, we can find a finite number of them covering the support of f. Without loss of generality we can suppose that the support of f is contained in $K_{0,\psi}$ for some ψ. Then there exists n, depending on f, such that $f = \sum_{j=1}^{p_n} b_i \chi_{K_{n,\psi_j}}$. Then one can easily verify that $F_1 f = F_2 f$. Since each continuous compactly supported function is the uniform limit of a sequence of finite linear combination of characteristic functions of sets $K_{n,\psi}$, we conclude that F_1 and F_2 define the same functional on $C_c(\Lambda)$. Finally the Riesz representation lemma implies that $m_1 = m_2$. □

LEMMA 2.2. *Let c and c' be chambers of A_0. Each isometry $\psi : \{c, c'\} \to \mathbf{X}$ extends uniquely to an isometry $\psi' : \mathrm{conv}(c, c') \to \mathbf{X}$.*

PROOF. We need to show that for each minimal gallery joining c and c' there is a unique minimal gallery of the same (reduced) type joining $\psi'(c)$ and $\psi'(c')$. Since ψ' send minimal galleries to minimal galleries and is type preserving it is sufficient to prove that if there is a gallery of a given reduced type joining two given chambers this is unique. But this last statement is a well known property of buildings, [12, Section 3]. □

The following Proposition will allow us to prove the existence of m. For simplicity in all this section we suppose that our convex sets contain at least one chamber.

PROPOSITION 2.3. *Let $C \subseteq C'$ be a non-empty, finite and convex subsets of A_0. Each isometry $\psi : C \to \mathbf{X}$ extends in a finite number of ways $N(C, C')$ to an isometry $\psi' : C' \to \mathbf{X}$. Moreover $N(C, C')$ does not depend on ψ.*

PROOF. By Zorn's lemma it suffices to extend the domain of ψ to a strictly larger convex subset of chambers. We can find a chamber $d \in C$ and $i \in I$ such that $d' \notin C$ where d' denotes the chamber of A_0 distinct from and i-adjacent to d. Let a be any chamber in $\mathbf{X}$ distinct from and i-adjacent to $\psi(d)$. We extend ψ by letting $\psi' = \psi$ on C and $\psi'(d') = a$. From Lemma 2.2 it follows that ψ' is uniquely determined on $C' = \mathrm{conv}(C, d')$. Moreover $N(C, C') = q_i$ and then it does not depend on ψ. □

We define the measure m by setting

$$m(K_{0,\psi}) = 1$$

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and since for each isometry $\psi : c_0 \to \mathbf{X}$ and for each $n \geq 1$ we have the equality

$$\{\lambda \in \Lambda : \lambda(c_0) = \psi(c_0)\} = \bigcup_{\substack{\psi' \in \Lambda_n \\ \psi'(c_0) = \psi(c_0)}} K_{n,\psi'}$$

where the union is disjoint, we set

$$m(K_{n,\psi}) = \frac{1}{N(C_0, C_n)}.$$

Let now C be a compact convex set and $\psi : C \to \mathbf{X}$ be an isometry. We set $K_{C,\psi} = \{\lambda \in \Lambda : \lambda|_C = \psi\}$.

COROLLARY 2.4. *The measure $m(K_{C,\psi})$ of the compact set $K_{C,\psi}$ does not depend on ψ.*

PROOF. There exists n such that $C \subseteq C_n$. Then the assertion follows from the equality $\{\lambda \in \Lambda : \lambda|_C = \psi\} = \bigcup_{\substack{\psi' \in \Lambda_n \\ \psi'|_C = \psi}} K_{n,\psi'}$. □

REMARK 2.5. Note that if $c_0 \in C$ then $m(K_{C,\psi}) = 1/N(c_0, C)$.

LEMMA 2.6. *The measure m is invariant with respect to W.*

PROOF. Let $w \in W$ and $c = wc_0$. Let C be a finite convex set containing c_0 and c and $\psi : C \to \mathbf{X}$ be an isometry. We get

$$m(K_{c,\psi}) = N(c, C)m(K_{C,\psi}) = \frac{N(c, C)}{N(c_0, C)} m(K_{0,\psi}) = \frac{N(c, C)}{N(c_0, C)}.$$

We will prove that the quotient $N(c, C)/N(c_0, C)$ is equal to 1. Clearly it does not depend on C, so we can choose $C = \operatorname{conv}(c_0, c)$. If $(i_1, \ldots, i_d)$ is the type of a minimal gallery joining c_0 and c then the number of ways to extend an isometry of $\{c_0\}$ into $\mathbf{X}$ to an isometry of C into $\mathbf{X}$ is equal to the number of galleries of type $(i_1, \ldots, i_d)$ starting at $\psi(c_0)$. This number is equal to $q_{i_1} \cdots q_{i_d}$ and is the same as the number of galleries of type $(i_d, \ldots, i_1)$ starting at $\psi(c)$. So condition (i) of Proposition 2.1 holds with c_0 replaced by c. Moreover by Corollary 2.4 it follows that m satisfies condition (ii) of Proposition 2.1 with C_n replaced by wC_n. Then the W-invariance of m follows because m is uniquely determined by (i) and (ii). □

LEMMA 2.7. *W acts properly on Λ.*

PROOF. One can easily verify that for every $\lambda \in \Lambda$ the stabilizer W_λ of λ in W is finite (indeed it is trivial) and λ has a neighborhood U such that $wU \cap U = \emptyset$ for all $w \in W \setminus W_\lambda$. Since W is discrete this is exactly the definition of proper action. □

The fact that W acts properly on Λ implies that $W\backslash\Lambda$ is a locally compact, separable, Hausdorff topological space. In the following proposition we state, without proof, a classical result about disintegration of measures known since about 1950 (see for example [5, Section 3]). The formulation given here has been communicated to me by Professor Tim Steger [10].

PROPOSITION 2.8. *Let Λ be a locally compact, separable, Hausdorff topological space. Let m be a regular and positive measure on Λ. Suppose that we have a group W which acts properly on Λ and preserving m. Then there exists a unique regular and positive measure m_Q on $W\backslash\Lambda$ such that for each $f \in C_c(\Lambda)$ we get*

$$\int_{\lambda\in\Lambda} f(\lambda)dm(\lambda) = \int_{W\lambda\in W\backslash\Lambda}\left(\sum_{w\in W} f(w\lambda)\right) dm_Q(W\lambda). \tag{2.1}$$

REMARK 2.9. Let $\pi : \Lambda \to W\backslash\Lambda$ be the natural projection, and $K \subseteq \Lambda$ be an open compact set such that $\pi|_K$ is injective. If f is the characteristic function of K then $\sum_{w\in W} f(w\lambda)$ as a function defined on $W\backslash\Lambda$ is the characteristic function of $\pi(K)$. Then equality (2.1) gives

$$m(K) = m_Q(\pi(K)).$$

Let now $C \subset A_0$ be a non-empty, finite, convex set, $\psi : C \to \mathbf{X}$ be an isometry and $K = K_{C,\psi}$. If we identify $W\backslash\Lambda$ with $\mathscr{A}$ we have

$$\pi(K) = \{A \in \mathscr{A} : \psi(C) \in A\}.$$

Since C contains at least one chamber the restriction of π on K is injective and then

$$m_Q(\{A \in \mathscr{A} : \psi(C) \in A\}) = m(K).$$

Generally if C' is a finite convex set in some apartment of $\mathbf{X}$ we can find a finite convex set $C \subset A_0$ and an isometry $\psi : C \to \mathbf{X}$ such that $\psi(C) = C'$. So we can compute the measure of $\mathscr{A}_{C'} = \{A \in \mathscr{A} : C' \in A\}$ for each finite convex set $C' \subseteq \mathbf{X}$.

