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Exotic Fusion Systems Related to Sporadic Simple Groups

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Abstract

We describe several exotic fusion systems related to the sporadic simple groups at odd primes. More generally, we classify saturated fusion systems supported on Sylow 3-subgroups of the Conway group Co_1 and the Thompson group F_3, and a Sylow 5-subgroup of the Monster M, as well as a particular maximal subgroup of the latter two p-groups. This work is supported by computations in MAGMA.

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1. *Introduction*

A fusion system over a finite p-group S is a category satisfying certain conditions modeled on properties of finite groups and the internal actions associated to their Sylow p-subgroups. The typical example of a fusion system arises just like this: as the *p-fusion category* of a finite group. In this case, certain additional conditions are satisfied which may be abstracted as additional axioms, defining the class of *saturated* fusion systems. However, not all saturated fusion systems can be realised as the p-fusion category of some finite group, giving rise to *exotic* fusion systems.

Over the course of this work, we completely classify all saturated fusion systems supported on Sylow 3-subgroups of the Conway group Co_1 and the Thompson group F_3, and a Sylow 5-subgroup of the Monster M. In addition, we also classify saturated fusion systems supported on a particular maximal subgroup of a Sylow 3-subgroup of F_3, and of a Sylow 5-subgroup of M. Of particular interest in this determination is the occurrence of several exotic fusion systems supported on these p-groups. In total we uncover sixteen new exotic systems up to isomorphism, seven of which are simple, giving a rich source of reasonably complicated examples.

We have not yet considered the implications of these new exotic fusion systems to any of the areas in which fusion systems have application (see [6] for a survey), and have studied them purely for their interesting structural properties, and for their appearance in other ongoing classification programs concerning fusion systems. Since exotic fusion systems themselves are still poorly understood, at this moment a considerable amount of attention is just focused on determining new families of examples with the ultimate goal of discerning exotic fusion systems from those occurring as p-fusion categories of finite groups, without having to rely on heavy machinery from finite group theory.

Our first main result is the following, and is proved via Proposition 4·14, Theorem 4·15 and Theorem 4·16:

THEOREM A. *Let $\mathcal{F}$ be a saturated fusion system on a 3-group S with S isomorphic to a Sylow 3-subgroup of Co_1. If $O_3(\mathcal{F}) = \{1\}$ then $\mathcal{F}$ is isomorphic to the 3-fusion category of Co_1, $Sp_6(3)$ or $Aut(Sp_6(3))$.*

We point out that the 3-fusion system of Co_1 has been identified by work of Oliver [**33**, theorem A] but from a different starting point than what is considered in this paper. We remark that the proofs of [**33**, theorem A] and Theorem A do not depend on each other, however they both reduce to a situation where one has strong information about the local actions in the fusion system. At this point, either paper could use the other's result but yet again, different (and complementary) approaches are taken to prove the uniqueness of the fusion system of Co_1.

We now move on to the construction of some exotic fusion systems. We use the same methodology to prove Theorem B and Theorem C, although the arguments vary slightly depending on the structure of the underlying p-group S. The author first encountered the systems in Theorem B while classifying certain fusion systems which contain only two essential subgroups [**45**]. These systems arise as a fusion theoretic generalisation of weak BN-pairs of rank 2, a collection of amalgams classified by work of Delgado and Stellmacher [**16**]. In Theorem B one of the exotic systems we uncover arises as a fusion system "completion" of an amalgam of F_3-type, as defined in [**16**]. In the case of the group F_3, the corresponding amalgam generates the entire group. This is in contrast to the fusion system case, where the 3-fusion category of F_3 requires another set of 3-local actions, corresponding to the maximal subgroups of F_3 of shape $3^5 : 2.\mathrm{Sym}(6)$, to be properly generated.

THEOREM B. *Let $\mathcal{F}$ be a saturated fusion system on a 3-group S with S isomorphic to a Sylow 3-subgroup of F_3. If $O_3(\mathcal{F}) = \{1\}$ then either $\mathcal{F}$ is isomorphic to the 3-fusion category of F_3; or $\mathcal{F}$ is isomorphic to one of two exotic examples. In all cases, $\mathcal{F}$ is simple.*

THEOREM C. *Let $\mathcal{F}$ be a saturated fusion system on a 5-group S with S isomorphic to a Sylow 5-subgroup of M. If $O_5(\mathcal{F}) = \{1\}$ then either $\mathcal{F}$ is isomorphic to the 5-fusion category of M; or $\mathcal{F}$ is isomorphic to one of three exotic examples. Two of the three exotic fusion systems are simple.*

Theorem B is proved as Theorem 5·18 while Theorem C is proved as Theorem 6·24 and Theorem 6·25. We note that the process by which we construct some of the systems in Theorem B and Theorem C can also be applied to the 2-fusion category of J_3. In this application, one obtains three proper saturated subsystems, all of which contain no non-trivial normal 2-subgroups. However, unlike the odd prime cases, the subsystems recovered are realizable by finite groups. Indeed, these subsystems are isomorphic to the 2-fusion categories of J_2, $\mathrm{PSL}_3(4) : 2$ and $\mathrm{PGL}_3(4) : 2$, as demonstrated in [**30**, theorem 4·8].

Interestingly, we record that some of the exotic fusion systems described in Theorem B and Theorem C contain a unique proper non-trivial strongly closed subgroup which does not support a normal fusion subsystem. In both cases, this strongly closed subgroup is the centraliser of the second center of the Sylow p-subgroup, and is also essential in the fusion system. This is another instance where fusion systems seem to depart from the conventions of finite simple groups. As witnessed in [**18**, corollary 1·4], if G is a finite simple group with a non-trivial strongly closed subgroup A then $N_G(A)$ controls strong G-fusion in $S \in \mathrm{Syl}_p(G)$ and so $\mathcal{F}_S(G)$ is not simple in this instance.

Where we have a proper non-trivial strongly closed subgroup T, we are able to descend to exotic subsystems supported on T, and we speculate that this may be an illustration of a more generic method to construct exotic subsystems of exotic fusion systems. The examples we obtain in the theorems below arise in this fashion.

THEOREM D. *Let S be isomorphic to a Sylow 3-subgroup of F_3. Then, up to isomorphism, there are two saturated fusion system supported on $C_S(Z_2(S))$ in which $C_S(Z_2(S))$ is not normal. Both of these systems are exotic and only one is simple.*

THEOREM E. *Let S be isomorphic to a Sylow 5-subgroup of M. Then, up to isomorphism, there are nine saturated fusion system supported on $C_S(Z_2(S))$ in which $C_S(Z_2(S))$ is not normal. All of these systems are exotic and two are simple.*

The exotic fusion systems described in Theorem D and Theorem E are reminiscent of the exotic fusion systems supported on p-groups of maximal class, as determined in [**23**]. There, in almost all cases where $\mathcal{F}$ is an exotic fusion system, there is a class of essential subgroups which are *pearls*: essential subgroups isomorphic to p^2 or p_+^{1+2}. It is clear that for a fusion system $\mathcal{F}$ with a pearl P, $O^{p'}(\mathrm{Out}_{\mathcal{F}}(P)) \cong \mathrm{SL}_2(p)$ and so these occurrences are strongly connected to certain pushing up configurations in local group theory. In our case, the analogous set of essential subgroups P are of the form $p^4 \times C_P(O^{p'}(\mathrm{Out}_{\mathcal{F}}(P)))$ where $O^{p'}(\mathrm{Out}_{\mathcal{F}}(P)) \cong \mathrm{SL}_2(p^2)$, and in one of our cases $C_P(O^{p'}(\mathrm{Out}_{\mathcal{F}}(P)))$ is non-trivial. We speculate that both the systems containing pearls and our examples are part of a much larger class of exotic fusion systems which arise as the odd prime counterparts to "obstructions to pushing up" in the sense of Aschbacher [**3**]. A clear understanding of this would go some way to explaining the dearth of exotic fusion systems at the prime 2.

With this work, we move closer to classifying all saturated fusion systems supported on Sylow p-subgroups of the sporadic simple groups, for p an odd prime, complementing several other results in the literature. Indeed, all that remains is the study of saturated fusion systems on Sylow 3-subgroups of the Fischer groups, the Baby Monster and the Monster. For the reader's convenience, we tabulate the known results with regards to fusion systems on Sylow p-subgroups of sporadic simple groups in Table 1.

We remark that, perhaps aside from the Sylow 3-subgroup of Fi_{22}, the remaining cases are large and complex enough that it is laborious and computationally expensive to verify any results using the fusion systems package in MAGMA [**8, 36**]. Throughout this work, we lean on a small portion of these algorithms for the determination of the essentials subgroups of the saturated fusion systems under investigation (as in Proposition 4·3), although the techniques used in [**35, 43**] could be employed here instead. We record that several of the main theorems have been verified using the full potential of this MAGMA package. However, we believe it is important to provide handwritten arguments in order to exemplify some of the more interesting structural properties of the fusion systems described within, while simultaneously elucidating some of the computations performed implicitly by the MAGMA package. For the sake of brevity, the MAGMA code we use is not included here and has instead been relegated to an alternate version of this paper [**44**, appendix A].

Table 1. Fusion systems on non-abelian Sylow p-subgroups of sporadic groups for p odd

Simple Group	$\|S\|$	Reference	#Exotic Systems Supported
$^2F_4(2)'$, J_2, J_4, M_{12}, M_{24}, Ru, He	3^3	[40]	0
J_3	3^5	[36]	0
Co_1	3^9	Section 4	0
Co_2, McL	3^6	[7]	0
Co_3	3^7	[36]	0
Fi_{22}	3^9	–	Open
Fi_{23}, B	3^{13}	–	Open
Fi_{24}'	3^{16}	–	Open
Suz, Ly	3^7	[36]	0
HN	3^6	[36]	0
Th	3^{10}	Section 5	2
M	3^{20}	–	Open
Co_1	5^4	[25, 31, 32]	24
Co_2, Co_3, Th, HS, McL, Ru	5^3	[40]	0
HN, Ly, B	5^6	[35]	0
M	5^9	Section 6	4
Fi_{24}', He, O'N	7^3	[40]	3
M	7^6	[35]	27
J_4	11^3	[40]	0
M	13^3	[40]	0

Our notation and terminology for finite groups is a jumble of conventions from [4, 20, 27], and we hope that our usage will be clear from context. With regards to notation concerning the sporadic simple groups, we will generally follow the Atlas [13] with the exception of Thompson's sporadic simple group, which we refer to as F_3 instead of the usual Th, except in Table 1. We make this choice to emphasise the connection with "amalgams of type F_3" as defined in [16]. For fusion systems, we almost entirely follow the conventions in [5].

2. *Preliminaries: groups*

We start with some elementary observations regarding the Thompson subgroup of a finite p-group and the related notion of failure to factorise modules. For a more in depth account of this phenomena, see [27, section 9·2].

Definition 2·1. Let S be a finite p-group. Set $\mathcal{A}(S)$ to be the set of all elementary abelian subgroup of S of maximal rank. Then the Thompson subgroup of S is defined as $J(S) := \langle A \mid A \in \mathcal{A}(S)\rangle$.

PROPOSITION 2·2. *Let S be a non-trivial finite p-group. Then the following hold*:

(i) J(S) is a non-trivial characteristic subgroup of S;

(ii) $\Omega_1(C_S(J(S))) = \Omega_1(Z(J(S))) = \bigcap_{A\in\mathcal{A}(S)} A$; *and*

(iii) if $J(S) \le T \le S$, *then* $J(S) = J(T)$.

See [27, 9·2·8] for parts (i) and (iii). Additionally, by part (d) of that result, we see that $\Omega_1(C_S(J(S))) \le \Omega_1(Z(J(S)))$. Since $Z(J(S)) \le C_S(J(S))$, it is clear that $\Omega_1(C_S(J(S))) = \Omega_1(Z(J(S)))$.

Let $a \in \bigcap_{A \in \mathcal{A}(S)} A$. Then a has order p and $[a, A] = \{1\}$ for all $A \in \mathcal{A}(S)$. By definition, $[a, J(S)] = \{1\}$ so that $a \in \Omega_1(C_S(J(S)))$ and $\bigcap_{A \in \mathcal{A}(S)} A \le \Omega_1(C_S(J(S)))$. Now, for $x \in C_S(J(S))$ of order p, we have that $x \le C_S(J(S)) \le C_S(A)$ for all $A \in \mathcal{A}(S)$. Hence, $x \in \Omega_1(C_S(A))$ for all $A \in \mathcal{A}(S)$. But now, $\langle x \rangle A$ is elementary abelian of order at least as large as A and by the definition of $\mathcal{A}(S)$, we have that $x \in A$. Therefore, $x \in \bigcap_{A \in \mathcal{A}(S)} A$ and $\Omega_1(C_S(J(S))) = \bigcap_{A \in \mathcal{A}(S)} A$, completing the proof of (ii).

Definition 2·3. Let G be a finite group, V a GF(p)G-module and $A \le G$. If

(i) $A/C_A(V)$ is an elementary abelian p-group;

(ii) $[V, A] \ne \{1\}$; and

(iii) $|V/C_V(A)| \le |A/C_A(V)|$

then V is a failure to factorise module (abbrev. FF-module) for G and A is an offender on V.

We will also make liberal use of several coprime action results, often without explicit reference.

PROPOSITION 2·4 ***(Coprime Action).*** *Suppose that a group G acts on a group A coprimely, and B is a G-invariant subgroup of A. Then the following hold*:

(i) $C_{A/B}(G) = C_A(G)B/B$;

(ii) $[A, G] = [A, G, G]$;

(iii) $A = [A, G]C_A(G)$ *and if A is abelian then* $A = [A, G] \times C_A(G)$; *and*

(iv) *if G acts trivially on* $A/\Phi(A)$*, then G acts trivially on A.*

Proof. See, for instance, [27, chapter 8].

In conclusion (iv) in the statement above, one can say a little more. The following is a classical result of Burnside, but the version we use follows from [20, (I·5·1·4)].

LEMMA 2·5 (Burnside). *Let S be a finite p-group. Then* $C_{Aut(S)}(S/\Phi(S))$ *is a normal p-subgroup of Aut*(S).

LEMMA 2·6. *Let E be a finite p-group and* $Q \le A \le Aut(E)$. *Suppose there exists a normal chain* $\{1\} = E_0 \trianglelefteq E_1 \trianglelefteq E_2 \trianglelefteq \ldots \trianglelefteq E_m = E$ *of subgroups such that for each* $\alpha \in A$, $E_i\alpha = E_i$ *for all* $0 \le i \le m$. *If for all* $1 \le i \le m$, *Q centralises* E_i/E_{i-1}, *then* $Q \le O_p(A)$.

Proof. See [20, (I·5·3·3)].

LEMMA 2·7 (A×B-Lemma). *Let AB be a finite group which acts on a p-group V. Suppose that B is a p-group,* $A = O^p(A)$ *and* $[A, B] = \{1\} = [A, C_V(B)]$. *Then* $[A, V] = \{1\}$.

Proof. See [4, (24·2)].

Our final results in this section with regards to groups and modules concern the identification of some local actions within the groups Co_1, $Sp_6(3)$ and M.