REMARK 2.10. For each affine building $\mathbf{X}$, from the definition of m_Q it follows immediately that if c is any chamber in $\mathbf{X}$ then $m_Q(\mathscr{A}_c) = 1$. More generally let σ be any simplex in $\mathbf{X}$ and E be any apartment in $lk(\sigma)$. We can identify $\mathscr{C}(A_0)$ with W and then there exists an isometry ψ such that $E = \psi(W_\sigma)$ and $m_Q(\mathscr{A}_E) = 1/N(1, W_\sigma)$. If w_σ denotes the longest element in W_σ, (that is the unique element which is opposite to 1), then $W_\sigma = \mathrm{conv}(1, w_\sigma)$. We recall that the reduced expressions for w_σ are in bijective correspondence with the minimal galleries joining 1 with w_σ. If $s_{i_1} \cdots s_{i_{l(w_\sigma)}}$ is a reduced expression for w_σ then there exists one (and only one) gallery of type

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$(i_1, \ldots, i_{l(w_\sigma)})$ joining 1 with w_σ and this implies that $N(1, W_\sigma) = q_{i_1} \cdots q_{i_{l(w_\sigma)}}$. For each set $\mathscr{E}'_\sigma \subseteq \{A \cap lk(\sigma) : A \in \mathscr{A}_\sigma\}$ let $\mathscr{A}'_\sigma = \{A \in \mathscr{A}_\sigma : A \cap lk(\sigma) \in \mathscr{E}'_\sigma\} \subseteq \mathscr{A}_\sigma$. As $\mathscr{A}'_\sigma = \cup_{E \in \mathscr{E}'_\sigma} \mathscr{A}_E$, then, denoting by $|\cdot|$ the cardinality of a given set, we get

$$m_Q(\mathscr{A}'_\sigma) = \frac{|\mathscr{E}'_\sigma|}{N(1, W_\sigma)}. \tag{2.2}$$

Note that if σ has corank two and $lk(\sigma)$ has parameters $\{q_s, q_t\}$ then $N(1, W_\sigma)$ is equal to $q_s^{(l(w_\sigma)+1)/2} q_t^{(l(w_\sigma)-1)/2}$ if $l(w_\sigma)$ is odd and to $(q_s q_t)^{l(w_\sigma)/2}$ if $l(w_\sigma)$ is even. Indeed, if $\ldots stst$ is the longest word in W_σ this follows because there exists exactly one gallery of type $(\ldots, s, t, s, t)$ joining any two opposite chambers. Note also that, as there exists one (and only one) of type $(\ldots, t, s, t, s)$ joining the same chambers, then if $l(w_\sigma)$ is odd, exchanging the role of q_s and q_t we have that $q_s = q_t$ (that is a rank-two spherical building is necessarily homogeneous if its diameter is odd).

3. Inversion of the Radon transform

We define a *strip* $\mathscr{S}$ in a given apartment as the intersection $\alpha \cap \alpha'$ of two half-apartments α and α' such that α contains the opposite of α'.

Let $\mathscr{S}$ be a strip and let $\partial\alpha \cup \partial\alpha'$ its boundary. Each wall l not parallel to $\partial\alpha$ (equivalently to $\partial\alpha'$) splits $\mathscr{S}$ in two subsets B and B' such that $\mathscr{C}(B) \cap \mathscr{C}(B') = \emptyset$ and $\mathscr{C}(B) \cup \mathscr{C}(B') = \mathscr{C}(\mathscr{S})$. We say that B and B' are the *opposite half-strips* originating from $\partial l \cap \mathscr{S}$. Let B and B' be opposite half-strips and denote by BB' the strip $\mathscr{S}$ such that $\mathscr{S} = B \cup B'$. Similar conventions will hold for unions of half-apartments and so on

If g is a complex-valued function defined on $\mathscr{C}$ set $\Psi g(\mathscr{S}) = \sum_{c \in \mathscr{S}} g(c)$ for each strip $\mathscr{S}$, whenever the right-hand side is meaningful.

LEMMA 3.1. *Let $\mathscr{S}$ be a strip in* **X**. *Then there exist half-apartments $\alpha, \beta, \gamma, \delta$, with interiors disjoint from $\mathscr{S}$ and from each other, such that $\alpha\mathscr{S}\gamma$, $\beta\mathscr{S}\delta$, $\alpha\beta$, $\gamma\delta$ are apartments; if $g \in L^1(\mathscr{C}(A))$ for each apartment $A \in \mathscr{A}$, then*

$$2\Psi g(\mathscr{S}) = Rg(\alpha\mathscr{S}\gamma) + Rg(\beta\mathscr{S}\delta) - Rgf(\alpha\beta) - Rg(\gamma\delta). \tag{3.1}$$

PROOF. Let $\mathscr{S} = \alpha' \cap \alpha''$. We can take for α the opposite of α' and for δ the opposite of α''. Then the existence of β and γ is guaranteed by the hypothesis that **X** is thick. We have $Rg(\alpha\mathscr{S}\gamma) = \sum_{c \in \alpha} g(c) + \sum_{c \in \mathscr{S}} g(c) + \sum_{c \in \beta} g(c)$, and so forth, so the right-hand side of (3.1) yields $2\sum_{c \in \mathscr{S}} g(c)$ which is equal to the left-hand side. □

Case I: **X** with diagram of type $\widetilde{A}_2$

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LEMMA 3.2. *Let c be any chamber in* $\mathbf{X}$ *and denote by* σ_i, $i = 0, 1, 2$, *its edges. Then there exist half-strips* B_{ij}, $i, j = 0, 1, 2$ *and* $j \neq i$ *with interiors disjoint from* c *and disjoint from each other, such that* B_{ij} *originates from* σ_i *and* $B_{ij}c$ *is a strip originating from* σ_j *and such that* $B_{ij}cB_{ji}$ *and* $B_{ij_1}B_{ij_2}$, *for* $j_1 \neq j_2$ *are strips. If* $g \in L^1(\mathscr{C}(A))$ *for each* $A \in \mathscr{A}$ *then*

$$3g(c) = \sum_{i<j} \Psi g(B_{ij}cB_{ji}) - \sum_{i\neq j_1\neq j_2} \Psi g(B_{ij_1}B_{ij_2}). \tag{3.2}$$

PROOF. We can suppose that σ_i has cotype i. Let A be any apartment containing c and let $\partial\Phi_i$ be the wall in A corresponding to σ_i. Let x_i be the vertex of c of type i and note that the subgroup of W which stabilizes x_i in A is generated by s_{i^+} and s_{i^-} where $i^\pm = i \pm 1 \bmod 3$. Since the element $s_i s_{i^+} s_{i^-} s_{i^+}$ has infinite order then $\partial\Phi_{i^+i^-i^+}$ is the (unique) wall in A containing x_i and parallel to $\partial\Phi_i$. Consider the strips $\mathscr{S}_i = \Phi_i^+ \cap \Phi_{i^+i^-i^+}^+$ and let $B'_{i^+i^-} = \mathscr{S}_i \cap \Phi_{i^+}^-$ and $B_{i^-i^+} = \mathscr{S}_i \cap \Phi_{i^-}^-$ and note that since $c = \bigcap_{i\in I} \Phi_i^+$ then $\mathscr{S}_i = B'_{i^+i^-}cB_{i^-i^+}$. Let Π_{i^+} be any half-apartment in $\mathbf{X}$ whose interior is disjoint from $\Phi_{i^+}^+$ and $\Phi_{i^+}^-$ and such that $\partial\Pi_{i^+} = \partial\Phi_{i^+}$. Hence $\Pi_{i^+} \cup \Phi_{i^+}^+$ and $\Pi_{i^+} \cup \Phi_{i^+}^-$ are apartments.

Consider the map $\rho_{c,A}|_{\Pi_{i^+}\cup\Phi_{i^+}^+} : \Pi_{i^+} \cup \Phi_{i^+}^+ \to A$. It is the unique isomorphism of $\Pi_{i^+} \cup \Phi_{i^+}^+$ onto A fixing $cB_{i^-i^+}$. Denote by $B_{i^+i^-}$ the half-strip in Π_{i^+} such that its image under $\rho_{c,A}$ is the half-strip $B'_{i^+i^-}$. Then $B_{i^+i^-}cB_{i^-i^+}$ is a strip in $\Pi_{i^+} \cup \Phi_{i^+}^+$ isomorphic and isometric to $\mathscr{S}_i$ which we denote by $\mathscr{S}_i'$.

Let $j = i^-$ and note that the reflection s_{i^+} maps $B'_{i^+i^-}$ onto $\mathscr{S}_j \cap \Phi_{j^-}^+ = B'_{j^+j^-}c$. Consider the isomorphism of $\Pi_{i^+} \cup \Phi_{i^+}^-$ onto A which is equal to the identity on $\Phi_{i^+}^-$ and is equal to $s_{i^+} \circ \rho_{c,A}$ on Π_{i^+}. It maps $B_{i^+i^-}B_{j^-j^+}$ isometrically onto $\mathscr{S}_j$. Then $B_{i^+i^-}B_{j^-j^+}$ is a strip in $\Pi_{i^+} \cup \Phi_{i^+}^-$ whose interior is disjoint from c and which we denote by $\mathscr{S}_i''$. Considering the strips $\mathscr{S}_i'$ and $\mathscr{S}_i''$, $i = 0, 1, 2$, as we have that $\Psi g(\mathscr{S}_i') = \sum_{d\in B_{i^+i^-}} g(d) + g(c) + \sum_{d\in B_{i^-i^+}} g(d)$, and so forth, we get that the right hand side of (3.2) is equal to the left hand-side and this proves the assertion. (In Figure 2 the half-strips $B_{i^+i^-}$ are dashed to mean that they do not lie in the same apartment containing the half-strips $B_{i^-i^+}$.) □

Let $a(c)$ be the right-hand side of (3.2). Then, for each $c \in \mathscr{C}$, Lemma 3.1 allows an expression of $a(c)$ and then of $g(c)$ in terms of Rg. So g is uniquely determined by Rg.