LEMMA 2·8. *Suppose that G is a finite group with $O_3(G)=\{1\}$ and V is a faithful $GF(3)G$-module of dimension 6. Assume that for $S\in Syl_3(G)$, $S\cong 3^{1+2}_+$, $G=O^{3'}(G)$ and there is an elementary abelian subgroup $A\leq S$ of order 9 with $|V/C_V(A)|=|V/C_V(a)|=3^3$ for all $a\in A^\#$. Then $G\cong PSL_3(3)$ or $2.M_{12}$.*

Proof. Let G be a minimal counterexample with respect to $|G|$. By [**27**, 8·3·4(a)], $O_{3'}(G)=\langle C_{O_{3'}(G)}(a)\mid a\in A^\#\rangle$ and since $C_V(a)=C_V(A)$ for all $a\in A^\#$, we have that $O_{3'}(G)$ normalises $C_V(A)$. Set $T:=\langle A^{AO_{3'}(G)}\rangle$ so that $C_V(A)=C_V(T)\leq C_V(O_{3'}(T))$. By coprime action again, $V=[V,O_{3'}(T)]\times C_V(O_{3'}(T))$. But now,

$$C_{[V,O_{3'}(T)]}(A)\leq [V,O_{3'}(T)]\cap C_V(A)=[V,O_{3'}(T)]\cap C_V(T)=\{1\}$$

and as A is a 3-group, we must have that $[V,O_{3'}(T)]=\{1\}$. Since G acts faithfully on V, we infer that $O_{3'}(T)=\{1\}$. Then, as $A\leq T\leq AO_{3'}(G)$ and $T\cap O_{3'}(G)\leq O_{3'}(T)=\{1\}$, we conclude that $A=T$ is normalised by $O_{3'}(G)$. In particular, $[A,O_{3'}(G)]=\{1\}$.

Since $O_3(G)=\{1\}$ and $F^*(G)$ is self-centralising in G, we have shown that G contains a component, K say, whose order is divisible by 3. Then $E:=\langle K^G\rangle$ is normalised by S and so we deduce that it contains $Z(S)$. Note that since $m_3(S)=2$ and $O_3(E)\leq O_3(G)=\{1\}$, E contains at most two components of G whose orders are divisible by 3. Indeed, since S is a 3-group, we see that S normalises these components. If E contains exactly two components of G whose orders are divisible by 3, K_1 and K_2 say, then $K_i\cap S\trianglelefteq S$ for $i\in\{1,2\}$ so that $Z(S)\leq K_1\cap K_2\leq Z(E)$. Hence, $Z(S)\leq O_3(Z(E))\leq O_3(G)$, a contradiction.

Thus, $E=K$ is quasisimple. Now, $K=\langle Z(S)^K\rangle=\langle Z(S)^G\rangle$ and so K is a component of $H:=\langle Z(S)^G\rangle S$ so that $H=O^{3'}(H)$ is almost-quasisimple. Note that $O_3(H)$ is trivial for otherwise $Z(S)\leq O_3(H)\cap K\leq O_3(K)\leq O_3(G)$ and since $O_3(G)=\{1\}$, this is a contradiction. Hence, by minimality, either $H\cong \mathrm{PSL}_3(3)$ or $2.\mathrm{M}_{12}$; or $G=H$. In the former case, we deduce that $H=K\trianglelefteq G$ and since $S\leq H$ and $G=O^{3'}(G)$, we have that $G=H$.

Hence, a minimal counterexample of this lemma is almost quasisimple. Now, $|A|^2=3^4>3^3=|V/C_V(A)|$ so that V is a *2F-module* for G with *offender* A in the language of [**24**]. By [**24**, Table 1], G is isomorphic to either a group of Lie type in characteristic 3 or $2.\mathrm{M}_{12}$. The groups of Lie type in characteristic 3 with Sylow 3-subgroup isomorphic to S are well known (see [**21**, (3·3)]), and so we have that $G\cong 2.\mathrm{M}_{12}$, $\mathrm{PSL}_3(3)$ or $\mathrm{SU}_3(3)$. Now, $\mathrm{SU}_3(3)$ has only one non-trivial module of dimension 6 over GF(3), the *natural module*. But for this module, we have that $|C_V(B)|=3^2$ for any subgroup B of the Sylow 3-subgroup which has order 9.

In the following proposition, MAGMA is used to verify certain calculations. The actual code itself may be found in [**44**, appendix A].

LEMMA 2·9. *Suppose that $Q\cong 5^{1+6}_+$, $G\leq Out(Q)$ and write $V=Q/Z(Q)$ and $S\in Syl_5(G)$. Suppose the following hold*:

(i) S is elementary abelian of order 25;

(ii) $G=\langle S^G\rangle$;

(iii) $O_5(G)=\{1\}$; and

(iv) $|C_V(S)|=5$ and $|C_V(s)|=25$ for all $s\in S^\#$.

Then $G\cong 2.J_2$.

Proof. Since Q is extraspecial, $O_5(G)=\{1\}$ and $G=O^{5'}(G)$, applying [**47**] we have that G is isomorphic to a subgroup of $\mathrm{Sp}_6(5)$ and $Q/Z(Q)$ may be identified with the natural module for $\mathrm{Sp}_6(5)$ in this action. We appeal to [9, Table 8·28, Table 8·29] for the list of maximal subgroups of $\mathrm{Sp}_6(5)$. These are

$$2.\mathrm{J}_2,\ \mathrm{Sp}_2(5)\circ\mathrm{GO}_3(5),\ \mathrm{GU}_3(5).2, \mathrm{Sp}_2(5^3).3,\ \mathrm{Sp}_2(5)^3{:}\mathrm{Sym}(3),\ \mathrm{Sp}_2(5)\times\mathrm{Sp}_4(5),$$

$$5^6{:}\mathrm{GL}_3(5),\ 5^{3+4}{:}(\mathrm{GL}_2(5)\times\mathrm{Sp}_2(5)) \text{ and } 5_+^{1+4}{:}(C_4\times\mathrm{Sp}_4(5)).$$

Aiming for a contradiction, assume throughout that $G\not\cong 2.\mathrm{J}_2$.

We compute that the maximal subgroups in which a Sylow 5-subgroup fixes a subspace of dimension 1 are $2.\mathrm{J}_2$, $\mathrm{Sp}_2(5)\circ\mathrm{GO}_3(5)$, $5^6{:}\mathrm{GL}_3(5)$, $5^{3+4}{:}(\mathrm{GL}_2(5)\times\mathrm{Sp}_2(5))$ and $5_+^{1+4}{:}(C_4\times\mathrm{Sp}_4(5))$. We refer to these subgroups as $M_1,\ldots,M_5$ respectively. In M_2, one can compute that there is a 5-element which fixes a subspace of dimension 3 and as a Sylow 5-subgroup of M_2 has order 25, G cannot be isomorphic to a subgroup of M_2. If G is isomorphic to a subgroup of M_3, then as $O_5(G)=\{1\}$, G projects as a subgroup of $\mathrm{GL}_3(5)$. But every subgroup of $\mathrm{GL}_3(5)$ which has a Sylow 5-subgroup of order 25 has a normal 5-subgroup, a contradiction.

Similarly, if G is isomorphic to a subgroup of M_4, then G is isomorphic to a subgroup of $\mathrm{GL}_2(5)\times\mathrm{Sp}_2(5)$. Indeed, since $G=\langle S^G\rangle$, $|S|=25$ and $O_5(G)=\{1\}$, it follows that $G\cong\mathrm{SL}_2(5)\times\mathrm{Sp}_2(5)$. Hence, $GO_5(M_4)=O^{5'}(M_4)$. Let $L\leq G$ be such that $L\trianglelefteq G$ and $L\cong\mathrm{SL}_2(5)\cong\mathrm{Sp}_2(5)$. Then L contains a Sylow 2-subgroup T of $LO_5(M_4)\trianglelefteq G$. By a calculation (see [44, appendix A]), we have that $C_{GO_5(M_4)}(T)\cong 2\times\mathrm{SL}_2(5)$. Since $C_G(T)\cong 2\times\mathrm{SL}_2(5)$, we have that $C_{GO_5(M_4)}(T)=C_G(T)$. However, for $R\in\mathrm{Syl}_5(C_{GO_5(M_4)}(T))$, we have that $|C_V(R)|=5^5$, a clear contradiction.

If G is isomorphic to a subgroup of M_5 then G is isomorphic to a subgroup of $C_4\times\mathrm{Sp}_4(5)$. Since $G=\langle S^G\rangle$, we see that G is isomorphic to a subgroup of $\mathrm{Sp}_4(5)$. Using MAGMA (see [44, appendix A]), since $O_5(G)=\{1\}$, $|S|=25$ and $G=\langle S^G\rangle$, we calculate that $G\cong\mathrm{SL}_2(25)$ or $\mathrm{SL}_2(5)\times\mathrm{SL}_2(5)$. Moreover, the center of $\mathrm{Sp}_4(5)$ is equal to the center of a Sylow 2-subgroup of $\mathrm{Sp}_4(5)$ and it follows from computations that G centralises the center of a Sylow 2-subgroup of $L_5:=O^{5'}(M_5)$, which we denote by T. Then $G=G'\leq C_{L_5}(T)'\cong\mathrm{Sp}_4(5)$ and so G is contained in a specified complement to $O_5(M_5)$ in L_5. But then we calculate for all such candidates for G that $|C_V(S)|=5^4$, a contradiction.

Hence, G is isomorphic to a proper subgroup of $M_1\cong 2.\mathrm{J}_2$. But, appealing to [13] for a list of maximal subgroups of J_2, the only maximal subgroups of $2.\mathrm{J}_2$ which have a Sylow 5-subgroup of order 25 also have a normal 5-subgroup, a contradiction.

3. *Preliminaries: fusion systems*

We now let S be a finite p-group and $\mathcal{F}$ a saturated fusion system on S, referring to [5, 14] for standard terminology and results regarding fusion systems. We use the remainder of this section to reaffirm some important concepts regarding fusion systems which pertain to this work, and mention some vital results from other sources in the literature.

We begin with the notion of isomorphism for fusion systems.

Definition 3·1. Let $\mathcal{F}$ be a saturated fusion system on a p-group S and let $\alpha: S\to T$ be a group isomorphism. Define $\mathcal{F}^\alpha$ to be the fusion system on T with

$$\mathrm{Hom}_{\mathcal{F}^\alpha}(P,Q)=\{\alpha^{-1}\gamma\alpha \mid \gamma\in\mathrm{Hom}_{\mathcal{F}}(P\alpha^{-1},Q\alpha^{-1}\}$$

for $P,Q\leq T$.

We then say that a fusion system $\mathcal{E}$ over a p-group T is isomorphic to $\mathcal{F}$, written $\mathcal{E} \cong \mathcal{F}$, if there a group isomorphism $\alpha{:}S \to T$ with $\mathcal{E} = \mathcal{F}^{\alpha}$.

Remark. We note that our definition of isomorphism coincides with morphisms defined in [**5**, definition II·2·2] which are surjective and have trivial kernel.

Importantly, for G a finite group, $S \in \mathrm{Syl}_p(G)$ and K a normal p'-subgroup of G, writing $\overline{G} := G/K$, we have that $\mathcal{F}_S(G) \cong \mathcal{F}_{\overline{S}}(\overline{G})$. This is often viewed as one of the main attractions for working with fusion systems in place of finite groups.

We recall that $\mathcal{F}$ is *realizable* if there is a finite group G and $S \in \mathrm{Syl}_p(G)$ such that $\mathcal{F} = \mathcal{F}_S(G)$, and $\mathcal{F}$ is *exotic* otherwise. By the above observation, if we aim to show that $\mathcal{F}$ is realised by a finite group G, then we may as well assume that $O_{p'}(G) = \{1\}$.

Notation. Let $\mathcal{F}$ be a fusion system and let $\mathcal{F}_1, \mathcal{F}_2$ be fusion subsystems of $\mathcal{F}$. That is, $\mathcal{F}_i$ is a subcategory of $\mathcal{F}$ which is itself a fusion system. Write $\langle \mathcal{F}_1, \mathcal{F}_2 \rangle_S$ for the smallest subsystem of $\mathcal{F}$ supported on S which contains both $\mathcal{F}_1$ and $\mathcal{F}_2$.

At various points, we may also write $\langle \mathcal{M}_1, \mathcal{M}_2, \ldots \rangle_S$ where $\mathcal{M}_i$ is some set of morphisms contained in $\mathcal{F}$ and by this we mean the smallest subsystem of $\mathcal{F}$ supported on S which contains $\mathcal{M}_i$ for all i. We also mix the two conventions e.g. $\langle \mathcal{F}_1, \mathcal{M}_1, \mathcal{M}_2 \rangle_S$ is the smallest subsystem of $\mathcal{F}$ supported on S containing $\mathcal{F}_1$, $\mathcal{M}_1$ and $\mathcal{M}_2$.

We emphasise that saturation is not imposed here. So even if $\mathcal{F}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ are saturated, then $\langle \mathcal{F}_1, \mathcal{F}_2 \rangle_S$ need not be saturated.

We denote the set of $\mathcal{F}$-centric subgroups of $\mathcal{F}$ by $\mathcal{F}^c$ and the fully $\mathcal{F}$-normalised, $\mathcal{F}$-centric-radical subgroups of $\mathcal{F}$ by $\mathcal{F}^{frc}$, referring to [**5**, definition I·2·4, definition I·3·1] for the appropriate definitions. We present the following result as a lemma, but in truth it may be considered as part of the definition of saturation of a fusion system.

LEMMA 3·2. *Let $\mathcal{F}$ be a saturated fusion system on a p-group S. For a fully $\mathcal{F}$-centralised subgroup P of S and R a subgroup of $N_S(P)$ strictly containing P, the morphisms in $N_{Aut_{\mathcal{F}}(P)}(Aut_R(P))$ lift to $\mathcal{F}$-automorphisms of R.*

Proof. Since P is fully $\mathcal{F}$-normalised and $\mathcal{F}$ is saturated, P is *receptive*, as defined in [**5**, definition I·2·2]. Hence, for $\alpha \in N_{\mathrm{Aut}_{\mathcal{F}}(P)}(\mathrm{Aut}_R(P))$ and $N_\alpha := \{g \in N_S(P) | {}^{\alpha}c_g \in \mathrm{Aut}_S(P)\}$, we have $P < R \leq N_\alpha \leq N_S(P)$ such that there is $\widehat{\alpha} \in \mathrm{Hom}_{\mathcal{F}}(N_\alpha, S)$ with $R\widehat{\alpha} = R$ and $\widehat{\alpha}|_P = \alpha$. Indeed, $\widehat{\alpha}$ restricts to $\overline{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(R)$, as desired.

Often, the morphisms we choose to lift in Lemma 3·2 can chosen to lift all the way to certain *essential subgroups* of $\mathcal{F}$.

Definition 3·3. Let $\mathcal{F}$ be a saturated fusion system on S and let $E < S$. Then E is essential in $\mathcal{F}$ if E is fully $\mathcal{F}$-normalised, $\mathcal{F}$-centric and has the property that $\mathrm{Out}_{\mathcal{F}}(E)$ contains a strongly p-embedded subgroup.

We denote by $\mathcal{E}(\mathcal{F})$ the essential subgroups of $\mathcal{F}$.

LEMMA 3·4. *We have that $\mathcal{E}(\mathcal{F}) \subseteq \mathcal{F}^{frc}$.*

Proof. See [**5**, proposition I·3·3(a)].

In later sections, our treatment of saturated fusion systems will focus specifically on the actions associated to essential subgroups, and the morphisms lifted to them. The reasoning behind this is that a saturated fusion system is completely determined by this information. This observation is contained in the following theorem.

THEOREM 3·5 (Alperin – Goldschmidt Fusion Theorem). *Let $\mathcal{F}$ be a saturated fusion system over a p-group S and let $\mathcal{E}^0(\mathcal{F})$ be a set of representatives of the $\mathcal{F}$-conjugacy classes of $\mathcal{E}(\mathcal{F})$. Then*

$$\mathcal{F} = \langle Aut_{\mathcal{F}}(Q), Aut_{\mathcal{F}}(S) \mid Q \in \mathcal{E}^0(\mathcal{F})\rangle_S.$$

Proof. See [**5**, theorem I·3·5] and [**14**, proposition 7·25].

We will make frequent use of the following lemma which comes as a result of Lemma 3·2 and the Alperin–Goldschmidt theorem.

LEMMA 3·6. *Let $\mathcal{F}$ be a saturated fusion system on S and let $Q \le S$ be $\mathcal{F}$-centric and fully $\mathcal{F}$-normalised.*

(i) If there is $E \in \mathcal{E}(\mathcal{F})$ such that for all $P \in Q^{\mathcal{F}}$ we have that P is properly contained in E, P is properly contained in no other essentials, and E is $Aut_{\mathcal{F}}(S)$-invariant, then for $Q < R \le N_S(Q)$ and $\alpha \in N_{Aut_{\mathcal{F}}(Q)}(Aut_R(Q))$ there is $\widehat{\alpha} \in Aut_{\mathcal{F}}(E)$ with $\widehat{\alpha}|_Q = \alpha$. In particular, if $Q \notin \mathcal{E}(\mathcal{F})$ then $Aut_E(Q) \trianglelefteq Aut_{\mathcal{F}}(Q)$ and Q is not $\mathcal{F}$-radical.

(ii) If P is not properly contained in any essential subgroup of $\mathcal{F}$ for all $P \in Q^{\mathcal{F}}$, then for $Q < R \le N_S(Q)$ and $\alpha \in N_{Aut_{\mathcal{F}}(Q)}(Aut_R(Q))$ there is $\widehat{\alpha} \in Aut_{\mathcal{F}}(S)$ with $\widehat{\alpha}|_Q = \alpha$. In particular, if $Q \notin \mathcal{E}(\mathcal{F})$ then for any $Aut_{\mathcal{F}}(S)$-invariant subgroup B which contains Q, $Aut_B(Q) \trianglelefteq Aut_{\mathcal{F}}(Q)$ and Q is not $\mathcal{F}$-radical.

Proof. Let $Q < R \le N_S(Q)$ with Q $\mathcal{F}$-centric and $\alpha \in N_{\mathrm{Aut}_{\mathcal{F}}(Q)}(\mathrm{Aut}_R(Q))$. In particular, Q is fully $\mathcal{F}$-centralised.

Suppose that there is $E \in \mathcal{E}(\mathcal{F})$ such that E is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant and for all $P \in Q^{\mathcal{F}}$, E is the unique essential subgroup of $\mathcal{F}$ which properly contains P. By Lemma 3·2, there is $\widetilde{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(R)$ with $\widetilde{\alpha}|_Q = \alpha$. By the Alperin–Goldschmidt theorem, we may write $\widetilde{\alpha} = (\alpha_1 \circ \ldots \circ \alpha_n)|_R$ where $\alpha_i \in \mathrm{Aut}_{\mathcal{F}}(F)$ for $F \in \mathcal{E}(\mathcal{F}) \cup \{S\}$. Now, Q is properly contained in exactly one essential subgroup (namely E), and as $R > Q$ we must have that $\alpha_1 \in \mathrm{Aut}_{\mathcal{F}}(S)$ or $\alpha_1 \in \mathrm{Aut}_{\mathcal{F}}(E)$. Notice that if $R \le E$, then as E is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant, we may replace α_1 by $\alpha_1|_E \in \mathrm{Aut}_{\mathcal{F}}(E)$ and $Q\alpha_1 < R\alpha_1 \le E$. If $R \not\le E$, then $\alpha_1 \in \mathrm{Aut}_{\mathcal{F}}(S)$ and $R\alpha_1 \not\le E$.

Now, $Q\alpha_1 < R\alpha_1$ and $Q\alpha_1$ is properly contained in exactly one essential subgroup, and so $\alpha_2 \in \mathrm{Aut}_{\mathcal{F}}(S)$ or $\alpha_2 \in \mathrm{Aut}_{\mathcal{F}}(E)$. Again, if $R\alpha_1 \le E$ (so that $R \le E$) then as E is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant, we may arrange that $\alpha_2 \in \mathrm{Aut}_{\mathcal{F}}(E)$. Otherwise, $\alpha_2 \in \mathrm{Aut}_{\mathcal{F}}(S)$. Continuing in this fashion, we see that either $R \le E$ and we may take $\widehat{\alpha} = \alpha_1 \circ \ldots \circ \alpha_n \in \mathrm{Aut}_{\mathcal{F}}(E)$; or $R \not\le E$ and $\widehat{\alpha} = \alpha_1 \circ \ldots \circ \alpha_n \in \mathrm{Aut}_{\mathcal{F}}(S)$. In the latter case, since $Q \le E$, we have that $\widehat{\alpha}|_E$ is still a lift of α, and so the first statement of (i) holds. In particular, in either case we see that α normalises $\mathrm{Aut}_E(Q)$.

Assume now that $Q \notin \mathcal{E}(\mathcal{F})$. Then by [**5**, proposition I·3·3], $\mathrm{Aut}_{\mathcal{F}}(Q)$ is generated by maps $\alpha \in N_{\mathrm{Aut}_{\mathcal{F}}(Q)}(\mathrm{Aut}_R(Q))$ for some $Q < R \le N_S(Q)$. But all such maps normalise $\mathrm{Aut}_E(Q)$ and as $Q < E$ and Q is $\mathcal{F}$-centric, $\{1\} < \mathrm{Out}_E(Q) \trianglelefteq \mathrm{Out}_{\mathcal{F}}(Q)$ and Q is not $\mathcal{F}$-radical. This completes the proof of (i).

For the proof of (ii), we follow the same proof scheme as for (i). However, this time we note that as each $\mathcal{F}$-conjugate of Q is not properly contained in any essential subgroup of $\mathcal{F}$, the Alperin–Goldschmidt theorem promises that $\widetilde{\alpha}$ is a composition of restrictions of maps in $\mathrm{Aut}_{\mathcal{F}}(S)$ and so we deduce that $\widehat{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(S)$. In particular, $\widehat{\alpha}$ normalises any

$\mathrm{Aut}_{\mathcal{F}}(S)$-invariant subgroup of S containing Q. Applying [5, proposition I·3·3], $\mathrm{Aut}_{\mathcal{F}}(Q)$ is generated by maps $\alpha \in N_{\mathrm{Aut}_{\mathcal{F}}(Q)}(\mathrm{Aut}_R(Q))$ for some $Q < R \le N_S(Q)$ and all such maps lift to maps which normalise any $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant subgroup B of S containing Q. Since these maps also normalise Q, they must normalise $N_B(Q)$ and we deduce that $\mathrm{Aut}_B(Q) \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(Q)$. As Q is $\mathcal{F}$-centric, and taking $B = S$, we have that $\{1\} < \mathrm{Out}_S(Q) \trianglelefteq \mathrm{Out}_{\mathcal{F}}(Q)$ and Q is not $\mathcal{F}$-radical. This completes the proof of (ii), and so completes the proof of the lemma.

Throughout the later portions of this work, we will often employ computational methods to determine a list of potential essential subgroups of a fusion system $\mathcal{F}$ supported on a given p-group S via the fusion systems package in MAGMA [36, 37].

Roughly speaking, the algorithm first determines a list a subgroups of S which are self-centralising in S, a prerequisite to being essential. Since the groups with a strongly p-embedded subgroup are "known", the isomorphism type of $N_S(E)/E$ for a potential essential subgroup E should have a prescribed form too. Then further checks are carried out which verify that certain internal conditions in E hold which necessarily hold if E is essential in some saturated fusion system supported on S. These checks and more are described in [36].

The following result is a useful tool for identifying automisers of essential subgroups.

THEOREM 3·7. *Suppose that E is an essential subgroup of a saturated fusion system $\mathcal{F}$ over a p-group S, and assume that there are $\mathrm{Aut}_{\mathcal{F}}(E)$-invariant subgroups $U \le V \le E$ such that $E = C_S(V/U)$ and V/U is an FF-module for $G := \mathrm{Out}_{\mathcal{F}}(E)$. Then, writing $L := O^{p'}(G)$ and $W := V/U$, we have that $L/C_L(W) \cong SL_2(p^n)$, $C_L(W)$ is a p'-group and $W/C_W(L)$ is a natural $SL_2(p^n)$-module for some $n \in \mathbb{N}$.*

Proof. Since $E = C_S(W)$, we infer that $\mathrm{Inn}(E) = C_{\mathrm{Aut}_S(E)}(W)$ so that $C_G(W)$ is a p'-group. In particular, $G/C_G(W)$ has a strongly p-embedded subgroup and so too does $L/C_L(W) \cong LC_G(W)/C_G(W) = O^{p'}(G/C_G(W))$ by [26, remark 3·5]. Then W is an FF-module for $L/C_L(W)$ and we apply [26, theorem 5·6] to obtain the result.

The next two results of this section are pivotal in creating exotic fusion systems from p-fusion categories while maintaining saturation. The first of these techniques we refer to as "pruning."

LEMMA 3·8. *Suppose that $\mathcal{F}$ is a saturated fusion system on S and P is an $\mathcal{F}$-essential subgroup of S. Let $\mathcal{C}$ be a set of $\mathcal{F}$-class representatives of $\mathcal{F}$-essential subgroups with $P \in \mathcal{C}$. Assume that if $Q < P$ then Q is not S-centric. Then $\mathcal{G} = \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(E) \mid E \in \mathcal{C} \setminus \{P\}\rangle_S$ is saturated. Furthermore, $\mathcal{E}(\mathcal{G}) = \mathcal{E}(\mathcal{F}) \setminus \{P^{\mathcal{F}}\}$.*

Proof. We apply [36, lemma 6·4], taking $K = H_{\mathcal{F}}(P)$ where $H_{\mathcal{F}}(P)$ denotes the subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$ which is generated by $\mathcal{F}$-automorphisms of P which is extend to $\mathcal{F}$-isomorphisms between strictly larger subgroups of S. By that result, the fusion system $\langle \mathcal{G}, K\rangle_S$ is saturated. However, for $P < R \le S$ we have that $\mathrm{Hom}_{\mathcal{F}}(R, S) = \mathrm{Hom}_{\mathcal{G}}(R, S)$ and we conclude that $K \subseteq \mathcal{G}$ so that $\mathcal{G}$ is saturated.

Also included in [36, lemma 6·4] is the statement that $P \notin \mathcal{E}(\mathcal{G})$. Since $\mathrm{Aut}_{\mathcal{G}}(E) = \mathrm{Aut}_{\mathcal{F}}(E)$ for all $E \in \mathcal{E}(\mathcal{F}) \setminus \{P^{\mathcal{F}}\}$, we ascertain that $\mathcal{E}(\mathcal{G}) = \mathcal{E}(\mathcal{F}) \setminus \{P^{\mathcal{F}}\}$.

PROPOSITION 3·9. *Let $\mathcal{F}_0$ be a saturated fusion system on a finite p-group S. Let $V \le S$ be a fully $\mathcal{F}_0$-normalised subgroup, set $H = \mathrm{Out}_{\mathcal{F}_0}(V)$ and let $\widetilde{\Delta} \le \mathrm{Out}(V)$ be such that H is a strongly p-embedded subgroup of $\widetilde{\Delta}$. For Δ the full preimage of $\widetilde{\Delta}$ in $\mathrm{Aut}(V)$, write $\mathcal{F} = \langle \mathcal{F}_0, \Delta\rangle_S$. Assume further that*

(i) *V is $\mathcal{F}_0$-centric and minimal under inclusion amongst all $\mathcal{F}$-centric subgroups*; *and*

(ii) *no proper subgroup of V is $\mathcal{F}_0$-essential.*

Then $\mathcal{F}$ is saturated.

Proof. See [**10**, proposition 5·1] or [**42**, theorem C].

We recall the notion of normaliser fusion systems from [**5**, section I·6], noting that for P a fully $\mathcal{F}$-normalised subgroup, $N_{\mathcal{F}}(P)$ is a saturated fusion subsystem of $\mathcal{F}$. We say P is *normal* in $\mathcal{F}$ if $\mathcal{F} = N_{\mathcal{F}}(P)$ and we denote by $O_p(\mathcal{F})$ the unique largest normal subgroup of $\mathcal{F}$. The following proposition connects normal subgroups of $\mathcal{F}$, strongly closed subgroups of $\mathcal{F}$ in the sense of [**5**, definition I·4·1], and the essential subgroups of $\mathcal{F}$.

PROPOSITION 3·10. *Let $\mathcal{F}$ be a saturated fusion system over a p-group S. Then the following are equivalent for a subgroup $Q \leq S$*:

(i) $Q \trianglelefteq \mathcal{F}$;

(ii) *Q is strongly closed in $\mathcal{F}$ and contained in every centric radical subgroup of $\mathcal{F}$*; *and*

(iii) *Q is contained in each essential subgroup, Q is $Aut_{\mathcal{F}}(E)$-invariant for any essential subgroup E of $\mathcal{F}$ and Q is $Aut_{\mathcal{F}}(S)$-invariant.*

Moreover, if Q is an abelian subgroup of S, then $Q \trianglelefteq \mathcal{F}$ if and only if Q is strongly closed in $\mathcal{F}$.

Proof. See [**5**, proposition I·4·5] and [**5**, corollary I·4·7].

Fundamental to our analysis of fusion systems is the application of a plethora of known results from finite group theory. Particularly, given a fully normalised subgroup Q, we wish to understand the actions induced by $N_{\mathcal{F}}(Q)$ and to do this, we wish to work in a finite group which models the behaviour of this normaliser subsystem.

THEOREM 3·11 (Model Theorem). *Let $\mathcal{F}$ be a saturated fusion system over a p-group S. Assume that there is $Q \leq S$ which is $\mathcal{F}$-centric and normal in $\mathcal{F}$. Then the following hold*:

(i) *there is a model for $\mathcal{F}$. That is, there is a finite group G with $S \in Syl_p(G)$, $F^*(G) = O_p(G)$ and $\mathcal{F} = \mathcal{F}_S(G)$*;

(ii) *if G_1 and G_2 are two models for $\mathcal{F}$, then there is an isomorphism $\phi{:}G_1 \to G_2$ such that $\phi|_S = Id_S$*;

(iii) *for any finite group G with $S \in Syl_p(G)$, $F^*(G) = Q$ and $Aut_G(Q) = Aut_{\mathcal{F}}(Q)$, there is $\beta \in Aut(S)$ such that $\beta|_Q = Id_Q$ and $\mathcal{F}_S(G) = \mathcal{F}^\beta$. Thus, there is a model for $\mathcal{F}$ which is isomorphic to G.*

Proof. See [**5**, theorem I·4·9].

As with finite groups, we desire a more global sense of normality in fusion systems, not just restricted to p-subgroups. That is, we are interested in subsystems of a fusion system $\mathcal{F}$ which are *normal*. We use the notion of normality provided in [**5**, definition I·6·1], noting that this condition is stronger than some of other definitions in the literature.

By [**15**, theorem A], a proper, non-trivial normal subsystem of $\mathcal{F}$ with respect to one of the accepted definitions of normality gives rise to a proper, non-trivial normal subsystem of $\mathcal{F}$ with respect to the other accepted definitions. Thus, we can unambiguously declare $\mathcal{F}$ to be *simple* if it has no proper, non-trivial normal subsystems and so, for our purposes, the distinction between the definitions of normality is unimportant.

Of particular importance in our case is the normal subsystem $O^{p'}(\mathcal{F})$ of $\mathcal{F}$, and more generally, the saturated subsystems of $\mathcal{F}$ of *index prime to p*, as in [5, definition I·7·3]. The following result characterises some of the most important properties of these subsystems.