Case II: $\mathbf{X}$ with diagram of type $\widetilde{C}_2$

Let c be any chamber in $\mathbf{X}$ and let x_0 be its vertex with the property that $lk(x_0)$ is isomorphic to a spherical building with diagram of type $A_1 \times A_1$. Suppose that x_0 has type 0, and denote by σ_0 the edge of c opposite to x_0.

LEMMA 3.3. *Let c' be a chamber distinct from c and 0-adjacent to it. Let σ_1 and*

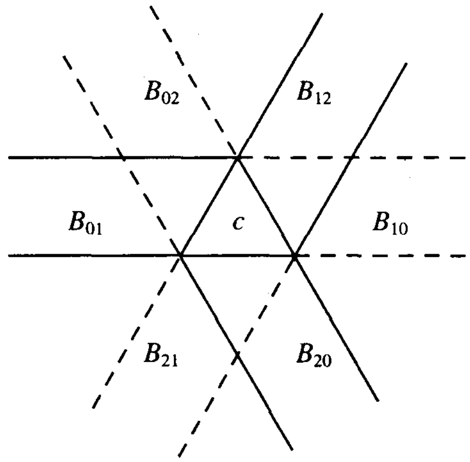

FIGURE 2.

σ_2, *(respectively* σ_1' *and* σ_2'*), be the edges of* c *(respectively* c'*) of cotype* 1 *and* 2. *Then there exist half-strips* B_1, $\widetilde{B}_1$, B_2, $\widetilde{B}_2$, *with interiors disjoint from* c, c' *and from each other, such that* B_1, $\widetilde{B}_1$ *originate from* σ_1 *and* B_2, $\widetilde{B}_2$ *originate from* σ_2', *and such that* $B_1cc'B_2$, $\widetilde{B}_1cc'\widetilde{B}_2$, $B_1\widetilde{B}_1$, $B_2\widetilde{B}_2$ *are strips; if* $g \in L^1(\mathscr{C}(A))$ *for each* $A \in \mathscr{A}$ *then*

$$(3.3)\quad 2(g(c)+g(c')) = \Psi g(B_1cc'B_2) + \Psi g(\widetilde{B}_1cc'\widetilde{B}_2) - \Psi g(B_1\widetilde{B}_1) - \Psi g(B_2\widetilde{B}_2).$$

PROOF. We can argue as in the proof of Lemma 3.2. The details are left to the interested reader. □

Denote by $a(c, c')$ the right-hand side of (3.3) and let c'' be a chamber 0-adjacent to c and different from c and c'. Lemma 3.3 applied to c, c'' and then to c', c'' allows an expression of $g(c)$ in terms of Ψ. Indeed

$$(3.4)\qquad 4g(c) = a(c, c') + a(c, c'') - a(c', c'').$$

Finally, for each $c \in \mathscr{C}$, Lemma 3.1 allows us to express the right hand side of (3.4) by a finite linear combination of values of Rg.

Case III: **X** with diagram of type $\widetilde{G}_2$.

Label the vertices of **X** in such a way that the type 0 (respectively 1, 2) corresponds to the vertices v such that $lk(v)$ has diagram of type A_2 (respectively $lk(v)$ has diagram of type $A_1 \times A_1$, $lk(v)$ has diagram of type G_2). Note that the affine Coxeter group $W(\widetilde{A}_2)$ of type $\widetilde{A}_2$ is a subgroup (of index six) of the affine Coxeter group $W(\widetilde{G}_2)$ of

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type $\widetilde{G}_2$ associated to **X**. So the set of walls, and then the set of half-apartments of any apartment of **X** contains the set of walls, (respectively the set of half-apartments) of the complex associated to $W(\widetilde{A}_2)$. (From the tiling induced by $W(\widetilde{G}_2)$ we can recover the tiling induced by $W(\widetilde{A}_2)$ considering pairs of distinct 0-adjacent chambers.)

Let c and c' be any two distinct 0-adjacent chambers and let σ_1 and σ_2 (respectively $\sigma_1{}'$ and $\sigma_2{}'$) the edges of c (respectively c') of cotype 1 and 2. Set $l_1 = \sigma_1$, $l_2 = \sigma_1'$ and $l_3 = \sigma_2 \cup \sigma_2{}'$, then arguing as in the case of a building of type $\widetilde{A}_2$ we get the following lemma.

LEMMA 3.4. *Define c and c' as above. Then there exist half-strips B_{ij}, $i, j = 1, 2, 3$ and $j \neq i$ with interiors disjoint from c, c' and from each other, such that B_{ij} originates from l_i and $B_{ij}cc'$ is a strip originating from l_j and such that $B_{ij}cc'B_{ji}$ and $B_{ij_1}B_{ij_2}$ are strips: if $g \in L^1(\mathscr{C}(A))$ for each $A \in \mathscr{A}$ then*

$$g(c) + g(c') = \frac{1}{3}\left(\sum_{i<j} \Psi g(B_{ij}cc'B_{ji}) - \sum_{i \neq j_1 \neq j_2} \Psi g(B_{ij_1}B_{ij_2})\right). \tag{3.5}$$

Consider now another distinct chamber c'' 0-adjacent to c. Then Lemma 3.4 applied to c, c'' and then to c', c'' allows an expression of $g(c) + g(c'')$ and of $g(c') + g(c'')$ in terms of Ψ. Proceeding as in the case of a building of type $\widetilde{C}_2$ we can recover the value of g in c in terms of Rg.

We have just proved the following proposition.

PROPOSITION 3.5. *Let* **X** *be any rank three affine building. The operator R is injective on the space of functions that are summable on the set of chambers of each apartment of* **X**.

4. Inversion formulas

From now on, given any chamber c in **X** we will denote by A_0 a fixed apartment containing c and by ρ the retraction of **X** on A_0 with center c. Given a chamber d and a simplex σ of **X** we will denote by $\operatorname{proj}_\sigma d$ (*projection* of d on σ) the unique chamber of $lk(\sigma)$ nearest to d, (that is, the first element of any stretched gallery from σ to d), [9, Corollary 3.9].

PROPOSITION 4.1. *Let $\sigma \subset c$ and $A \in \mathscr{A}_\sigma$. Then*

$$\mathscr{C}(\rho(A)) = \bigcup_{d \in \mathscr{C}(A) \cap lk(\sigma)} \operatorname{proj}_\sigma^{-1}(\rho(d)) \cap \mathscr{C}(A_0).$$

PROOF. For each chamber $d \in A \cap lk(\sigma)$ let L_d be the convex subcomplex $\mathrm{proj}_\sigma^{-1}(d) \cap \mathscr{C}(A)$ ([11, Proposition 2.32]). Note that

$$\mathscr{C}(A) = \bigcup_{d \in A \cap lk(\sigma)} L_d,$$

where the union is disjoint. We will show that

$$\rho(L_d) = \mathrm{proj}_\sigma^{-1} \rho(d) \cap \mathscr{C}(A_0).$$

From this our assertion will follow. Note that, as the retraction ρ maps every stretched gallery starting from σ, onto a stretched gallery with same origin, one has that

$$\rho(L_d) \subseteq \mathrm{proj}_\sigma^{-1} \rho(d) \cap \mathscr{C}(A_0).$$

To prove that equality holds we argue as follows. Letting $\rho_{d,A}$ the retraction on A with center d, then the set $L_d \cup \{c\}$ is isometric to $L_d \cup \{\rho_{d,A}(c)\} \subset A$ and this implies ([9, Theorem 3.6]) that there exists an apartment A' containing L_d and c. Let $\varphi : A' \to A_0$ be the unique isomorphism fixing $A' \cap A_0$; then the image of L_d under φ is equal to $\mathrm{proj}_\sigma^{-1} \varphi(d) \cap \mathscr{C}(A_0)$. Therefore, as φ and ρ agree pointwise on L_d, we have proved the proposition. □

Let A be any apartment in $\mathbf{X}$. As A is the collection of its chambers, the 'position' of A with respect to c is completely known if for each chamber $d \in A$ we know the value at d of the function δ_c or, equivalently, if we look at the image of A under ρ. In view of [11, Proposition 2.32], the content of Proposition 4.1 is that to know the image of A under ρ it is sufficient to know the image of $A \cap lk(\sigma)$ under the restriction of ρ to $lk(\sigma)$.