LEMMA 3·12. *Fix a saturated fusion system $\mathcal{F}$ over a p-group S, and set $\mathcal{E}_0 := \langle O^{p'}(Aut_{\mathcal{F}}(P)) \mid P \leq S\rangle_S$, as a (not necessarily saturated) fusion system on S. Define*

$$Aut^0_{\mathcal{F}}(S) := \langle \alpha \in Aut_{\mathcal{F}}(S) \mid \alpha|_P \in Hom_{\mathcal{E}_0}(P, S),\ some\ \ P \in \mathcal{F}^c\rangle$$

and let $\mathcal{E}$ be a saturated fusion system on S of index prime to p in $\mathcal{F}$. Then

(i) *$Aut^0_{\mathcal{F}}(S) \leq Aut_{\mathcal{E}}(S) \leq Aut_{\mathcal{F}}(S)$ and each group L with $Aut^0_{\mathcal{F}}(S) \leq L \leq Aut_{\mathcal{F}}(S)$ gives rise to a unique saturated fusion subsystem of index prime to p in $\mathcal{F}$;*

(ii) *$\mathcal{E} \trianglelefteq \mathcal{F}$ if and only if $Aut_{\mathcal{E}}(S) \trianglelefteq Aut_{\mathcal{F}}(S)$; and*

(iii) *there is a unique minimal saturated subsystem $O^{p'}(\mathcal{F}) \trianglelefteq \mathcal{F}$ of index prime to p, and $Aut_{O^{p'}(\mathcal{F})}(S) = Aut^0_{\mathcal{F}}(S)$.*

In particular, $\mathrm{Aut}^0_{\mathcal{F}}(S) = \mathrm{Aut}_{\mathcal{F}}(S)$ implies that $\mathcal{F} = O^{p'}(\mathcal{F})$, and $O^{p'}(O^{p'}(\mathcal{F})) = O^{p'}(\mathcal{F})$.

Proof. See [5, theorem I·7·7].

If $\mathcal{E}$ is a saturated subsystem of index prime to p in $\mathcal{F}$ with $[\mathrm{Aut}_{\mathcal{F}}(S){:}\mathrm{Aut}_{\mathcal{E}}(S)] = r$, then we say that $\mathcal{E}$ has index r in $\mathcal{F}$.

We provide a short lemma characterising essential subgroups in saturated subsystems of index prime to p.

LEMMA 3·13. *Let $\mathcal{F}$ be a saturated fusion system on S and let $\mathcal{B}$ be a saturated fusion subsystem of $\mathcal{F}$ of index prime to p. Then $\mathcal{E}(\mathcal{F}) = \mathcal{E}(\mathcal{B})$.*

Proof. By [5, lemma I·7·6(a)], the centric subgroups of $\mathcal{F}$ and $\mathcal{B}$ coincide.

It is clear that any fully $\mathcal{F}$-normalised subgroup of S is also fully $\mathcal{B}$-normalised. Suppose that P is a fully $\mathcal{B}$-normalised subgroup of S which is not fully $\mathcal{F}$-normalised and choose $Q \leq S$ a fully $\mathcal{F}$-normalised $\mathcal{F}$-conjugate of P. Choose $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$. By [5, lemma I·7·6(a)] the "Frattini condition" holds and so there is $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ and $\phi_0 \in \mathrm{Hom}_{\mathcal{B}}(P, Q\alpha^{-1})$ such that $\phi = \phi_0 \circ \alpha$. Since $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ we have that $|N_S(Q)| = |N_S(Q\alpha^{-1})|$ and since P is fully $\mathcal{B}$-normalised $|N_S(P)| \geq |N_S(Q\alpha^{-1})|$, yielding a contradiction. Hence, any fully $\mathcal{B}$-normalised subgroup is also fully $\mathcal{F}$-normalised.

Finally, since $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P)) \leq \mathrm{Aut}_{\mathcal{B}}(P) \leq \mathrm{Aut}_{\mathcal{F}}(P)$ for all $P \leq S$, $\mathrm{Out}_{\mathcal{F}}(P)$ has a strongly p-embedded subgroup if and only if $O^{p'}(\mathrm{Out}_{\mathcal{F}}(P))$ does, and we conclude that $\mathcal{E}(\mathcal{F}) = \mathcal{E}(\mathcal{B})$.

We close this section with a result concerning strongly closed subgroups of fusion systems, and how they might be used to verify the exoticity of certain saturated fusion systems. In the analogous definition for finite groups, a result of Foote, building on work of Goldschmidt, promises that when $p = 2$, the only simple groups G which contain a proper non-trivial strongly closed subgroup $T < S \in \mathrm{Syl}_2(G)$ are $\mathrm{PSU}_3(2^n)$ and $\mathrm{Sz}(2^n)$. Work of Flores and Foote [18] complements the result in the odd prime case, using the classification of finite simple groups. From their results, we deduce the following consequence ready for use in fusion systems.

THEOREM 3·14. *Suppose that $\mathcal{F}$ is a saturated fusion system over a p-group S and A is a proper non-trivial strongly closed subgroup chosen minimally with respect to adhering to these conditions. Assume that no normal subsystem of $\mathcal{F}$ is supported on A. Then $\mathcal{F}$ is exotic.*

Proof. Assume that $\mathcal{F}$ and A satisfy the hypotheses of the lemma, and suppose that there is a finite group G with $\mathcal{F} = \mathcal{F}_S(G)$. We may as well choose G such that $O_{p'}(G) = \{1\}$. Then A is a proper non-trivial strongly closed subgroup of G. Following [18], let $\mathcal{O}_A(G)$ be the largest normal subgroup N of G such that $A \cap N \in \mathrm{Syl}_p(N)$. Then $\mathcal{O}_A(G) \cap A$ is a strongly closed subgroup of G. By the minimality of A, and using that $O_{p'}(G) = \{1\}$, we deduce that either $A \in \mathrm{Syl}_p(\mathcal{O}_A(G))$ or $\mathcal{O}_A(G) = \{1\}$. In the former case, we have that $\mathcal{F}_A(\mathcal{O}_A(G)) \trianglelefteq \mathcal{F}_S(G) = \mathcal{F}$, a contradiction. Hence, $\mathcal{O}_A(G) = \{1\}$. Applying [18, theorem 1·1] when $p = 2$ and [18, theorem 1·3] when p is odd, we conclude that A is elementary abelian. But then by Proposition 3·10 we have that $A \trianglelefteq \mathcal{F}$ so that $\mathcal{F}_A(A)$ is a normal subsystem of $\mathcal{F}$ supported on A, another contradiction. Hence, no such G exists and $\mathcal{F}$ is exotic.

This result provides an alternate check on exoticity distinct from the techniques currently used in the literature, albeit still relying on the classification of finite simple groups.

4. *Fusion Systems on a Sylow 3-subgroup of Co_1*

In this section, we classify all saturated fusion systems supported on a 3-group S which is isomorphic to a Sylow 3-subgroup of the sporadic simple group Co_1, validating Theorem A. Utilising the Atlas [13], we extract the following 3-local maximal subgroups from $G := \mathrm{Co}_1$:

$$M_1 \cong 3^6 : 2.\mathrm{M}_{12}$$

$$M_2 \cong 3_+^{1+4}{:}\mathrm{Sp}_4(3).2$$

$$M_3 \cong 3^{3+4} : 2.(\mathrm{Sym}(4) \times \mathrm{Sym}(4))$$

and remark that for a given $S \in \mathrm{Syl}_3(G)$, M_i can be chosen such that $S \in \mathrm{Syl}_3(M_i)$. We record that $|S| = 3^9$ and $J(S) = O_3(M_1)$ (where $J(S)$ is as defined in Definition 2·1). We denote $\mathbf{J} := O_3(M_1)$, $\mathbf{Q} := O_3(M_2)$ and $\mathbf{R} := O_3(M_3)$.

In addition, S is isomorphic to a Sylow 3-subgroup of $\mathrm{Sp}_6(3)$ and in this isomorphism we recognise the subgroups $E_1, E_2, E_3 \leq S$ whose images correspond to the unipotent radicals of the minimal parabolic subgroups of $\mathrm{Sp}_6(3)$. Indeed, E_1, E_2, E_3 are also essential subgroups of $\mathcal{F}_S(\mathrm{Co}_1)$ such that

$$N_G(E_1) = M_1 \cap M_2 \cong 3_+^{1+4}.3^3{:}(2 \times \mathrm{GL}_2(3))$$

$$N_G(E_2) = M_1 \cap M_3 \cong 3^6.3^2{:}(2 \times \mathrm{GL}_2(3))$$

$$N_G(E_3) = M_2 \cap M_3 \cong 3^{3+4}.3{:}(2 \times \mathrm{GL}_2(3))$$

In an abuse of notation, we suppress the isomorphism between S and a Sylow 3-subgroup of $\mathrm{Sp}_6(3)$ and let E_1, E_2, E_3 be subgroups of Co_1 or of $\mathrm{Sp}_6(3)$ where appropriate.

We also note the following characterizations of E_1, E_2 and E_3 from their embeddings in S.

(i) E_1 is the unique subgroup of S of order 3^8 such that $\Phi(E_1) = \Phi(Y)$, where Y is the preimage in S of $Z(S/\mathbf{J})$ and has order 3^7.

(ii) $E_2 = C_S(Z_2(S))$.

(iii) E_3 is the unique subgroup X of S of order 3^8 which is not equal to E_1 but satisfies $\mho^1(X) = Z(S)$.

In particular, E_1, E_2 and E_3 are characteristic subgroups of S, and so too is $\mathbf{R} = E_2 \cap E_3$. In what follows, we take several liberties with the determination of various characteristic subgroups of the E_i, but all of these properties are easily verified by computer (e.g. using MAGMA and taking S to be a Sylow 3-subgroup of $\mathrm{Sp}_6(3)$).

PROPOSITION 4·1. *Let $\mathcal{F} = \mathcal{F}_S(Sp_6(3))$. Then $\mathcal{E}(\mathcal{F}) = \{E_1, E_2, E_3\}$.*

Proof. This is a consequence of the Borel–Tits theorem [21, corollary 3·1·6].

We record one final subgroup of G. Let $X \trianglelefteq M_1$ with $M_1/X \cong \mathrm{M}_{12}$ and consider the maximal subgroup $H \cong \mathrm{Alt}(4) \times \mathrm{Sym}(3)$ of M_1/X. Define E_4 to be the largest normal 3-subgroup of the preimage of H in M_1 so that

$$N_G(E_4) = N_{M_1}(E_4) \cong 3^6.3{:}(\mathrm{SL}_2(3) \times 2).$$

Then E_4 is an essential subgroup of $\mathcal{F}_S(\mathrm{Co}_1)$, E_4 is not contained in any other essential subgroup of $\mathcal{F}_S(\mathrm{Co}_1)$ and $[N_G(S){:}N_{N_G(S)}(E_4)] = 6$.

We note that non-trivial elements of $E_1/\mathbf{J}$ and $E_2/\mathbf{J}$ comprise of elements of type 3A in $N_G(\mathbf{J})/\mathbf{J} \cong 2.\mathrm{M}_{12}$ and non-trivial elements of $E_4/\mathbf{J}$ correspond to elements of type 3B in ATLAS terminology [13]. In particular, for $x \in R \in E_4^{\mathcal{F}}$ with $x \notin \mathbf{J}$, x acts on $\mathbf{J}$ unlike any element of E_1 or E_2.

PROPOSITION 4·2 *Let $\mathcal{F} = \mathcal{F}_S(Co_1)$. Then $\mathcal{E}(\mathcal{F}) = \{E_1, E_2, E_3, E_4^{\mathcal{F}}\}$.*

Proof. See [41].

We now move onto to the classification of all saturated fusion systems on S. Throughout we suppose that $\mathcal{F}$ is a saturated fusion system on a 3-group S such that S is isomorphic to a Sylow 3-subgroup of Co_1.

We utilise the fusion systems package in MAGMA [36, 37] to verify the following proposition. The code and outputs are included in [44, appendix A].

PROPOSITION 4·3. *$\mathcal{E}(\mathcal{F}) \subseteq \{E_1, E_2, E_3, E_4^{\mathcal{F}}\}$.*

For the duration of this section, we will frequently use that $\mathbf{J} = J(S) = J(E_i)$ is a characteristic subgroup of E_i for $i \in \{1, 2, 4\}$. This follows from Proposition 2·2 (iii).

LEMMA 4·4. *Suppose that $O^{3'}(Out_{\mathcal{F}}(\mathbf{J})) \cong PSL_3(3)$. Then $E_4^{\mathcal{F}} \cap \mathcal{E}(\mathcal{F}) = \emptyset$.*

Proof. We note first that E_4 is contained in no other essential subgroup of $\mathcal{F}$ and so by the Alperin–Goldschmidt theorem, $\{E_4^{\mathcal{F}}\} = \{E_4^{\mathrm{Aut}_{\mathcal{F}}(S)}\}$. In particular, if any $\mathcal{F}$-conjugate of E_4 is essential, then every $\mathcal{F}$-conjugate of E_4 is. Since $\mathbf{J}$ is invariant under $\mathrm{Aut}_{\mathcal{F}}(S)$ we may as well assume, aiming for a contradiction, that $E_4 \in \mathcal{E}(\mathcal{F})$.

Since $\mathbf{J} = J(E_4)$, we have that $N_{\mathcal{F}}(E_4) \leq N_{\mathcal{F}}(\mathbf{J})$ and so $N_{\mathcal{F}}(E_4) = N_{N_{\mathcal{F}}(\mathbf{J})}(E_4)$. In particular, if $E_4 \in \mathcal{E}(\mathcal{F})$ then $E_4 \in \mathcal{E}(N_{\mathcal{F}}(\mathbf{J}))$. By the uniqueness of models provided by

Theorem 3·11, for H a model of $N_{\mathcal{F}}(\mathbf{J})$, we have that $N_{\mathcal{F}}(E_4) = \mathcal{F}_{N_S(E_4)}(N_H(E_4))$ so that $\mathrm{Out}_{\mathcal{F}}(E_4) = N_H(E_4)/E_4$. Since $O^{3'}(H/\mathbf{J}) \cong \mathrm{PSL}_3(3)$, we have that $N_H(E_4) \leq N_H(S)$ and so $N_H(E_4)/E_4$ does not have a strongly 3-embedded subgroup, a contradiction.

LEMMA 4·5. *If $E_4 \in \mathcal{E}(\mathcal{F})$, then $\{E_1, E_2\} \subseteq \mathcal{E}(\mathcal{F})$. Moreover, $E_4 \in \mathcal{E}(\mathcal{F})$ if and only if $O^{3'}(Out_{\mathcal{F}}(\mathbf{J})) \cong 2.M_{12}$.*

Proof. Suppose that $\mathcal{F}$ is a saturated fusion system on S with $E_4 \in \mathcal{E}(\mathcal{F})$. Then, as $\mathbf{J} = J(E_4)$, $N_{\mathcal{F}}(E_4) \leq N_{\mathcal{F}}(\mathbf{J})$ and so E_4 is also essential in $N_{\mathcal{F}}(\mathbf{J})$. Since $E_4 \ntrianglelefteq S$ and $|E_4/\mathbf{J}| = 3$, Proposition 3·10 implies that $\mathbf{J} = O_3(N_{\mathcal{F}}(\mathbf{J}))$. By Theorem 3·11 there is a finite group H with $S \in \mathrm{Syl}_3(H)$, $N_{\mathcal{F}}(\mathbf{J}) = \mathcal{F}_S(H)$ and $F^*(H) = \mathbf{J}$. Then, $O^{3'}(H)/\mathbf{J}$ is determined by Lemma 2·8. Using that $E_4 \in \mathcal{E}(\mathcal{F})$ and applying Lemma 4·4, we conclude that $O^{3'}(H)/\mathbf{J} \cong 2.\mathrm{M}_{12}$.

Suppose now that $O^{3'}(\mathrm{Out}_{\mathcal{F}}(\mathbf{J})) \cong 2.\mathrm{M}_{12}$ and again set H to be a model for $N_{\mathcal{F}}(\mathbf{J})$ so that $O^{3'}(H)/\mathbf{J} \cong 2.\mathrm{M}_{12}$. We examine the maximal subgroups of $2.\mathrm{M}_{12}$ as can be found in the Atlas [**13**], and identify them with their preimage in $O^{3'}(H)$. Then there are three classes of maximal 3-local subgroups H_1, H_2, H_3, and we may arrange in each case that $S \cap H_i \in \mathrm{Syl}_3(H_i)$. These groups have the same shape as $N_G(E_1)$, $N_G(E_2)$ and $N_G(E_4)$ respectively. Indeed, in each case, $|N_S(O_3(H_i))/O_3(H_i)| = 3$ and so we deduce that $H_i/O_3(H_i)$ contains a strongly 3-embedded subgroup for $i \in \{1, 2, 3\}$. Since $\mathbf{J} = J(O_3(H_i))$, we have that $O^{3'}(\mathrm{Out}_{\mathcal{F}}(O_3(H_i))) = O^{3'}(\mathrm{Out}_H(O_3(H_i))) = O^{3'}(\mathrm{Out}_{H_i}(O_3(H_i)))$ contains a strongly 3-embedded subgroup for all $i \in \{1, 2, 3\}$. Moreover, each $O_3(H_i)$ is fully $\mathcal{F}$-normalised and as $\mathbf{J}$ is $\mathcal{F}$-centric, so too is $O_3(H_i)$ for $i \in \{1, 2, 3\}$. Applying Proposition 4·3, we have that $O_3(H_1), O_3(H_2), O_3(H_3)$ are equal to E_1, E_2 and $E_4\alpha$ for some $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$. Hence, $E_1, E_2, E_4 \in \mathcal{E}(\mathcal{F})$, as required.

As a consequence of the above lemma, we have the following observation. Let $R \leq S$ be such that $|R/\mathbf{J}| = 3$ and $R \ntrianglelefteq S$. The for a saturated fusion system $\mathcal{F}$ on S, if R is $\mathcal{F}$-essential then $R \in E_4^{\mathcal{F}}$ and $R/\mathbf{J}$ corresponds to a subgroup of order 3 generated by an element of type 3B in $O^{3'}(\mathrm{Out}_{\mathcal{F}}(\mathbf{J})) \cong 2.\mathrm{M}_{12}$. Moreover, under these conditions and for $x \in S \setminus \mathbf{J}$ such that $x\mathbf{J}$ is an element of type 3B in $O^{3'}(\mathrm{Out}_{\mathcal{F}}(\mathbf{J})) \cong 2.\mathrm{M}_{12}$, we have that $\langle x \rangle \mathbf{J}$ is an essential subgroup of $\mathcal{F}$ which is $\mathcal{F}$-conjugate to E_4.

LEMMA 4·6. *Suppose that $E_1 \in \mathcal{E}(\mathcal{F})$. Then $O^{3'}(Out_{\mathcal{F}}(E_1)) \cong SL_2(3)$, both $E_1/\mathbf{J}$ and $\Phi(E_1)/Z(S)$ are natural $SL_2(3)$-modules for $O^{3'}(Out_{\mathcal{F}}(E_1))$, and $\mathbf{J}/\Phi(E_1)$ is a natural $\Omega_3(3)$-module for $O^{3'}(Out_{\mathcal{F}}(E_1))/Z(O^{3'}(Out_{\mathcal{F}}(E_1))) \cong PSL_2(3)$.*

Proof. Assume that $E_1 \in \mathcal{E}(\mathcal{F})$. We calculate that $Z(S) = Z(E_1)$ is of order 3, and $\Phi(E_1) = \mathbf{J} \cap \mathbf{Q}$ is elementary abelian of order 3^3 with $C_S(\Phi(E_1)) = \mathbf{J}$. Let $K := C_{\mathrm{Aut}_{\mathcal{F}}(E_1)}(\Phi(E_1))$ so that $\mathrm{Aut}_{\mathbf{J}}(E_1) \in \mathrm{Syl}_3(K)$ and K normalises $\mathrm{Inn}(E_1)$. In particular, $[K, \mathrm{Inn}(E_1)] \leq K \cap \mathrm{Inn}(E_1) = \mathrm{Aut}_{\mathbf{J}}(E_1)$ and K centralises the quotient $E_1/\mathbf{J}$. Now, as $\mathbf{J}$ is elementary abelian, $K/C_K(\mathbf{J})$ is a $3'$-group and centralises $Z(E_1) = C_{\mathbf{J}}(\mathrm{Inn}(E_1)) \leq \Phi(E_1)$. Applying the A×B-lemma, with $K|_{\mathbf{J}}$, $\mathrm{Inn}(E_1)|_{\mathbf{J}}$ and $\mathbf{J}$ in the roles of A, B and V we deduce that K centralises $\mathbf{J}$, and so K centralises the chain $\{1\} \trianglelefteq \mathbf{J} \trianglelefteq E_1$. By Lemma 2·6, K is a 3-group. Thus, $K = \mathrm{Aut}_{\mathbf{J}}(E_1)$ and so we infer that $\mathrm{Aut}_{\mathcal{F}}(E_1)/K$ acts faithfully on $\Phi(E_1)$. Since $\mathrm{Aut}_S(E_1)$ centralises $Z(E_1) = Z(S)$ and $\mathrm{Inn}(E_1) = C_{\mathrm{Aut}_S(E_1)}(\Phi(E_1)/Z(S))$, we conclude that $O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_1))$ acts faithfully on $\Phi(E_1)/Z(S)$ of order 3^2. Then as $O_3(O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_1))) = \{1\}$, considering subgroups of $\mathrm{SL}_2(3)$ yields that $O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_1)) \cong \mathrm{SL}_2(3)$ and $\Phi(E_1)/Z(S)$ is its natural module.

We note that for $r \in O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_1))$, if r centralises $E_1/\mathbf{J}$, then as $[E_1, \Phi(E_1)] = Z(S)$, we have by the three subgroups lemma that $[r, \Phi(E_1), E_1] = \{1\}$ so that r centralises $\Phi(E_1)/Z(S)$, a contradiction. Hence, $O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_1)) \cong \mathrm{SL}_2(3)$ and $E_1/\mathbf{J}$ is its natural module. Set $V := \mathbf{J}/\Phi(E_1)$ of order 3^3. Then for $T = Z(O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_1)))$ we have by coprime action that $V = [V, T] \times C_V(T)$. However, $\mathrm{Out}_S(E_1)$ acts indecomposably on V and we conclude that $V = [V, T]$ or $V = C_V(T)$ is an irreducible 3-dimensional $\mathrm{SL}_2(3)$-module. Thus, $V = C_V(T)$ is a natural $\Omega_3(3)$-module for $O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_1))/Z(O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_1))) \cong \mathrm{PSL}_2(3)$.

LEMMA 4·7. *Suppose that $E_2 \in \mathcal{E}(\mathcal{F})$. Then $O^{3'}(Out_{\mathcal{F}}(E_2)) \cong SL_2(3)$, $Z(E_2)$ is of order 3^3, $|\boldsymbol{J}/\Phi(E_2)| = 3$ and both $E_2/\boldsymbol{J}$ and $\Phi(E_2)/Z(E_2)$ are natural $SL_2(3)$-modules for $O^{3'}(Out_{\mathcal{F}}(E_2))$.*

Proof. Assume that $E_2 \in \mathcal{E}(\mathcal{F})$. One can calculate that that $\Phi(E_2) = [E_2, \mathbf{J}]$ is of order 3^5 and is contained in $\mathbf{J}$, and $|Z(E_2)| = 3^3$ and $|Z_2(S)| = 3^2$. By Proposition 2·2 (iii), $\mathbf{J} = J(E_2)$ and $\mathbf{J}/\Phi(E_2)$ is of order 3 and centralised by S. Hence, $O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_2))$ acts trivially on $\mathbf{J}/E_2$ and so must act faithfully on $E_2/\mathbf{J}$ of order 3^2 by Lemma 2·5 and coprime action. Since $O_3(O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_2))) = \{1\}$, we deduce that $O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_2)) \cong \mathrm{SL}_2(3)$ and $E_2/\mathbf{J}$ is its natural module. Letting $r \in O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_2))$, if r centralised $\Phi(E_2)/Z(E_2)$ then by coprime action, $[r, \mathbf{J}, E_2] = \{1\}$. Moreover, since $[E_2, \mathbf{J}, r] \le Z(E_2)$ we conclude by the three subgroups lemma that $[E_2, r, \mathbf{J}] \le Z(E_2)$. But $\mathbf{J}$ is abelian so that $[E_2, r, \mathbf{J}] = [[E_2, r]\mathbf{J}, \mathbf{J}] = [E_2, \mathbf{J}] = \Phi(E_2)$, a contradiction. Hence, $\Phi(E_2)/Z(E_2)$ is a 2-dimensional faithful module for $O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_2))$ and so is a natural $\mathrm{SL}_2(3)$-module.

LEMMA 4·8. *Suppose that $E_3 \in \mathcal{E}(\mathcal{F})$. Then $O^{3'}(Out_{\mathcal{F}}(E_3)) \cong SL_2(3)$, $\boldsymbol{R}$ is normalised by $Aut_{\mathcal{F}}(E_3)$, and $\boldsymbol{R}/\Phi(E_3)$ and $\Phi(E_3)/Z(\boldsymbol{R})$ are natural $SL_2(3)$-modules for $O^{3'}(Out_{\mathcal{F}}(E_3))$.*

Proof. Assume that $E_3 \in \mathcal{E}(\mathcal{F})$. One may calculate that $Z_2(S) = Z_2(E_3)$ and so $C_{E_3}(Z_2(E_3)) = E_2 \cap E_3 = \mathbf{R} \trianglelefteq \mathrm{Out}_{\mathcal{F}}(E_3)$. Since S centralises $E_3/\mathbf{R}$, we must have that $O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_3))$ centralises $E_3/\mathbf{R}$. Moreover, one can calculate that $\Phi(E_3)$ is of order 3^5 and so by Lemma 2·5 and coprime action, $O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_3))$ acts faithfully on $\mathbf{R}/\Phi(E_3)$ which has order 3^2. As in Lemma 4·6, we conclude that $O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_3)) \cong \mathrm{SL}_2(3)$ and $\mathbf{R}/\Phi(E_3)$ is a natural module.

Let $r \in O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_3))$ of $3'$-order. We note that $Z(S) < Z_2(E_3) = Z_2(S) < Z(\mathbf{R}) = Z(E_2)$ and $Z(E_2)$ has order 3^3. It follows that $O^3(O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_3)))$ acts trivially on $Z(\mathbf{R})$. Assume that r centralises $\Phi(E_3)/Z(\mathbf{R})$. Then by coprime action r centralises $\Phi(E_3)$. One can calculate that $C_{E_3}(\Phi(E_3))) = Z_2(E_3) \le \Phi(E_3)$ so that $[r, E_3] = \{1\}$ by coprime action and the three subgroups lemma, a contradiction. Hence, $\Phi(E_3)/Z(\mathbf{R})$ is a 2-dimensional faithful module for $O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_3))$ and so is a natural $\mathrm{SL}_2(3)$-module.

PROPOSITION 4·9. *Assume that $\mathcal{E}(\mathcal{F}) \subseteq \{E_i\}$ for some $i \in \{1, 2, 3\}$. Then one of the following occurs:*

(i) *$\mathcal{F} = N_{\mathcal{F}}(S)$; or*

(ii) *$\mathcal{F} = N_{\mathcal{F}}(E_i)$ where $O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_i)) \cong \mathrm{SL}_2(3)$ for some $i \in \{1, 2, 3\}$.*

Proof. If $\mathcal{E}(\mathcal{F}) = \emptyset$, then outcome (i) is satisfied by the Alperin–Goldschmidt theorem. Thus, we may assume that E_i is the unique essential subgroup of $\mathcal{F}$. Indeed, we must have that E_i

is invariant under $\mathrm{Aut}_{\mathcal{F}}(S)$ and so $E_i \trianglelefteq \mathcal{F}$. Then Lemma 4·6, Lemma 4·7 and Lemma 4·8 complete the proof in case (ii).

LEMMA 4·10. *Assume that $E_1 \in \mathcal{E}(\mathcal{F})$. Then there is a unique Aut(S)-conjugate of $\mathbf{Q}$ which is Aut$_{\mathcal{F}}(E_1)$-invariant and Aut$_{\mathcal{F}}(S)$-invariant.*

Proof. Assume that $E_1 \in \mathcal{E}(\mathcal{F})$. By Lemma 4·6, $O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_1)) \cong \mathrm{SL}_2(3)$ normalises $\mathbf{J}$ and $\Phi(E_1)$, and $V := \mathbf{J}/\Phi(E_1)$ is a irreducible 3-dimensional module for $O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_1))/T \cong \mathrm{PSL}_2(3)$ where $T = C_{O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_1))}(V)$. For $U := E_1/\Phi(E_1)$ we have that U/V has order 9 and as T acts non-trivially on U by coprime action, we deduce that $C_U(T) = V$ and $U = [U,T] \times V$ where $[U,T]$ is a natural $\mathrm{SL}_2(3)$-module.

For X the preimage in E_1 of $[U,T]$, we have that $X \trianglelefteq S$ and $X \cap \mathbf{J} = \Phi(E_1)$. Moreover, since $X/\Phi(E_1)$ is an irreducible module for $\mathrm{Out}_{\mathcal{F}}(E_1)$, we deduce that $|\Omega_1(X)| \neq 3^4$. With this information, we calculate that there are 3 subgroups of E_1 satisfying these properties including X. Furthermore, since E_1, $\Phi(E_1)$ and $\mathbf{J}$ are all characteristic subgroups of S, we have that $X\alpha$ also satisfies these properties for all $\alpha \in \mathrm{Aut}(S)$, and we calculate that under the action of Aut(S), all 3 subgroups of E_1 are conjugate (see [**44**, appendix A] for the explicit code for these calculations). Finally, since $\mathbf{Q}$ satisfies these properties, we conclude that there is $\alpha \in \mathrm{Aut}(S)$ such that $X = \mathbf{Q}\alpha$. By the module decomposition of U above, X is the unique such Aut(S)-conjugate of $\mathbf{Q}$ which is $\mathrm{Aut}_{\mathcal{F}}(E_1)$-invariant.

By definition, $\mathcal{F}^{\alpha^{-1}}$ is a saturated fusion system on S which is isomorphic to $\mathcal{F}$, for $\alpha \in \mathrm{Aut}(S)$. Furthermore, it follows from the above lemma that there is $\alpha \in \mathrm{Aut}(S)$ such that $\mathbf{Q}$ is the unique subgroup of S in its Aut(S)-conjugacy class which is both $\mathrm{Aut}_{\mathcal{F}^{\alpha^{-1}}}(E_1)$-invariant and $\mathrm{Aut}_{\mathcal{F}^{\alpha^{-1}}}(S)$-invariant. Since we are only interested in investigating the possibilities of $\mathcal{F}$ up to isomorphism, we may as well assume for the remainder of this section that $\mathbf{Q}$ is $\mathrm{Aut}_{\mathcal{F}}(E_1)$-invariant whenever $E_1 \in \mathcal{E}(\mathcal{F})$. Indeed, $\mathbf{Q}$ is the preimage in E_1 of $[E_1/\Phi(E_1), Z(O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_1)))]$.

PROPOSITION 4·11. *Suppose that $\{E_1, E_2\} \subseteq \mathcal{E}(\mathcal{F})$. Then either:*

(i) $E_4^{\mathcal{F}} \cap \mathcal{E}(\mathcal{F}) = \emptyset$, $\mathcal{E}(N_{\mathcal{F}}(\mathbf{J})) = \{E_1, E_2\}$ and $O^{3'}(\mathrm{Out}_{\mathcal{F}}(\mathbf{J})) \cong \mathrm{PSL}_3(3)$; or

(ii) $E_4 \in \mathcal{E}(\mathcal{F})$, $\mathcal{E}(N_{\mathcal{F}}(\mathbf{J})) = \{E_1, E_2, E_4^{\mathcal{F}}\}$ and $O^{3'}(\mathrm{Out}_{\mathcal{F}}(\mathbf{J})) \cong 2.\mathrm{M}_{12}$.