If $\Phi \subseteq A_0$ is a convex chamber subcomplex, we set

$$\mathscr{A}^\Phi = \{A \in \mathscr{A} : \rho(A) = \Phi\}.$$

In particular Φ can be Φ_r^+ or Φ_r^- for some reflection r in W. In the sequel, to simplify the notation, we will write Φ_r for Φ_r^-.

We will radialize with respect to m_Q the discrete inversion formulas of Section 3. To this end it will be necessary to derive case by case the measure of suitable subsets of $\mathscr{A}$. We now prove some technical lemmas.

LEMMA 4.2. *Let $(\mathbf{X}, \mathscr{A})$ be a spherical building with diagram of type A_1, with $S = \{s\}$ and parameter q_s. Then*

$$|\mathscr{A}^{\Phi_s}| = \frac{q_s(q_s - 1)}{2}. \tag{4.1}$$

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PROOF. As $\mathbf{X}$ is a rank-one building, each apartment consists of a pair of distinct, s-adjacent chambers. Moreover $\Phi_s = \{s\}$. Then the requested number of apartments is equal to the number of unordered pairs (d_0, d_1) of distinct chambers such that $\delta_c(d_0) = \delta_c(d_1) = s$, and this number turns out be equal the right-hand side of (4.1). □

REMARK 4.3. Let $\mathbf{X}$ be any building. Denote by $\Gamma(w_0, \ldots, w_n) = \Gamma_{i_1,\ldots,i_n}(w_0, \ldots, w_n)$, $w_k \in W$ for $0 \le k \le n$, the collection of simple galleries $(d_0, \ldots, d_n)$ of reduced type $(i_1, \ldots, i_n)$, such that $\delta_c(d_k) = w_k$ for all $0 \le k \le n$. For each $d \in \mathscr{C}$ such that $\delta_c(d) = w_k$, denote by $a(w_k, i_{k+1}, w_{k+1})$ the number of distinct chambers d' i_{k+1}-adjacent to d such that $\delta_c(d') = w_{k+1}$. Then an easy induction shows that

$$|\Gamma(w_0, \ldots, w_n)| = \phi|\delta_c^{-1}(w_0)| \prod_{0\le k\le n-1} a(w_k, i_{k+1}, w_{k+1})$$

where

$$\phi = \begin{cases} \frac{1}{2} & \text{if } n \text{ is odd, } w_j = w_{n-j} \text{ for all } j \text{ and } (i_1, \ldots, i_n) = (i_n, \ldots, i_1); \\ 1 & \text{otherwise.} \end{cases}$$

Recall that $w_{k+1} \in \{w_k, w_k s_{i_k}\}$ and note that from Proposition 1.2 it follows immediately that

$$a(w_k, i_{k+1}, w_{k+1}) = \begin{cases} 1 & \text{if} \quad w_{k+1} = w_k s_{i_k} \quad \text{and} \quad l(w_k s_{i_k}) = l(w_k) - 1; \\ q_{i_{k+1}} & \text{if} \quad w_{k+1} = w_k s_{i_k} \quad \text{and} \quad l(w_k s_{i_k}) = l(w_k) + 1; \\ q_{i_{k+1}} - 1 & \text{if} \quad w_{k+1} = w_k. \end{cases}$$

REMARK 4.4. Note that if $w = s_{i_1} \cdots s_{i_l}$ is a reduced expression for $w \in W$ then $|\delta_c^{-1}(w)|$ is the same as the number of minimal galleries of type $(i_1, \ldots, i_l)$ starting at c.

LEMMA 4.5. *Let* $(\mathbf{X}, \mathscr{A})$ *be a spherical building with diagram of type* A_2, *with* $S = \{s, t\}$ *and parameter* q. *Then the sets* $\mathscr{A}^{\Phi_{sts}}$, *(note that* $sts = tst$*),* $\mathscr{A}^{\Phi_{sts}\cap\Phi_s}$ *and* $\mathscr{A}^{\Phi_{sts}\cap\Phi_t}$ *are not empty and the following equalities hold:*

$$|\mathscr{A}^{\Phi_{sts}}| = \frac{(q-1)q^3}{2}, \tag{4.2}$$

$$|\mathscr{A}^{\Phi_s\cap\Phi_{sts}}| = |\mathscr{A}^{\Phi_t\cap\Phi_{sts}}| = \frac{(q-1)^2q^3}{2}. \tag{4.3}$$

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PROOF. Let $(d_0, \dots, d_n)$ any simple gallery of reduced type and length $n = \operatorname{diam} \mathbf{X}$ and denote by $\gamma'(d_0, d_n) = (d_0, d'_1, \dots, d'_{n-1}, d_n)$ the unique gallery which exhausts $\operatorname{conv}\{d_0, d_n\}$. Starting with a chamber d_0 such that $\delta_c(d_0) \in \Phi$ we first construct all galleries $(d_0, \dots, d_n)$, (for $n = 3$), such that $\delta_c(d_i) \in \Phi$, $1 \le i \le n$, then we look at the image under δ_c of $\operatorname{conv}\{d_0, d_n\}$.

Let $\Phi = \Phi_{sts}$ and choose d_0 such that $\delta_c(d_0) = st$. Then as any apartment in $\mathbf{X}$ contains exactly one gallery of type (s, t, s) and exactly one gallery of type (t, s, t), we can suppose, without loss of generality, that $(d_0, \dots, d_3)$ has type (t, s, t). Then starting from d_0 and applying Proposition 1.2 repeatedly, we see that for such a gallery to have its term d_i such that $\delta_c(d_i) \in \Phi_{sts} = \{st, sts, ts\}$, $1 \le i \le 3$, only the following two possibilities can occur: either $(d_0, \dots, d_3) \in \Gamma(st, st, sts, ts)$, or $(d_0, \dots, d_3) \in \Gamma(st, st, sts, sts)$. If $(d_0, \dots, d_3) \in \Gamma(st, st, sts, ts)$ then, being $d'_1 \underset{s}{\sim} d_0$ and $l(\delta_c(d_0)s) = l(\delta_c(d_0)) + 1$, we have that $\delta_c(d'_1) = sts$. While, being $d'_2 \underset{s}{\sim} d_3$ and $l(\delta_c(d_3)s) = l(\delta_c(d_3)) - 1$ we get $\delta_c(d'_2) = t$ or $\delta_c(d'_2) = \delta_c(d_3)$. In the first case it cannot be $d'_1 \underset{t}{\sim} d'_2$. Then necessarily $\delta_c(d'_2) = ts$. Note that in this case δ_c maps $\operatorname{conv}\{d_0, d_3\}$ onto Φ_{sts} (see Figure 3(a)). Analogously if $(d_0, \dots, d_3) \in \Gamma(st, st, sts, sts)$ one can prove that necessarily $\delta_c(d'_1) = \delta_c(d'_2) = sts$ and in this case δ_c maps $\operatorname{conv}\{d_0, d_3\}$ onto $\{st, sts\} = \Phi_s \cap \Phi_{sts}$. (see Figure 3(b))

As each apartment in $\mathscr{A}^{\Phi_{sts}}$ and in $\mathscr{A}^{\Phi_{sts} \cap \Phi_s}$ contains a chamber d_0 such that $\delta_c(d_0) = st$ in this way we can construct all apartments in $\mathscr{A}^{\Phi_{sts}}$ and in $\mathscr{A}^{\Phi_{sts} \cap \Phi_s}$. Note that the correspondence between $\Gamma_{t,s,t}(st, st, sts, ts)$ (respectively $\Gamma_{t,s,t}(st, st, sts, sts)$) and $\mathscr{A}^{\Phi_{sts}}$ (respectively $\mathscr{A}^{\Phi_{sts} \cap \Phi_s}$) is 2-1. In fact each apartment in $\mathscr{A}^{\Phi_{sts}}$ (respectively in $\mathscr{A}^{\Phi_{sts} \cap \Phi_s}$) contains two distinct galleries in $\Gamma_{t,s,t}(st, st, sts, ts)$ (respectively in $\Gamma_{t,s,t}(st, st, sts, sts)$): indeed (d_0, d_1, d_2, d_3) and (d_1, d_0, d'_1, d'_2). Therefore by Remark 4.3 we get (4.2) and part of (4.3).

Let now d_0 be such that $\delta_c(d_0) = ts$, and suppose that $\gamma(d_0, d_3)$ has type (s, t, s). This case can be considered as being symmetric to the preceding case, so we obtain again apartments in $\mathscr{A}^{\Phi_{sts}}$ by choosing d_1 such that $\delta_c(d_1) = ts$ and d_3 such that $\delta_c(d_3) = st$. We obtain apartments in $\mathscr{A}^{\Phi_{sts} \cap \Phi_t}$ by choosing d_1 as above and d_3 such that $\delta_c(d_2) = \delta_c(d_3) = sts$. This completes the proof of (4.3). □

LEMMA 4.6. *Let* $(\mathbf{X}, \mathscr{A})$ *be a spherical building with diagram of type* B_2, *with* $S = \{s, t\}$ *and parameters* q_s, q_t. *Then the sets* $\mathscr{A}^{\Phi_{sts}}$ *and* $\mathscr{A}^{\Phi_{sts} \cap \Phi_t}$ *are not empty and we have*

$$|\mathscr{A}^{\Phi_{sts}}| = (q_s q_t)^2 \frac{(q_t - 1)}{2}. \tag{4.4}$$

If $q_t > 2$ *then the set* $\mathscr{A}^{\Phi_{sts} \cap \Phi_t}$ *is the union of two disjoint subset* $(\mathscr{A}^{\Phi_{sts} \cap \Phi_t})'$ *and*

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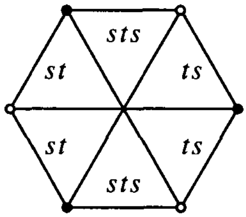

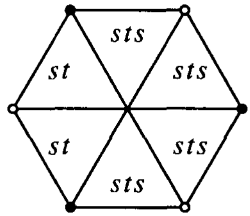

FIGURE 3.

$(\mathscr{A}^{\Phi_{sts}\cap\Phi_t})''$ such that

(4.5) $$|(\mathscr{A}^{\Phi_{sts}\cap\Phi_t})'| = (q_s q_t)^2 \frac{(q_t - 1)^2}{4},$$

(4.6) $$|(\mathscr{A}^{\Phi_{sts}\cap\Phi_t})''| = (q_s q_t)^2 \frac{(q_s - 1)(q_t - 1)(q_t - 2)}{2}.$$

The same formulas hold exchanging the roles of s and t.

PROOF. We argue as in the proof of Lemma 4.5 and use the same notation: starting from any $w \in W$ (the dihedral group of order eight), such that $l(stsw) < l(w)$ (respectively $l(stsw) < l(w)$ and $l(tw) < l(w)$) we first construct all simple galleries $(d_0, \dots, d_n)$, for $n = 4$, such that $\delta_c(d_0) = w$ and that for $0 \le i \le 4$, $\delta_c(d_i)$ satisfies the same inequality or inequalities.

Let $w = tst$ (this choice is convenient as $w \in \Phi_{sts} \cap \Phi_t$), and suppose that $(d_0, \dots, d_4)$ has type (t, s, t, s). Then applying at each step Proposition 1.2 and denoting by $\underline{w}$ the longest element $stst = tsts$ in W, one can verify that only the following four cases occur:

(1) $(d_0, \dots, d_4) \in \Gamma_{t,s,t,s}(tst, tst, \underline{w}, sts, st)$
(2) $(d_0, \dots, d_4) \in \Gamma_{t,s,t,s}(tst, tst, \underline{w}, \underline{w}, tst)$
(3) $(d_0, \dots, d_4) \in \Gamma_{t,s,t,s}(tst, tst, \underline{w}, sts, sts)$
(4) $(d_0, \dots, d_4) \in \Gamma_{t,s,t,s}(tst, tst, \underline{w}, \underline{w}, \underline{w})$

and that, respectively, for $\gamma'(d_0, d_4)$ we get:

(1′) $\gamma'(d_0, d_4) \in \Gamma_{s,t,s,t}(tst, \underline{w}, sts, st, st)$
(2′) $\gamma'(d_0, d_4) \in \Gamma_{s,t,s,t}(tst, \underline{w}, \underline{w}, tst, tst)$
(3′) $\gamma'(d_0, d_4) \in \Gamma_{s,t,s,t}(tst, \underline{w}, \underline{w}, \underline{w}, sts)$
(4′) either $\gamma'(d_0, d_4) \in \Gamma_{s,t,s,t}(tst, \underline{w}, sts, sts, \underline{w})$ or $\gamma'(d_0, d_4) \in \Gamma_{s,t,s,t}(tst, \underline{w}, \underline{w}, \underline{w}, \underline{w})$.

As $\Phi_{sts} = \{tst, \underline{w}, sts, st\}$, only case (1) leads to apartments in $\mathscr{A}^{\Phi_{sts}}$, and (4.4) follows as the correspondence between this set and the set of galleries as in (1) is a

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1-2 correspondence.

If $(d_0, \ldots, d_4)$ is as in (2) then δ_c maps $\operatorname{conv}\{d_0, d_4\}$ onto $\{tst, \underline{w}\} = \Phi_t \cap \Phi_{sts}$. Denote by $(\mathscr{A}^{\Phi_{sts}\cap\Phi_t})'$ the set of apartments like $\operatorname{conv}\{d_0, d_4\}$ and note that there is a 4-1 correspondence between $\Gamma_{t,s,t,s}(tst, tst, \underline{w}, \underline{w}, tst)$ and $(\mathscr{A}^{\Phi_{sts}\cap\Phi_t})'$. From this we obtain (4.5).

Suppose now that $(d_0, \ldots, d_4)$ is as in (3). In this case δ_c maps $\operatorname{conv}\{d_0, d_4\}$ onto $\{tst, \underline{w}, sts\} = \Phi_{sts} \cap \Phi_{tst}$. If we denote by $(\mathscr{A}^{\Phi_{sts}\cap\Phi_{tst}})'$ the set of apartments that we obtain from such galleries, then we get $|(\mathscr{A}^{\Phi_{sts}\cap\Phi_{tst}})'| = |\Gamma_{t,s,t,s}(tst, tst, \underline{w}, sts, sts)|$, as each apartment in $(\mathscr{A}^{\Phi_{sts}\cap\Phi_{tst}})'$ contains only one gallery in $\Gamma_{t,s,t,s}(tst, tst, \underline{w}, sts, sts)$.

Finally suppose that $(d_0, \ldots, d_4)$ is as in (4). In this case the image under δ_c of $\operatorname{conv}\{d_0, d_4\}$ is not uniquely determined by $(d_0, \ldots, d_4)$, the two cases in (4′) being possible. In the first case a careful look at the whole set of chambers shows that $\operatorname{conv}\{d_0, d_4\} \in (\mathscr{A}^{\Phi_{sts}\cap\Phi_{tst}})'$. Note also that in no other case, except the one already examined, does δ_c map $\operatorname{conv}\{d_0, d_4\}$ onto $\Phi_{sts} \cap \Phi_{tst}$. So $(\mathscr{A}^{\Phi_{sts}\cap\Phi_{tst}})' = \mathscr{A}^{\Phi_{sts}\cap\Phi_{tst}}$. In the second case we have again that δ_c maps $\operatorname{conv}\{d_0, d_4\}$ onto $\Phi_t \cap \Phi_{sts}$, but, as the cardinality of the fiber $\delta_c^{-1}(\underline{w}) \cap \operatorname{conv}\{d_0, d_4\}$ is different from the other case, the set of apartments that we obtain is disjoint from $(\mathscr{A}^{\Phi_{sts}\cap\Phi_t})'$. Denote by $(\mathscr{A}^{\Phi_{sts}\cap\Phi_t})''$ this new set of apartments. As each apartment in $(\mathscr{A}^{\Phi_{sts}\cap\Phi_t})''$ contains two galleries as in (4) while each apartment in $\mathscr{A}^{\Phi_{sts}\cap\Phi_{tst}}$ contains only one such gallery, then (4.6) follows from the equality

$$2|(\mathscr{A}^{\Phi_{sts}\cap\Phi_t})''| = |\Gamma_{t,s,t,s}(tst, tst, \underline{w}, \underline{w}, \underline{w})| - |\Gamma_{t,s,t,s}(tst, tst, \underline{w}, sts, sts)|.$$

To conclude our proof note that, by symmetry, we obtain formulas like (4.4), (4.5) and (4.6) by exchanging the roles of s and t. □

REMARK 4.7. If A is a fixed apartment in any building $\mathbf{X}$ with $c \in \mathscr{C}(A)$ as fundamental chamber and if $\{\Phi_i\}_{i\in I}$ is the set of negative half-apartments in A corresponding to the set of simple reflections, then writing $d \in \bigcap_{i\in I} \Phi_i$ is the same as stating that d is opposite to c [9, Theorem 2.16].