Moreover, in each case, if $E_3 \notin \mathcal{E}(\mathcal{F})$ then $\mathcal{F} = N_{\mathcal{F}}(\mathbf{J})$.

Proof. By Proposition 4·3, $\mathcal{E}(N_{\mathcal{F}}(\mathbf{J})) \subseteq \{E_1, E_2, E_3, E_4^{\mathcal{F}}\}$. We note that as $\mathbf{J} = J(E_1) = J(E_2) = J(E_4)$, $\mathrm{Out}_{\mathcal{F}}(E_i) = \mathrm{Out}_{N_{\mathcal{F}}(\mathbf{J})}(E_i)$ for $i \in \{1,2,4\}$. Furthermore, since E_i is self-centralising in S and fully normalised in $\mathcal{F}$, we see that $E_i \in \mathcal{E}(N_{\mathcal{F}}(\mathbf{J}))$ if and only if $E_i \in \mathcal{E}(\mathcal{F})$ for $i \in \{1,2,4\}$. Since $\mathbf{J} \not\leq E_3$, we necessarily have that $E_3 \notin \mathcal{E}(N_{\mathcal{F}}(\mathbf{J}))$ by Proposition 3·10.

Suppose that $\{E_1, E_2\} \subseteq \mathcal{E}(\mathcal{F})$. Let X be the largest subgroup normalised by $\mathrm{Aut}_{\mathcal{F}}(E_1)$ and $\mathrm{Aut}_{\mathcal{F}}(E_2)$. Since $\mathbf{J} = J(E_1) = J(E_2)$, we have that $\mathbf{J} \leq X \leq E_1 \cap E_2$. Furthermore, by Lemma 4·6, $E_1/\mathbf{J}$ is irreducible under $\mathrm{Aut}_{\mathcal{F}}(E_1)$ and we deduce that $X = \mathbf{J}$ and $\mathbf{J} = O_3(N_{\mathcal{F}}(\mathbf{J}))$. Indeed, $\mathrm{Out}_{\mathcal{F}}(\mathrm{J})$ satisfies the hypothesis of Lemma 2·8 and we deduce that $O^{3'}(\mathrm{Out}_{\mathcal{F}}(\mathbf{J})) \cong \mathrm{PSL}_3(3)$ or $2.\mathrm{M}_{12}$. In the former case, we have by Lemma 4·4 that $E_4^{\mathcal{F}} \cap \mathcal{E}(\mathcal{F}) = \emptyset$, and so (i) holds. In the latter case, we have by Lemma 4·5 that $E_4 \in \mathcal{E}(\mathcal{F})$, and so (ii) holds. Finally, since $\mathbf{J} = J(S)$ and $\mathbf{J}$ is invariant under $\mathrm{Aut}_{\mathcal{F}}(S)$, Proposition 3·10 and Proposition 4·3 imply that if $E_3 \notin \mathcal{E}(\mathcal{F})$ then $\mathcal{F} = N_{\mathcal{F}}(\mathbf{J})$.

PROPOSITION 4·12. *Suppose that* $\{E_1, E_3\} \subseteq \mathcal{E}(\mathcal{F})$*. Then* $O^{3'}(Out_{\mathcal{F}}(\mathbf{Q})) \cong Sp_4(3)$*,* $\mathcal{E}(N_{\mathcal{F}}(\mathbf{Q})) = \{E_1, E_3\}$ *and if* $E_2 \notin \mathcal{E}(\mathcal{F})$ *then* $\mathcal{F} = N_{\mathcal{F}}(\mathbf{Q})$*.*

Proof. Suppose that $\{E_1, E_3\} \subseteq \mathcal{E}(\mathcal{F})$ and let X be the largest subgroup of S normalised by both $\mathrm{Aut}_{\mathcal{F}}(E_1)$ and $\mathrm{Aut}_{\mathcal{F}}(E_3)$. Then $X \le E_1 \cap E_3$ so that $\mathbf{J} \not\le X$. Since $\mathrm{Aut}_{\mathcal{F}}(E_1)$ acts irreducibly on $\mathbf{J}/\Phi(E_1)$, by the choice of $\mathbf{Q}$ following Lemma 4·10 we have that $X \le \mathbf{Q}$. We note that $Z(S) = Z(E_1) = Z(E_3)$ so that $Z(S) \le X$.

Assume first that $X = Z(S)$, let G_i be a model for $N_{\mathcal{F}}(E_i)$, where $i \in \{1,3\}$, and G_{13} be a model for $N_{\mathcal{F}}(S)$. Since E_1 and E_3 are $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant, we can arrange that there are injective maps $\alpha_i{:}G_{13} \to G_i$ for $i \in \{1,3\}$. Furthermore, since $Z(S) \trianglelefteq G_1, G_3$, we may form injective maps $\alpha_i^*{:}G_{13}/Z(S) \to G_i/Z(S)$ so that the tuple $(G_1/Z(S), G_3/Z(S), G_{13}/Z(S), \alpha_1^*, \alpha_3^*)$ satisfies the hypothesis of [**16**, theorem A]. Since $|S/Z(S)| = 3^8$ and $|Z(S/Z(S))| = 3$, comparing with the outcomes provided by [**16**, theorem A], we have a contradiction.

Thus, $Z(S) < X$ and we deduce that $Z(S) < X \cap Z_2(E_1) \le \Phi(E_1)$. By Lemma 4·6, $\mathrm{Aut}_{\mathcal{F}}(E_1)$ is irreducible on $\Phi(E_1)/Z(S)$ and so we have that $\Phi(E_1) \le X$. If $X = \Phi(E_1)$ then $|X| = 3^3$ and $X \cap Z_2(S) > Z(S)$. Hence, $|X\Phi(E_3)/\Phi(E_3)| \le 3$ and as $\mathrm{Aut}_{\mathcal{F}}(E_3)$ acts irreducibly on $\mathbf{R}/\Phi(E_3)$ by Lemma 4·8 we deduce that $X \le \Phi(E_3)$. Similarly, $|XZ_2(S)/Z_2(S)| \le 3$ and as $\mathrm{Aut}_{\mathcal{F}}(E_3)$ acts irreducibly on $\Phi(E_3)/Z_2(S)$ by Lemma 4·8 we deduce that $X = Z_2(S)$, a contradiction since $X/Z(S)$ is a natural $\mathrm{SL}_2(3)$-module for $O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_1))$. Hence, $\Phi(E_1) < X$. Finally, since $X \le \mathbf{Q}$ and $\mathrm{Aut}_{\mathcal{F}}(E_1)$ acts irreducibly on $\mathbf{Q}/\Phi(E_1)$ by Lemma 4·6, we have that $X = \mathbf{Q}$.

We have that $O^{3'}(\mathrm{Out}_{\mathcal{F}}(\mathbf{Q}))$ acts faithfully on $\mathbf{Q}$. By [**47**], we deduce that $O^{3'}(\mathrm{Out}_{\mathcal{F}}(\mathbf{Q}))$ is isomorphic to a subgroup of $O^{3'}(\mathrm{Out}(\mathbf{Q})) \cong \mathrm{Sp}_4(3)$. Hence, $\mathrm{Out}_S(\mathbf{Q}) \in \mathrm{Syl}_3(\mathrm{Out}(\mathbf{Q}))$ and $O^{3'}(\mathrm{Out}_{\mathcal{F}}(\mathbf{Q}))$ is an overgroup of $\mathrm{Out}_S(\mathbf{Q})$ with no non-trivial normal 3-subgroups. By [**9**, Table 8·12], any maximal subgroup of $\mathrm{Sp}_4(3)$ which contains a Sylow 3-subgroup is a parabolic subgroup so has a normal 3-subgroups. Hence, $O^{3'}(\mathrm{Out}_{\mathcal{F}}(\mathbf{Q}))$ is contained in no maximal subgroups so that $O^{3'}(\mathrm{Out}_{\mathcal{F}}(\mathbf{Q})) \cong \mathrm{Sp}_4(3)$.

We note that the maximal abelian subgroups of $\mathbf{Q}$ have order 3^3 and so $\mathbf{QJ} = E_1$. In particular, $E_2 \not\ge \mathbf{Q} \not\le E_4$ and neither E_2 nor E_4 are essential in $N_{\mathcal{F}}(\mathbf{Q})$ by Proposition 3·10. Since E_1, E_3 are $\mathcal{F}$-centric, normal in S and satisfy $\mathrm{Out}_{\mathcal{F}}(E_i) = \mathrm{Out}_{N_{\mathcal{F}}(\mathbf{Q})}(E_i)$, we deduce that $E_1, E_3 \in \mathcal{E}(\mathcal{F})$ if and only if $E_1, E_3 \in \mathcal{E}(N_{\mathcal{F}}(\mathbf{Q}))$. By Lemma 4·5, if $E_4^{\mathcal{F}} \cap \mathcal{E}(\mathcal{F}) \ne \emptyset$, then $E_2 \in \mathcal{E}(\mathcal{F})$ and so by Proposition 4·3, if $E_2 \notin \mathcal{E}(\mathcal{F})$ then $\mathcal{E}(\mathcal{F}) = \{E_1, E_3\}$. In particular, since we have arranged that $\mathbf{Q}$ is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant by Lemma 4·10, applying Proposition 3·10 we see that $\mathcal{F} = N_{\mathcal{F}}(\mathbf{Q})$, completing the proof.

PROPOSITION 4·13. *Suppose that* $\{E_2, E_3\} \subseteq \mathcal{E}(\mathcal{F})$*. Then* $O^{3'}(Out_{\mathcal{F}}(\mathbf{R})) \cong \Omega_4^+(3) \cong SL_2(3) \bullet_{C_2} SL_2(3)$*,* $\mathcal{E}(N_{\mathcal{F}}(\mathbf{R})) = \{E_2, E_3\}$ *and if* $E_1 \notin \mathcal{E}(\mathcal{F})$ *then* $\mathcal{F} = N_{\mathcal{F}}(\mathbf{R})$*.*

Proof. Suppose that $\{E_2, E_3\} \subseteq \mathcal{E}(\mathcal{F})$. By Lemma 4·8, we have that $\mathbf{R} = E_2 \cap E_3$ is characteristic in E_3. Recall from Lemma 4·7 that for $V := E_2/\Phi(E_2)$ and $L := O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_2)) \cong \mathrm{SL}_2(3)$, $V = [V, L] \times C_V(L)$ where $[V,L]$ has order 3^2 and $C_V(L) = \mathbf{J}/\Phi(E_2)$.

We claim that $\mathbf{R}$ is the preimage of $[V,L]$ in E_2 and so is normalised by $\mathrm{Aut}_{\mathcal{F}}(E_2)$. First, observe that $[E_2, E_3]\Phi(E_2)/\Phi(E_2)$ has order 3 and is contained in $([V, L] \cap \mathbf{R})/\Phi(E_2)$. Since E_3 is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant, we deduce that either $\mathbf{R}$ is the preimage of $[V,L]$, or L centralises $\mathbf{R}/[E_2, E_3]\Phi(E_2)$. In the latter case, we deduce that $C_V(L) \le \mathbf{R}/\Phi(E_2)$ so that $\mathbf{J} \le \mathbf{R}$, a contradiction. Hence, $\mathbf{R}$ is the preimage in E_2 of $[V,L]$ and so is normalised by $\mathrm{Aut}_{\mathcal{F}}(E_2)$.

Since $\Phi(\mathbf{R}) = Z(\mathbf{R})$, $|\mathbf{R}/\Phi(\mathbf{R})| = 3^4$ and applying Lemma 2·5, we deduce that $O^{3'}(\mathrm{Out}_{\mathcal{F}}(\mathbf{R}))$ is isomorphic to a subgroup $\mathrm{SL}_4(3)$. Set $\overline{\mathbf{R}} = \mathbf{R}/\Phi(\mathbf{R})$. We note that $|C_{\overline{\mathbf{R}}}(\mathrm{Out}_S(\mathbf{R}))| = 3$ and that $\overline{\mathbf{R}} = \langle C_{\overline{\mathbf{R}}}(\mathrm{Out}_S(\mathbf{R}))^{\mathrm{Out}_{\mathcal{F}}(\mathbf{R})}\rangle$ by the actions of $N_{\mathrm{Out}_{\mathcal{F}}(\mathbf{R})}(\mathrm{Out}_{E_i}(\mathbf{R})) \cong \mathrm{Aut}_{\mathcal{F}}(E_i)/\mathrm{Aut}_{\mathbf{R}}(E_i)$ for $i \in \{2, 3\}$. In particular, $\mathrm{Out}_{\mathcal{F}}(\mathbf{R})$ stabilises no subspaces of $\overline{R}$ and $\overline{R}$ is indecomposable under $\mathrm{Out}_{\mathcal{F}}(\mathbf{R})$. Moreover, $|N_{O^{3'}(\mathrm{Out}_{\mathcal{F}}(\mathbf{R}))}(\mathrm{Out}_{E_i}(\mathbf{R}))|$ is divisible by 8 and we deduce that $O^{3'}(\mathrm{Out}_{\mathcal{F}}(\mathbf{R})) \ncong \mathrm{(P)SL}_2(9)$. Comparing with [9, Table 8·8], we deduce that $O^{3'}(\mathrm{Out}_{\mathcal{F}}(\mathbf{R}))$ is isomorphic to a subgroup of $\mathrm{SO}_4^+(3)$ or $\mathrm{Sp}_4(3)$. In the latter case, we check against the tables of maximal subgroups of $\mathrm{Sp}_4(3)$ [9, Table 8·12] and find no suitable candidates which contain $O^{3'}(\mathrm{Out}_{\mathcal{F}}(\mathbf{R}))$. In the former case, since $|\mathrm{SO}_4^+(3)|_3 = 3^2$ and comparing orders we deduce that $O^{3'}(\mathrm{Out}_{\mathcal{F}}(\mathbf{R})) \cong O^{3'}(\mathrm{SO}_4^+(3)) = \Omega_4^+(3)$, as desired.

Since $\mathbf{R} \trianglelefteq S$ is of order 3^7, contained in E_2 and does not contain $\mathbf{J}$ (for otherwise $\mathbf{J} \le E_3$), we see that $E_1 \not\ge \mathbf{R} \not\le E_4$ and neither E_1 nor E_4 are essential in $N_{\mathcal{F}}(\mathbf{R})$ by Proposition 3·10. Since E_2, E_3 are $\mathcal{F}$-centric, normal in S and satisfy $\mathrm{Out}_{\mathcal{F}}(E_i) = \mathrm{Out}_{N_{\mathcal{F}}(\mathbf{R})}(E_i)$, we deduce that $E_2, E_3 \in \mathcal{E}(\mathcal{F})$ if and only if $E_2, E_3 \in \mathcal{E}(N_{\mathcal{F}}(\mathbf{R}))$. By Lemma 4·5, if $E_4^{\mathcal{F}} \cap \mathcal{E}(\mathcal{F}) \neq \emptyset$, then $E_1 \in \mathcal{E}(\mathcal{F})$ and so by Proposition 4·3, if $E_1 \notin \mathcal{E}(\mathcal{F})$ then $\mathcal{E}(\mathcal{F}) = \{E_2, E_3\}$. Since E_2 and E_3 are characteristic subgroups of S, so too is $\mathbf{R}$. Hence, $\mathbf{R}$ is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant and so if $E_1 \notin \mathcal{E}(\mathcal{F})$, then applying Proposition 3·10, we have that $\mathcal{F} = N_{\mathcal{F}}(\mathbf{R})$, completing the proof.

Hence, as consequence of Proposition 4·3, Lemma 4·5 and Proposition 4·9-Proposition 4·13, we have proved the following result.