Let now $\mathbf{X}$ be a spherical building which is the direct product [9, p. 33] $\mathbf{X}_1 \times \cdots \times \mathbf{X}_n$ and let $A = A_1 \times \cdots \times A_n$ and $c = (c_1, c_2, \ldots, c_n)$ as above. Moreover, for $1 \le k \le n$ let I_k be the set of types of $\mathbf{X}_k$ and $\{\Phi_{i_k}\}_{i_k\in I_k}$ the corresponding set of negative half-apartments in A_k. As for a chamber $d = (d_1, \ldots, d_n)$ one has that $\mathbf{d}(c, d) = \operatorname{diam}\mathbf{X}$ if and only if $\mathbf{d}(c_k, d_k) = \operatorname{diam}\mathbf{X}_k$ for $1 \le k \le n$, then the following equality holds

$$|\mathscr{A}^{\bigcap_{i\in I}\Phi_i}| = |\mathscr{A}_1^{\bigcap_{i_1\in I_1}\Phi_{i_1}}| \cdots |\mathscr{A}_n^{\bigcap_{i_n\in I_n}\Phi_{i_n}}|.$$

REMARK 4.8. By Proposition 4.1, writing $\left\{A \cap lk(\sigma) : A \in \mathscr{A}_\sigma^{\Phi}\right\}$ is the same as writing $\{A \cap lk(\sigma) : A \in \mathscr{A}_\sigma\}^{\Phi\cap W_\sigma}$.

In the sequel we will simply write m for m_Q.

COROLLARY 4.9. *Let* **X** *an affine building with diagram of type* $\widetilde{A}_2$ *and parameter* q. *For each* $i, j = 0, 1, 2$ *such that* $i < j$ *the following equalities hold:*

$$m(\mathscr{A}^{\Phi_i}) = \frac{(q-1)}{2}, \tag{4.7}$$

$$m(\mathscr{A}^{\Phi_{iji}}) = \frac{(q-1)}{2}, \tag{4.8}$$

$$m(\mathscr{A}^{\Phi_i \cap \Phi_{iji}}) = m(\mathscr{A}^{\Phi_j \cap \Phi_{iji}}) = \frac{(q-1)^2}{2}. \tag{4.9}$$

PROOF. First note that if $\sigma = \partial\Phi \cap c$ then $\mathscr{A}^{\Phi} = \mathscr{A}^{\Phi}_{\sigma'}$ for each $\sigma' \subseteq \sigma$. Then, as $\sigma_i = \partial\Phi_i \cap c$ and as $lk(\sigma_i)$ is a building of rank one (and diameter one), (4.7) follows from (2.2) and from Lemma 4.2. Analogously, for each fixed i, j, if x is the vertex of c of type $I \setminus \{i, j\}$ then $lk(x)$ is of type A_2 and $\partial\Phi_i$, $\partial\Phi_j$, $\partial\Phi_{iji}$ are the three walls containing x. Therefore (4.8), (4.9) follow easily respectively from (4.2), (4.3) and from (2.2). □

COROLLARY 4.10. *Let* **X** *be an affine building with diagram of type* $\widetilde{B}_2$ *and suppose that the vertex of* c *labelled by* x_0 *is such that* $lk(x_0)$ *has diagram of type* $A_1 \times A_1$. *If* $i \in I \setminus \{0\}$, *then the following formulas hold:*

$$m(\mathscr{A}^{\Phi_i}) = \frac{(q_i-1)}{2}, \tag{4.10}$$

$$m(\mathscr{A}^{\Phi_{0i0}}) = \frac{(q_i-1)}{2}, \tag{4.11}$$

$$m((\mathscr{A}^{\Phi_{0i0} \cap \Phi_i})') = \frac{(q_i-1)^2}{4}, \tag{4.12}$$

$$m((\mathscr{A}^{\Phi_{0i0} \cap \Phi_i})'') = \frac{(q_0-1)(q_i-1)(q_i-2)}{2}. \tag{4.13}$$

Moreover

$$m(\mathscr{A}^{\Phi_1 \cap \Phi_2}) = \frac{(q_1-1)(q_2-1)}{4}, \tag{4.14}$$

$$m(\mathscr{A}^{\Phi_{010} \cap \Phi_{020}}) = \frac{q_0(q_1-1)(q_2-1)}{4}. \tag{4.15}$$

PROOF. To prove (4.10) we argue as in the proof of (4.7). Fix $i \in I \setminus \{0\}$ and let $j \in I \setminus \{0, i\}$. As $lk(x_j)$ is of type B_2 then (4.11), (4.12), (4.13) follow respectively from (4.4), (4.5), (4.6), with $s_0 = s$ and $s_i = t$. To prove (4.14) note that $\{s_i\}_{i=1,2}$ is the set of simple reflections in W_{x_0}. Then $\mathscr{A}^{\Phi_1\cap\Phi_2} = \mathscr{A}_{x_0}^{\Phi_1\cap\Phi_2}$ and

$$m(\mathscr{A}^{\Phi_1\cap\Phi_2}) = \left|\left\{A \cap lk(x_0) : A \in \mathscr{A}_{x_0}\right\}^{(\Phi_1\cap\Phi_2)\cap W_{x_0}}\right| (q_1q_2)^{-1}$$

As $lk(x_0)$ is the direct product of two spherical buildings of rank one then (4.14) follows from Remark 4.7 and from Lemma 4.2 (with $s = s_i$).

To prove (4.15) note that as s_0 interchanges Φ_0 and Φ_0^+ we have that $\Phi_{0i0} = s_0\Phi_i$, $i = 1, 2$. In other words, Φ_{010} and Φ_{020} are the negative half-apartments which correspond to the simple reflections in $s_0 W_{x_0} s_0 = W_{s_0x_0}$ and hence $\Phi_{010} \cap \Phi_{020} = \{d \in \mathscr{C}(A_0) : d(\operatorname{proj}_{s_0x_0} d, s_0c) = \operatorname{diam} lk(s_0x_0)\}$. Denote by c_k, $1 \le k \le q_0$, the generic chamber in $\mathbf{X}$ 0-adjacent to c, and let x_0^k be its vertex of type 0. Then $\rho(c_k) = s_0c$ and $\rho(x_0^k) = s_0x_0$. Now we show that for each $A \in \mathscr{A}^{\Phi_{010}\cap\Phi_{020}}$ there exists a unique $k \in \{1, \dots, q_0\}$ such that for each $d \in \mathscr{C}(A)$ $\operatorname{proj}_{\sigma_0} d = c_k$ and $x_0^k \in A$. First note that $\Phi_{010} \cap \Phi_{020} \subset \Phi_0$. In fact if $d \in \Phi_{010} \cap \Phi_{020}$ then $d \notin \Phi_0^+$, otherwise d would be opposite to s_0c, contradicting the fact that $\mathbf{X}$ is affine [11, Theorem 2.36]. Hence, if $A \in \mathscr{A}^{\Phi_{010}\cap\Phi_{020}}$, for each $d \in \mathscr{C}(A)$ then $\operatorname{proj}_{\sigma_0} d = c_k$ for some k, (otherwise $\operatorname{proj}_{\sigma_0} d = c$ and $\rho(d) \in \operatorname{proj}_{\sigma_0}^{-1} c \cap A_0 = \Phi_0^+$). Moreover if d and d' are two distinct chambers in A such that $\operatorname{proj}_{\sigma_0} d \neq \operatorname{proj}_{\sigma_0} d'$, then for the convexity of apartments we have that $\sigma_0 \in A$ and then $\rho(A) = \Phi_0$ which strictly contains $\Phi_{010} \cap \Phi_{020}$.

As the retraction ρ preserves i-adjacency and the distance from c it follows that

$$\left\{A \in \mathscr{A}_{x_0^k} : \forall d \in \mathscr{C}(A) \quad d(\operatorname{proj}_{x_0^k} d, c_k) = \operatorname{diam} lk(x_0^k)\right\} = \mathscr{A}_{x_0^k}^{\Phi_{010}\cap\Phi_{020}},$$

and

$$\mathscr{A}^{\Phi_{010}\cap\Phi_{020}} = \bigcup_{1\le k\le q_0} \mathscr{A}_{x_0^k}^{\Phi_{010}\cap\Phi_{020}},$$

where the union is disjoint.

Moreover for each $1 \le k \le q_0$

$$m(\mathscr{A}_{x_0^k}^{\Phi_{010}\cap\Phi_{020}}) = m(\mathscr{A}_{s_0x_0}^{\Phi_{010}\cap\Phi_{020}}) = m(\mathscr{A}_{x_0}^{\Phi_1\cap\Phi_2}),$$

and we get (4.15) by taking the sum over k. □

Let now $\mathbf{X}$ be a building with diagram of type $\widetilde{A}_2$. We introduce the bounded operator $M : L^\infty(\mathscr{A}) \to L^\infty(\mathscr{C})$ which we obtain by radializing with respect to m the inversion formula which follows from the application of Lemma 3.1 to the right-hand

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side of (3.2): for each $\varphi \in L^\infty(\mathscr{A})$ we set

$$M\varphi(c) = \left\{ \int_{\mathscr{A}^{A_0}} - \sum_{0\le i\le 2} \frac{1}{(q-1)} \int_{\mathscr{A}^{\Phi_i}} - \sum_{0\le i<j\le 2} \left[\frac{1}{3(q-1)} \int_{\mathscr{A}^{\Phi_{ij\,i}}} - \frac{1}{3(q-1)^2} \left(\int_{\mathscr{A}^{\Phi_i \cap \Phi_{ij\,i}}} + \int_{\mathscr{A}^{\Phi_j \cap \Phi_{ij\,i}}} \right) \right] \right\} \varphi(A) dm(A).$$

Formulas (4.7), (4.8), (4.9) show indeed that $M\varphi \in L^\infty(\mathscr{C})$ and that $\|M\| \le 4$.

LEMMA 4.11. *Let* $\mathbf{X}$ *be a building with diagram of type* $\widetilde{A}_2$ *and let* $d \in \mathscr{C}$. *If* $\rho(d) \in \Phi$ *and* $i, j \in I$, $i < j$ *then*

$$q^{\mathrm{d}(c,d)} m(\mathscr{A}^\Phi \cap \mathscr{A}_d) =$$

$$(4.16) \qquad = \begin{cases} 1 & \text{if } \Phi = A_0; \\ (q-1) & \text{if } \Phi = \Phi_i; \\ (q-1) & \text{if } \Phi = \Phi_{ij\,i}; \\ (q-1)^2 & \text{if } \Phi = \Phi_i \cap \Phi_{ij\,i} \text{ or } \Phi = \Phi_j \cap \Phi_{ij\,i},\ \rho(d) \notin \Phi_i \cap \Phi_j; \\ 2(q-1)^2 & \text{if } \Phi = \Phi_i \cap \Phi_{ij\,i} \text{ or } \Phi = \Phi_j \cap \Phi_{ij\,i},\ \rho(d) \in \Phi_i \cap \Phi_j. \end{cases}$$

PROOF. If $\Phi = A_0$ then, as $\mathscr{A}^{A_0} = \mathscr{A}_c$, the assertion follows immediately from the definition of m. In the other cases let $\sigma = \partial\Phi \cap c$. As $\rho(d) \in \Phi$ if and only if $\rho(\mathrm{proj}_\sigma) \in \Phi$, we can argue by induction on the length of a minimal gallery from σ to d. This suffices to prove the assertion for $d \in \mathscr{C}(lk(\sigma))$.

Denote by $n(\sigma, \Phi, d)$ the number of apartments in $\{A \cap lk(\sigma) : A \in \mathscr{A}^\Phi\}$ containing d, and let $w \in W_\sigma$ be such that $\delta_c(d) = w$. Then as for each $E \in \{A \cap lk(\sigma) : A \in \mathscr{A}^\Phi\}$

$$n(\sigma, \Phi, d) = \frac{\left|\{A \cap lk(\sigma) : A \in \mathscr{A}^\Phi\}\right| |\delta_c^{-1}(w) \cap \mathscr{C}(E)|}{|\delta_c^{-1}(w)|},$$

we have

$$m(\mathscr{A}^\Phi \cap \mathscr{A}_d) = \frac{n(\sigma, \Phi, d)}{N(1, W_\sigma)} = \frac{m(\mathscr{A}^\Phi)|\delta_c^{-1}(w) \cap \mathscr{C}(E)|}{|\delta_c^{-1}(w)|}.$$

As $\mathbf{X}$ is homogeneous we have that $|\delta_c^{-1}(w)| = q^{\mathrm{d}(c,d)}$. Then, case by case, by Corollary 4.9, to complete the proof we have to look only for the value of $|\delta_c^{-1}(w) \cap \mathscr{C}(E)|$. If $\Phi = \Phi_i$, then $\sigma = \sigma_i$. The assertion follows as δ_c maps each of the two chambers, of each apartment in $\{A \cap lk(\sigma) : A \in \mathscr{A}^\Phi\}$, onto $\{s_i\} = \Phi_i \cap W_{\sigma_i}$. If $\Phi = \Phi_{ij\,i}$, $\Phi_{ij\,i} \cap \Phi_i$ or $\Phi_{ij\,i} \cap \Phi_j$, then σ is equal to the vertex x of c of type $I \setminus \{i, j\}$. In each of these cases for each $w \in W_x \cap \Phi$ we can recover the value of

$|\delta_c^{-1}(w) \cap \mathscr{C}(E)|$ from the discussion in the proof of Lemma 4.5 where we have to put $s = s_i$ and $t = s_j$. Note only that writing $\rho(d) \in \Phi_i \cap \Phi_j$ is another way of stating that $\delta_c(d) = s_i s_j s_i = s_j s_i s_j$. □

PROPOSITION 4.12. *Let* $\mathbf{X}$ *be a building with diagram of type* $\widetilde{A_2}$ *and let* $g \in L^1(\mathscr{C}(A))$ *for each* $A \in \mathscr{A}$. *Then*

$$MRg = g. \tag{4.17}$$

PROOF. For fixed c in $\mathscr{C}$ we will prove (4.17) at c. If d is any chamber in $\mathbf{X}$ denote by $\mathbf{1}_d$ the Dirac function in d. As d ranges in $\mathscr{C}$, the set $\{\mathbf{1}_d\}$ is a basis for the space of functions that are summable on each apartment of $\mathbf{X}$. Then for the linearity of MR it will be sufficient to prove that for each $d \in \mathscr{C}$, (4.17) holds when $g = \mathbf{1}_d$.

If $d = c$ then we use (4.16) with $\Phi = A_0$ to prove that each side of (4.17) is equal to 1. Let now d be any chamber distinct from c, then, up to considering its image under the retraction ρ_{c,A_0}, we can suppose that $d \in \mathscr{C}(A_0)$. Therefore $d \in \Phi_k$ for some $k \in I$. Note that from [11, Proposition 2.32] this holds if and only if $\mathrm{proj}_{\sigma_k} d = s_k c$. Then, denoting by x_{k^-} and x_{k^+} the vertices of c respectively of type $k-1 \mod 3$ and $k+1 \mod 3$, as $\sigma_k = (x_{k^-}, x_{k^+})$ and as $\mathrm{proj}_{x_{k^\pm}}$ is contained in the convex hull of σ_k and d, we have that for $d \in \Phi_k$ only the following cases are possible:

(1) $\mathrm{proj}_{x_{k^-}} d = \mathrm{proj}_{\sigma_k} d = \mathrm{proj}_{x_{k^+}} d$;
(2) $\mathrm{proj}_{x_{k^-}} d = \mathrm{proj}_{\sigma_k} d \neq \mathrm{proj}_{x_{k^+}} d$;
(3) $\mathrm{proj}_{x_{k^-}} d \neq \mathrm{proj}_{\sigma_k} d = \mathrm{proj}_{x_{k^+}} d$;
(4) $\mathrm{proj}_{x_{k^-}} d \neq \mathrm{proj}_{\sigma_k} d$ and $\mathrm{proj}_{x_{k^+}} d \neq \mathrm{proj}_{\sigma_k} d$.

As s_k interchanges Φ_k and Φ_k^+, then applying again [11, Proposition 2.32] we obtain that

$$\mathrm{proj}_{x_{k^-}}^{-1}(s_k C) = s_k \Phi_{k^+}^+ \cap s_k \Phi_k^+ = \Phi_{kk^+k}^+ \cap \Phi_k$$

and

$$\mathrm{proj}_{x_{k^+}}^{-1}(s_k C) = s_k \Phi_{k^-}^+ \cap s_k \Phi_k^+ = \Phi_{kk^-k}^+ \cap \Phi_k.$$