PROPOSITION 4·14. *Suppose that $\mathcal{F}$ is a saturated fusion system on a 3-group S such that S is isomorphic to a Sylow 3-subgroup of Co_1. If $O_3(\mathcal{F}) = \{1\}$ then $\mathcal{E}(\mathcal{F}) = \{E_1, E_2, E_3\}$ or $\mathcal{E}(\mathcal{F}) = \{E_1, E_2, E_3, E_4^{\mathcal{F}}\}$.*

We now complete the classification of all saturated fusion systems supported on S. As evidenced in Proposition 4·12 and Proposition 4·13, the structure of $O^{3'}(\mathrm{Out}_{\mathcal{F}}(\mathbf{Q}))$ and $O^{3'}(\mathrm{Out}_{\mathcal{F}}(\mathbf{R}))$ is fairly rigid and the flexibility we exploit is in the possible choices of actions for $\mathrm{Out}_{\mathcal{F}}(\mathbf{J})$.

The identification of the fusion systems of $\mathrm{Sp}_6(3)$ and $\mathrm{Aut}(\mathrm{Sp}_6(3))$ is proved using a result of Onofrei [34] which identifies a *parabolic system* in $\mathcal{F}$. Further restrictions then identify $\mathrm{Sp}_6(3)$ from an associated chamber system. We remark that in the case of parabolic systems in groups, the definition is meant to abstractly capture a set of minimal parabolics containing a "Borel", in analogy with groups of Lie type in defining characteristic. We cannot hope to capture the rich theory of parabolic systems in groups (and fusion systems) here, but we refer to [29] for a survey of this area in the group theory case, and refer to [34] for the fusion system parallel.

THEOREM 4·15. *Suppose that $\mathcal{F}$ is a saturated fusion system on a 3-group S such that S is isomorphic to a Sylow 3-subgroup of Co_1. If $\mathcal{E}(\mathcal{F}) = \{E_1, E_2, E_3\}$ then $\mathcal{F} = \mathcal{F}_S(H)$ such that $H \cong Sp_6(3)$ or $Aut(Sp_6(3))$.*

Proof. Let $\mathcal{F}_{ij} := \langle N_{\mathcal{F}}(E_i), N_{\mathcal{F}}(E_j)\rangle_S$ for $i, j \in \{1, 2, 3\}$, noting that $N_{\mathcal{F}}(S) \le N_{\mathcal{F}}(E_i)$ for all $i \in \{1, 2, 3\}$. Then $\mathbf{J} \trianglelefteq \mathcal{F}_{12}$ and as $E_4 \notin \mathcal{E}(\mathcal{F})$, Proposition 4·11 along with the Alperin–Goldschmidt theorem imply that $\mathcal{F}_{12} = N_{\mathcal{F}}(\mathbf{J})$. Applying Proposition 4·12 we have that $\mathcal{F}_{13} = N_{\mathcal{F}}(\mathbf{Q})$, and Proposition 4·13 yields that $\mathcal{F}_{23} = N_{\mathcal{F}}(\mathbf{R})$.

Let $\alpha \in \mathrm{Hom}_{N_{\mathcal{F}}(E_i)\cap N_{\mathcal{F}}(E_j)}(P,Q)$ for $P,Q \le S$, $i \neq j$ and $i,j \in \{1,2,3\}$. Since $E_i \trianglelefteq N_{\mathcal{F}}(E_i) \cap N_{\mathcal{F}}(E_j)$, there is $\widehat{\alpha} \in \mathrm{Hom}_{N_{\mathcal{F}}(E_i)\cap N_{\mathcal{F}}(E_j)}(PE_i, QE_i)$ with $\widehat{\alpha}|_P = \alpha$. But $E_j \trianglelefteq N_{\mathcal{F}}(E_i) \cap N_{\mathcal{F}}(E_j)$ and so there is $\widetilde{\alpha} \in \mathrm{Hom}_{N_{\mathcal{F}}(E_i)\cap N_{\mathcal{F}}(E_j)}(PE_iE_j, QE_iE_j)$ with $\widetilde{\alpha}|_{PE_i} = \widehat{\alpha}$. Since $E_iE_j = S$, we have shown that for all $\alpha \in \mathrm{Hom}_{N_{\mathcal{F}}(E_i)\cap N_{\mathcal{F}}(E_j)}(P,Q)$, there is $\widetilde{\alpha} \in \mathrm{Aut}_{N_{\mathcal{F}}(E_i)\cap N_{\mathcal{F}}(E_j)}(S)$ with $\widetilde{\alpha}|_P = \alpha$. Hence, $S \trianglelefteq N_{\mathcal{F}}(E_i) \cap N_{\mathcal{F}}(E_j)$ so that $N_{\mathcal{F}}(S) = N_{\mathcal{F}}(E_i) \cap N_{\mathcal{F}}(E_j)$ whenever $i \neq j$. Hence, $\{\mathcal{F}_i : i \in \{1,2,3\}\}$ is a family of parabolic subsystems in the sense of [**34**, definition 5·1].

In fact, following [**34**, definition 7·4], $\mathcal{F}$ has a family of parabolic subsystems of type $\mathfrak{M}$, where $\mathfrak{M}$ is the diagram associated to $\mathcal{F}$ described in that definition. By Proposition 4·11, Proposition 4·12 and Proposition 4·13, $\mathfrak{M}$ is exactly the Dynkin diagram corresponding to the group $\mathrm{Sp}_6(3)$ and so is a spherical diagram. Then [**34**, proposition 7·5 (ii)] implies that $\mathcal{F}$ is the fusion system of a finite simple group G of Lie type in characteristic p extended by diagonal and field automorphisms. Then $N_{\mathcal{F}}(\mathbf{Q}) = \mathcal{F}_S(N_G(\mathbf{Q}))$ and as $O^{3'}(\mathrm{Out}_{\mathcal{F}}(\mathbf{Q})) \cong \mathrm{Sp}_4(3)$ acts irreducibly on $\mathbf{Q}/Z(S)$, we conclude that $N_G(\mathbf{Q}) = O^3(N_G(\mathbf{Q}))$ so that $G = O^3(G)$. Comparing with the structure of the Sylow 3-subgroups of the finite simple groups of Lie type (as can be found in [**21**, section 3·3]), we deduce that $\mathcal{F} = \mathcal{F}_S(G)$ where $\mathrm{Inn}(\mathrm{Sp}_6(3)) \le G \le \mathrm{Aut}(\mathrm{Sp}_6(3)))$.

THEOREM 4·16. *Suppose that $\mathcal{F}$ is a saturated fusion system on a 3-group S such that S is isomorphic to a Sylow 3-subgroup of Co_1. If $\mathcal{E}(\mathcal{F}) = \{E_1, E_2, E_3, E_4^{\mathcal{F}}\}$ then $\mathcal{F} \cong \mathcal{F}_S(Co_1)$.*

Proof. We observe first that $\mathcal{G} := \mathcal{F}_S(\mathrm{Co}_1)$ satisfies the hypothesis of the proposition and that $\mathcal{G} = \langle \mathrm{Aut}_{\mathcal{G}}(E_1), \mathrm{Aut}_{\mathcal{G}}(E_2), \mathrm{Aut}_{\mathcal{G}}(E_3), \mathrm{Aut}_{\mathcal{G}}(E_4), \mathrm{Aut}_{\mathcal{G}}(S)\rangle$ by the Alperin–Goldschmidt theorem. By Lemma 4·10, there is $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ such that $\mathbf{Q}\alpha$ is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant and $\mathrm{Aut}_{\mathcal{F}}(E_1)$-invariant. Since we are only interested in determining $\mathcal{F}$ up to isomorphism, we may replace $\mathcal{F}$ by $\mathcal{F}^{\alpha}$ and assume that $\mathbf{Q}$ is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant and $\mathrm{Aut}_{\mathcal{F}}(E_1)$-invariant. We have that $\mathbf{Q}$ is $\mathrm{Aut}_{\mathcal{G}}(E_1)$-invariant and $\mathrm{Aut}_{\mathcal{G}}(E_1)$-invariant by construction.

Since $\mathbf{J}$ is characteristic in E_1, E_2 and E_4, $\mathbf{Q} \trianglelefteq N_{\mathcal{F}}(E_3)$ and $\mathbf{Q} \trianglelefteq N_{\mathcal{G}}(E_3)$, we see that $\mathcal{G} = \langle N_{\mathcal{G}}(\mathbf{J}), N_{\mathcal{G}}(\mathbf{Q})\rangle_S$ and $\mathcal{F} = \langle N_{\mathcal{F}}(\mathbf{J}), N_{\mathcal{F}}(\mathbf{Q})\rangle_S$. Hence, upon showing that $N_{\mathcal{G}}(\mathbf{J}) = N_{\mathcal{F}}(\mathbf{J})$ and $N_{\mathcal{G}}(\mathbf{Q}) = N_{\mathcal{F}}(\mathbf{Q})$, we will have shown that $\mathcal{F} = \mathcal{G}$ and the proof will be complete.

Applying Proposition 4·11 and Proposition 4·12, since $E_4 \in \mathcal{E}(\mathcal{F})$, we have that $O^{3'}(\mathrm{Aut}_{\mathcal{F}}(\mathbf{J})) \cong 2.\mathrm{M}_{12}$ and $O^{3'}(\mathrm{Out}_{\mathcal{F}}(\mathbf{Q})) \cong \mathrm{Sp}_4(3)$. We may lift the $3'$-order morphisms in $N_{O^{3'}(\mathrm{Aut}_{\mathcal{F}}(\mathbf{J}))}(\mathrm{Aut}_S(\mathbf{J}))$ to morphisms in $\mathrm{Aut}_{\mathcal{F}}(S)$ by Lemma 3·2, which then restrict faithfully to morphisms of $\mathrm{Aut}_{\mathcal{F}}(\mathbf{Q})$ by Lemma 4·10. Similarly, any morphism in $N_{\mathrm{Aut}_{\mathcal{F}}(\mathbf{Q})}(\mathrm{Aut}_S(\mathbf{Q}))$ lift to morphisms in $\mathrm{Aut}_{\mathcal{F}}(S)$ by Lemma 3·2 and restrict faithfully to morphisms in $N_{\mathrm{Aut}_{\mathcal{F}}(\mathbf{J})}(\mathrm{Aut}_S(\mathbf{J}))$. Comparing the orders of the normaliser of a Sylow 3-subgroup of $2.\mathrm{M}_{12}$ with the normaliser of a Sylow 3-subgroup of $\mathrm{Out}(\mathbf{Q}) \cong \mathrm{Sp}_4(3).2$, and applying the Frattini argument, we deduce that $\mathrm{Aut}_{\mathcal{F}}(\mathbf{Q}) = \mathrm{Aut}(\mathbf{Q}) = \mathrm{Aut}_{\mathcal{G}}(\mathbf{Q}) \cong 3^4{:}(\mathrm{Sp}_4(3):2)$ and $\mathrm{Aut}_{\mathcal{F}}(\mathbf{J}) \cong 2.\mathrm{M}_{12}$. Since $\mathbf{J}$ admits $\mathrm{Aut}_{\mathcal{F}}(S)$ faithfully, we deduce that $|\mathrm{Aut}_{\mathcal{F}}(S)| = |\mathrm{Aut}_{\mathcal{G}}(S)|$. By Theorem 3·11, we conclude that there is $\beta \in \mathrm{Aut}(S)$ with $N_{\mathcal{F}^{\beta}}(\mathbf{Q}) = N_{\mathcal{G}}(\mathbf{Q})$. Since $\mathrm{Aut}_{\mathcal{G}}(S) = \mathrm{Aut}_{N_{\mathcal{G}}(\mathbf{Q})}(S) = \mathrm{Aut}_{N_{\mathcal{F}^{\beta}}(\mathbf{Q})}(S)$ and $|\mathrm{Aut}_{\mathcal{F}^{\beta}}(S)| = |\mathrm{Aut}_{\mathcal{F}}(S)| = |\mathrm{Aut}_{\mathcal{G}}(S)|$, we deduce that $\mathbf{Q}$ is $\mathrm{Aut}_{\mathcal{F}^{\beta}}(S)$-invariant. A similar argument reveals that $\mathbf{Q}$ is $\mathrm{Aut}_{\mathcal{F}^{\beta}}(E_1)$-invariant. As we are only interested in determining $\mathcal{F}$ up to isomorphism, we mat replace $\mathcal{F}$ by $\mathcal{F}^{\beta}$ so that $N_{\mathcal{G}}(\mathbf{Q}) = N_{\mathcal{F}}(\mathbf{Q})$, and $\mathbf{Q}$ is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant and $\mathrm{Aut}_{\mathcal{F}}(E_1)$-invariant.

Now, $N_{\mathcal{G}}(E_1) = N_{N_{\mathcal{G}}(\mathbf{Q})}(E_1) = N_{N_{\mathcal{F}}(\mathbf{Q})}(E_1) = N_{\mathcal{F}}(E_1)$. Then $N_{\mathcal{G}}(\mathbf{J}) \ge N_{\mathcal{G}}(E_1) \le N_{\mathcal{F}}(\mathbf{J})$ and by [**33**, proposition 2·11], it suffices to show that $\mathrm{Aut}_{\mathcal{G}}(\mathbf{J}) = \mathrm{Aut}_{\mathcal{F}}(\mathbf{J})$ and that the homomorphism $H^1(\mathrm{Out}_{\mathcal{G}}(\mathbf{J});\mathbf{J}) \to H^1(\mathrm{Out}_{N_{\mathcal{G}}(E_1)}(\mathbf{J});\mathbf{J})$ induced by restriction is surjective. For the latter condition, we calculate in MAGMA (see [**44**, appendix A]) that $H^1(\mathrm{Out}_{N_{\mathcal{G}}(E_1)}(\mathbf{J});\mathbf{J}) = \{1\}$ and so the homomorphism is surjective.

Let $K := \mathrm{Aut}_{N_{\mathcal{G}}(E_1)}(\mathbf{J})$, $X := \mathrm{Aut}_{\mathcal{G}}(\mathbf{J})$ and $Y := \mathrm{Aut}_{\mathcal{F}}(\mathbf{J})$ so that $K \leq X \cap Y \leq \mathrm{Aut}(\mathbf{J}) \cong \mathrm{GL}_6(3)$. We aim to show that $X = Y$. Since there is only one conjugacy class of groups isomorphic to $2.\mathrm{M}_{12}$ in $\mathrm{GL}_6(3)$, we may assume that there is $g \in \mathrm{Aut}(\mathbf{J})$ with $Y = X^g$ and $K \leq X \cap Y$. Hence, $K, K^g \leq Y \cong 2.\mathrm{M}_{12}$. Now, K is the unique overgroup of $T \in \mathrm{Syl}_3(X)$ of its isomorphism type whose largest normal 3-subgroup centralises only an element of order 3 in $\mathbf{J}$. Then, K^g is the unique overgroup of $T^g \in \mathrm{Syl}_3(X^g)$ with the same properties. Since $K \leq X^g = Y$, K is an overgroup of $P \in \mathrm{Syl}_3(X^g)$ with $O_3(K)$ centralizing only an element of order 3 in $\mathbf{J}$. Thus, for $m \in X^g$ with $P^m = T^g$, K^m and K^g are isomorphic overgroups of $T^g \in \mathrm{Syl}_3(X)$ and by uniqueness, we deduce that $K^m = K^g$. But now, $K = K^{gm^{-1}}$ so that $gm^{-1} \in N_{\mathrm{GL}_6(3)}(K)$ and $X^g = X^{gm^{-1}}$. However, one can calculate that $N_{\mathrm{GL}_6(3)}(K) = N_X(K)$ so that $Y = X^g = X$.