If d satisfies the equalities in (1) then $d \in \mathrm{proj}_{x_{k^-}}^{-1}(s_k C) \cap \mathrm{proj}_{x_{k^+}}^{-1}(s_k C)$ and then $d = s_k c = \cap_{i \in I} s_k \Phi_i^+$. Hence we can use (4.16) with $\Phi = A_0$ and $\Phi = \Phi_k$ to prove that each side of (4.17) vanishes. If d satisfies (2), then being $d \in s_k \Phi_{k^+}^+ \cap s_k \Phi_k$ and $d \notin s_k \Phi_{k^-}^+ \cap s_k \Phi_k$ we have that $d \in s_k \Phi_{k^-} \cap \Phi_k$, and then we use (4.16) with $\Phi = A_0$, Φ_k and Φ_{kk^-k} (considering the two cases $d \in \Phi_k \cap \Phi_{k^-}$ and $d \notin \Phi_k \cap \Phi_{k^-}$) to conclude that also in this case each side of (3.19) is equal to 0. Case (3) is symmetric to (2), so we obtain the same result by arguing as above with k^- replaced by k^+. Finally, the inequalities in (4) imply that $d \in \Phi_{kk^-k} \cap \Phi_{kk^+k}$. Observe that the wall $\partial\Phi_{kk\text{-}k}$ (respectively $\partial\Phi_{kk+k}$) is parallel to $\partial\Phi_{k+}$ (respectively $\partial\Phi_{k\text{-}}$). Hence $\Phi_{kk\text{-}k} \subset \Phi^+_{k+}$, (respectively $\Phi_{kk+k} \subset \Phi^+_{k\text{-}}$). Then if $d \in \Phi_{kk\text{-}k} \cap \Phi_{kk+k}$ it cannot happen that $d \in \Phi_k \cap \Phi_{k\text{-}}$ or $d \in \Phi_k \cap \Phi_{k+}$ and we use (4.16) with $\Phi = A_0, \Phi_k, \Phi_{kk\text{-}k}$ and Φ_{kk+k} ($d \notin \Phi_k \cap \Phi_{k\text{-}}$ and $d \notin \Phi_k \cap \Phi_{k+}$) to prove that each side of (4.17) equals 0 and hence to conclude the proof. □

Let now $\mathbf{X}$ be a building of type $\widetilde{B}_2$. In analogy with the previous case we introduce the bounded operator $M : L^\infty(\mathscr{A}) \to L^\infty(\mathscr{C})$ defined by

$$M\varphi(c) = \left\{ \int_{\mathscr{A}^{A_0}} - \sum_{i=1,2} \left[\frac{1}{(q_i-1)} \left(\int_{\mathscr{A}^{\Phi_i}} + \int_{\mathscr{A}^{\Phi_{0i0}}} \right) - \frac{1}{(q_i-1)^2} \int_{(\mathscr{A}^{\Phi_i \cap \Phi_{0i0}})'} \right] + \frac{1}{(q_1-1)(q_2-1)} \int_{\mathscr{A}^{\Phi_1 \cap \Phi_2}} + \frac{1}{q_0(q_1-1)(q_2-1)} \int_{\mathscr{A}^{\Phi_{010} \cap \Phi_{020}}} \right\} \varphi(A)\,dm(A).$$

Denote by T_i, $i = 0, 1, 2$ the operator defined on the space of complex-valued functions on $\mathscr{C}$, by

$$T_i f(c) = \frac{1}{q_i} \sum_{\substack{c' \underset{i}{\sim} c \\ c' \neq c}} f(c').$$

PROPOSITION 4.13. *For each $f \in L^1(\mathscr{C}(A))$ we get*

$$MRf = (1 + T_0)f. \tag{4.18}$$

Before proving Proposition 4.13 we need the following

LEMMA 4.14. *Let $d \in \mathscr{C}$ and $\rho(d) \in \Phi \subset A_0$ then*

$$|\delta_c^{-1}(\delta_c(d))|\, m(\mathscr{A}^\Phi \cap \mathscr{A}_d) = \begin{cases} 1 & \text{if } \Phi = A_0; \\ (q_i - 1) & \text{if } \Phi = \Phi_i,\ i = 1, 2; \\ (q_i - 1) & \text{if } \Phi = \Phi_{0i0},\ i = 1, 2; \\ (q_1 - 1)(q_2 - 1) & \text{if } \Phi = \Phi_1 \cap \Phi_2; \\ q_0(q_1 - 1)(q_2 - 1) & \text{if } \Phi = \Phi_{010} \cap \Phi_{020}. \end{cases} \tag{4.19}$$

Moreover if $\Phi = \Phi_i \cap \Phi_{0i0}$ then

$$|\delta_c^{-1}(\delta_c(d))|\, m((\mathscr{A}^\Phi)' \cap \mathscr{A}_D) = (q_i - 1)^2. \tag{4.20}$$

PROOF. We use essentially the same argument as in the proof of Lemma 4.11. So in the same way we prove the assertion when $\Phi = A_0$ and $\Phi = \Phi_i$. Similarly we can prove (4.19) when $\Phi = \Phi_{0i0}$ and (4.20) (apply Corollary 4.10 and see respectively the proof of (4.4) and (4.5) to find out the cardinality of the fiber $\delta_c^{-1}(w) \cap A$ for each $w \in \Phi$ and $A \in \mathscr{A}^\Phi$). When $\Phi = \Phi_1 \cap \Phi_2$ the assertion follows again from Corollary 4.9 and the fact that all four chambers in $W_{x_0} = \{1, s_1, s_2, s_1 s_2 = s_2 s_1\}$ retract onto $\{s_1 s_2\}$. When $\Phi = \Phi_{010} \cap \Phi_{020}$ then we know from Corollary 4.10 that for any fixed chamber d such that $\rho(d) \in \Phi$, there exist a unique chamber c_k, $k \in \{1, \dots, q_0\}$, 0-adjacent to c such that $\mathbf{d}(c_k, d) = \mathbf{d}(c, d) - 1$, (indeed $\operatorname{proj}_{\sigma_0} d$), and such that if x_0^k is its vertex of type 0, then

$$\mathscr{A}^\Phi \cap \mathscr{A}_d = \mathscr{A}^\Phi_{x_0^k} \cap \mathscr{A}_d.$$

Then by symmetry with the case $\Phi = \Phi_1 \cap \Phi_2$ the assertion follows as $|\delta_c^{-1}(\delta_c(d))| = q_0 |\delta_{c_k}^{-1}(\delta_{c_k}(d))|$. □

PROOF OF PROPOSITION 4.13. We proceed and use the same notation as in the proof of Proposition 4.12. Note that, as $\{c, s_0 c\} = \left(\cap_{i=1,2} \Phi_i^+\right) \cap \left(\cap_{i=1,2} \Phi_{0i0}^+\right)$ then $\mathscr{C}(A_0) \setminus \{c, s_0 c\} = \left(\cup_{i=1,2} \Phi_i\right) \cup \left(\cup_{i=1,2} \Phi_{0i0}\right)$. Note also that, as $\partial\Phi_1$ is parallel to $\partial\Phi_{020}$ and $\partial\Phi_2$ is parallel to $\partial\Phi_{010}$, then $\Phi_1 \cap \Phi_{020} = \Phi_2 \cap \Phi_{010} = \emptyset$. Then if $f = \mathbf{1}_c$ (respectively $f = \mathbf{1}_{s_0 c}$) we can use (4.19) only in the case $\Phi = A_0$ to prove that each side of (4.18) is equal to 1 (respectively to $(q_0)^{-1}$). Let $f = \mathbf{1}_d$, $d \in \mathscr{C}(A_0) \setminus \{c, s_0 c\}$ and suppose for example that $d \in \Phi_1$. Then, as one can easily verify, only the following three cases can occur:

(1) $d \in \Phi_2$;
(2) $d \in \Phi_{010}$;
(3) $d \in \Phi_2^+ \cap \Phi_{010}^+$.

In the first case we use (4.19) with $\Phi = \Phi_1$ and $\Phi = \Phi_1 \cap \Phi_2$ to prove that each side of (4.18) vanishes. In the second case we use (4.19) with $\Phi = \Phi_1$, $\Phi = \Phi_{010}$ and (4.20) with $i = 1$ and we obtain again that each side of (4.18) is equal to 0. Finally in the third case we can use (4.19) only in the case $\Phi = \Phi_1$ to conclude that also in this case each side of (4.18) is equal to 0.

Similarly one can argue in the other possible cases ($d \in \Phi_2, \dots$), and applying Lemma 4.14 one proves that (4.18) holds for $f = \mathbf{1}_d$ for each $d \in \mathscr{C}(A_0)$ (then in $\mathbf{X}$). By linearity and continuity, (4.18) follows for an arbitrary f. □

LEMMA 4.15. *For $i = 0, 1, 2$, $(1 + T_i)^{-1}$ is given by*

$$\frac{1}{2(q_i - 1)} [(2q_i - 1) - q_i T_i].$$

PROOF. First prove that $T_i^2 = (1/q_i) + (1 - 1/q_i) T_i$. Then the assertion follows by direct computation. □

We have just proved the following proposition.

PROPOSITION 4.16. *Let* **X** *be any building of type* $\widetilde{B}_2$. *Then the Radon transform* R *is inverted by*

$$\frac{1}{2(q_0-1)}\left[(2q_0-1)-q_0T_0\right]MRf = f. \tag{4.21}$$

REMARK 4.17. Applying the same techniques, an inversion formula of the same kind of (4.21) can be obtained for a building with diagram of type $\widetilde{G}_2$. Moreover the existence of simpler and more general inversion formulas can be proved [3] for 'large enough' values of parameters.

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