Remark. Suppose that $\mathcal{F} = \mathcal{F}_S(\mathrm{Co}_1)$ and set $\mathcal{F}_0 := \langle N_{\mathcal{F}}(E_1), N_{\mathcal{F}}(E_2), N_{\mathcal{F}}(E_3)\rangle_S$. The Alperin–Goldschmidt theorem yields that $\mathrm{Aut}_{\mathcal{F}}(E_4) \not\subset \mathcal{F}_0$ so that $\mathcal{F}_0 < \mathcal{F}$. By Proposition 4·12, we have that $N_{\mathcal{F}}(\mathbf{Q}) \leq \mathcal{F}_0$. Then as $O_3(\mathcal{F}_0) \leq O_3(N_{\mathcal{F}}(\mathbf{Q}))$, we conclude that if $O_3(\mathcal{F}_0) \neq \{1\}$ that $Z(S) = \Phi(\mathbf{Q}) \trianglelefteq \mathcal{F}_0$. But $Z(S) \ntrianglelefteq N_{\mathcal{F}}(E_2) = N_{\mathcal{F}_S(\mathrm{Co}_1)}(E_2)$ and so $O_3(\mathcal{F}_0) = \{1\}$. Hence, by Theorem 4·15 and Theorem 4·16, if $\mathcal{F}_0$ is saturated then $\mathcal{F}_0 \cong \mathcal{F}_S(G)$ where $G \in \{\mathrm{Sp}_6(3), \mathrm{Aut}(\mathrm{Sp}_6(3))\}$. But then $\mathrm{PSL}_3(3) \cong O^{3'}(\mathrm{Out}_{\mathcal{F}_0}(\mathbf{J})) \leq O^{3'}(\mathrm{Out}_{\mathcal{F}}(\mathbf{J})) \cong 2.\mathrm{M}_{12}$. But 13 divides $|\mathrm{PSL}_3(3)|$ and does not divide $|2.\mathrm{M}_{12}|$ and so we conclude that $\mathcal{F}_0$ is not saturated.

The above remark is of particular interest in the mission of classifying fusion systems which contain parabolic systems. In the case of the group $G := \mathrm{Co}_1$, the groups $N_G(E_i)$ for $i \in \{1, 2, 3\}$ all contain the "Borel" $N_G(S)$ and together generate G and so successfully form something akin to a parabolic system. Utilised above, work by Onofrei [34] parallels the group phenomena in fusion systems and provides conditions in which a parabolic system within a fusion system $\mathcal{F}$ gives rise to a parabolic system in the group sense. The resulting completion of the group parabolic system realises the fusion system and if certain additional conditions are satisfied, the fusion system is saturated.

Comparing with [34, definition 5·1], if $\mathcal{F}_0$ does not have a family of parabolic subsystems then the only possible condition we fail to satisfy for $\mathcal{F}_0$ is condition (F4). Indeed, the subsystem $\langle N_{\mathcal{F}}(E_1), N_{\mathcal{F}}(E_2)\rangle_S$ is *not* a saturated fusion system. Part of the reason this problem arises is that the 3-fusion category of $2.\mathrm{M}_{12}$ is isomorphic to the 3-fusion category of $\mathrm{PSL}_3(3)$ and, consequently, the image of E_4 is not essential in the quotient $N_{\mathcal{F}}(J(S))/J(S)$.

However, we still retain that

$$\langle N_{\mathcal{F}}(E_1), N_{\mathcal{F}}(E_2)\rangle_S \leq \mathcal{F}_S(\langle N_H(E_1), N_H(E_2)\rangle) = N_{\mathcal{F}}(J(S))$$

where $N_{\mathcal{F}}(J(S))$ is a saturated constrained fusion system with model H. Thus, we can still embed the models for $N_{\mathcal{F}}(E_1)$, $N_{\mathcal{F}}(E_2)$ uniquely in H and obtain a parabolic system of groups. Perhaps it is possible in all the situations we care about to create an embedding $\langle N_{\mathcal{F}}(E_1), N_{\mathcal{F}}(E_2)\rangle_S \leq \mathcal{F}_S(\langle G_1, G_2\rangle) \leq N_{\mathcal{F}}(U)$ where $N_{\mathcal{F}}(U)$ is constrained and G_1, G_2 are the models of $N_{\mathcal{F}}(E_1), N_{\mathcal{F}}(E_2)$. In such a circumstance, we should always be able to work in a group setting and can then force restrictions on the structures of $N_{\mathcal{F}}(E_i)$ for $i \in \{1, 2\}$.

Finally, we remark that the above example of Co_1 at the prime 3 is similar in spirit to the example of M_{24} at the prime 2 given in [23, p. 58].

5. *Fusion Systems related to a Sylow 3-subgroup of* F_3

We now investigate fusion systems supported on a 3-group S which is isomorphic to a Sylow 3-subgroup of the Thompson sporadic simple group F_3. For the exoticity checks in this section, we will use some terminology and results regarding the known finite simple groups. As a reference, we use [21]. Again, for structural results concerning S and its internal actions, we appeal to the Atlas [13]. We begin by noting the following 3-local maximal subgroups of F_3:

$$M_1 \cong 3^{2+3+2+2}{:}\mathrm{GL}_2(3)$$

$$M_2 \cong 3^{1+2+1+2+1+2}{:}\mathrm{GL}_2(3)$$

$$M_3 \cong 3^5{:}\mathrm{SL}_2(9).2$$

remarking that $|S| = 3^{10}$ and that for a given $S \in \mathrm{Syl}_3(\mathrm{F}_3)$, each M_i may be chosen so that $S \cap M_i \in \mathrm{Syl}_3(M_i)$. We make this choice for each M_i.

Set $E_i = O_3(M_i)$ so that $E_1 = C_S(Z_2(S))$ and $E_2 = C_S(Z_3(S)/Z(S))$ are characteristic subgroups of S, and so are $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant in any fusion system $\mathcal{F}$ on S. We obtain generators for M_1 and M_2 (and hence for S, E_1 and E_2) as in Proposition 5·17. For ease of notation, we fix $\mathcal{G} := \mathcal{F}_S(\mathrm{F}_3)$ for the remainder of this section.

PROPOSITION 5·1. *We have that* $\mathcal{G}^{frc} = \{E_1, E_2, E_3^S, S\}$. *In particular,* $\mathcal{E}(\mathcal{G}) = \{E_1, E_2, E_3^S\}$.

Proof. This follows from a combination of [2, Table 27] and [46].

We appeal to MAGMA (see [44, appendix A]) for the following result.

PROPOSITION 5·2. *Suppose that* $\mathcal{F}$ *is saturated fusion system on* S. *Then* $\mathcal{E}(\mathcal{F}) \subseteq \{E_1, E_2, E_3^S\}$.

We will need the following observation in the proofs of the coming results. Several aspects of this proof are verified computationally (see [44, appendix A]).

LEMMA 5·3. *Let* $\mathcal{F}$ *be a saturated fusion system on* S. *Then* $\{E_3^{\mathcal{F}}\} = \{E_3^S\}$, $E_1 = \langle E_3^S \rangle$ *and every* $\mathcal{F}$*-conjugate of* E_3 *contains* $Z_2(S)$, *is contained in* E_1 *and is not contained in* E_2. *Moreover, if any* $\mathcal{F}$*-conjugate of* E_3 *is essential in* $\mathcal{F}$ *then the following hold*:

- *(i)* *every* $\mathcal{F}$*-conjugate of* E_3 *is essential in* $\mathcal{F}$;
- *(ii)* $Z_2(S) \le [E_3, O^{3'}(\mathrm{Aut}_{\mathcal{F}}(E_3))]$, $O^{3'}(\mathrm{Aut}_{\mathcal{F}}(E_3)) \cong \mathrm{SL}_2(9)$ *and* $[E_3, O^{3'}(\mathrm{Aut}_{\mathcal{F}}(E_3))]$ *is a natural module for* $O^{3'}(\mathrm{Aut}_{\mathcal{F}}(E_3))$;
- *(iii)* $E_1 \in \mathcal{E}(\mathcal{F})$; *and*
- *(iv)* $O_3(\mathcal{F}) = \{1\}$.

Proof. Note that $[Z_2(S), E_3] = \{1\}$. One can see this in $\mathcal{G}$ for otherwise, since E_3 is elementary abelian, we would have that $Z_2(S) \not\le E_3$ and $[Z_2(S), E_3] \le Z(S)$, a contradiction since $\mathrm{Out}_{\mathcal{G}}(E_3) \cong \mathrm{SL}_2(9).2$ has no non-trivial modules exhibiting this behaviour. Since E_3 is self-centralising in S and $E_1 = C_S(Z_2(S))$, we deduce that $Z_2(S) \le E_3 \le E_1$. Now, $\Phi(E_1)$ is elementary abelian of order 3^5 and is not contained in E_3. Furthermore, $[E_3, \Phi(E_1)] \le [E_1, \Phi(E_1)] = Z_2(S) \le E_3$ so that $\Phi(E_1) \le N_S(E_3)$. Comparing with $\mathcal{G}$, we get that $N_S(E_3) = E_3\Phi(E_1) = N_{E_1}(E_3)$, $E_3 \cap \Phi(E_1)$ is of order 3^3 and $\Phi(E_1)$ induces an FF-action on E_3.

We verify computationally (see [44, appendix A]) that every elementary abelian subgroup A of order 3^5 which is contained in E_1 and has $|N_{E_1}(A)| = 3^7$ is S-conjugate to E_3. Moreover, for any such A we have that $E_1 = \langle A^S \rangle$. Since $E_2 \trianglelefteq S$ and $E_1 \not\leq E_2$, we have that $A \not\leq E_2$. We observe that $N_{E_1}(E_3) = N_S(E_3)$ and so any $\mathrm{Aut}_{\mathcal{F}}(S)$-conjugate of E_3 is S-conjugate to E_3. Similarly, we see that any $\mathrm{Aut}_{\mathcal{F}}(E_1)$-conjugate of E_3 is S-conjugate to E_3. Let R be an $\mathcal{F}$-conjugate of E_3 with $R = E_3\alpha$. By the Alperin–Goldschmidt theorem, we have that $\alpha = (\phi_1 \circ \ldots \phi_r)|_{E_3}$ where $\phi_i \in \mathrm{Aut}_{\mathcal{F}}(Q)$ where $Q \in \{E_1, E_2, S, E_3^{\mathcal{F}}\}$. Since S-conjugates of E_3 are never contained in E_2, it follows that $\alpha = (\phi_1 \circ \ldots \phi_r)_{E_3}$ where $\phi_i \in \mathrm{Aut}_{\mathcal{F}}(Q)$ where $Q \in \{E_1, S, E_3^{\mathcal{F}}\}$. By the above reasoning, we have that $R \leq E_1$ and $N_{E_1}(E)$ has order 3^7. Hence, R is S-conjugate to E_3 and $\{E_3^{\mathcal{F}}\} = \{E_3^S\}$.

Following the definition, it is clear that every S-conjugate of an essential subgroup is essential and so if any $\mathcal{F}$-conjugate of E_3 is essential in $\mathcal{F}$, then every $\mathcal{F}$-conjugate of E_3 is essential. Since both E_1 and E_2 are normal in S, we have shown that every $\mathcal{F}$-conjugate of E_3 contains $Z_2(S)$, is contained in E_1 and is not contained in E_2.

Assume that E_3 is essential in $\mathcal{F}$. Then for $L := O^{3'}(\mathrm{Aut}_{\mathcal{F}}(E_3))$, applying Theorem 3·7, we have that $L \cong \mathrm{SL}_2(9)$ and $E_3 = [E_3, L] \times C_{E_3}(L)$, where $[E_3, L]$ is a natural $\mathrm{SL}_2(9)$-module. It follows that $[\Phi(E_1), E_3] = Z_2(S)$ has order 9 and that $C_{E_3}(L) \cap Z_2(S) = \{1\}$. Let K be a Sylow 2-subgroup of $N_L(\mathrm{Aut}_S(E_3))$ so that K is cyclic of order 8 and acts irreducibly on $Z_2(S)$. Then if E_1 is not essential, using Lemma 3·6 and Proposition 5·2, the morphisms in K must lift to automorphisms of S. But then, upon restriction, the morphisms in K would normalise $Z(S)$, contradicting the irreducibility of $Z_2(S)$ under the action of K. Hence, $E_1 \in \mathcal{E}(\mathcal{F})$. Since $O_3(\mathcal{F}) \trianglelefteq S$ and, by Proposition 3·10, $O_3(\mathcal{F})$ is an $\mathrm{Aut}_{\mathcal{F}}(E_3)$-invariant subgroup of E_3, we conclude that $O_3(\mathcal{F}) = \{1\}$.

Throughout the remainder of this section, we set

$$\mathcal{H} = \langle \mathrm{Aut}_{\mathcal{G}}(E_1), \mathrm{Aut}_{\mathcal{G}}(E_2), \mathrm{Aut}_{\mathcal{G}}(S) \rangle_S$$

and

$$\mathcal{D} = \langle \mathrm{Aut}_{\mathcal{G}}(E_1), \mathrm{Aut}_{\mathcal{G}}(E_3), \mathrm{Aut}_{\mathcal{G}}(S) \rangle_S.$$

PROPOSITION 5·4. *$\mathcal{H}$ is a saturated fusion system with $\mathcal{H}^{frc} = \{E_1, E_2, S\}$.*

Proof. By applying Lemma 3·8 to $\mathcal{G}$ with $P = E_3$ we deduce that $\mathcal{H}$ is saturated. Moreover, by Lemma 5·3 we have that $\{E_3^S\} = \{E_3^{\mathcal{F}}\}$ and Lemma 3·8 reveals that $\mathcal{E}(\mathcal{H}) = \{E_1, E_2\}$.

Let R be a fully $\mathcal{H}$-normalised, radical, centric subgroup of S not equal to E_1, E_2 or S. Then some $\mathcal{H}$-conjugate of R must be contained in an $\mathcal{H}$-essential subgroup for otherwise, by Lemma 3·6, we infer that $\mathrm{Out}_S(R) \trianglelefteq \mathrm{Out}_{\mathcal{H}}(R)$ and R is not $\mathcal{H}$-radical. If an $\mathcal{H}$-conjugate of R is contained in a $\mathcal{G}$-conjugate of E_3 then since R is $\mathcal{H}$-centric, we would have that R is $\mathcal{G}$-conjugate to E_3 (and so would be S-conjugate to E_3). Then $\mathrm{Out}_S(R) \leq O^{3'}(\mathrm{Out}_{\mathcal{H}}(R)) \leq O^{3'}(\mathrm{Out}_{\mathcal{G}}(R)) \cong \mathrm{SL}_2(9)$. Since R is not $\mathcal{H}$-essential, it follows that $O^{3'}(\mathrm{Out}_{\mathcal{H}}(R))$ is contained in the unique maximal subgroup of $O^{3'}(\mathrm{Out}_{\mathcal{G}}(R))$ which contains $\mathrm{Out}_S(R)$ and so $\mathrm{Out}_S(R) \trianglelefteq O^{3'}(\mathrm{Out}_{\mathcal{H}}(R))$. Then the Frattini argument implies that $\mathrm{Out}_S(R) \trianglelefteq \mathrm{Out}_{\mathcal{H}}(R)$, a contradiction as R is $\mathcal{H}$-radical. Thus, no $\mathcal{H}$-conjugate of R is not contained in an $\mathcal{G}$-conjugate of E_3. Hence, by the Alperin–Goldschmidt theorem and using Proposition 5·2, since $\mathcal{H} = \langle \mathrm{Aut}_{\mathcal{G}}(E_1), \mathrm{Aut}_{\mathcal{G}}(E_2), \mathrm{Aut}_{\mathcal{G}}(S) \rangle_S$ and R is fully $\mathcal{H}$-normalised, R is fully $\mathcal{G}$-normalised and so is $\mathcal{G}$-centric. Finally, since $O_3(\mathrm{Out}_{\mathcal{G}}(R)) \leq O_3(\mathrm{Out}_{\mathcal{H}}(R)) = \{1\}$, we conclude that R is $\mathcal{G}$-centric-radical and comparing with Proposition 5·1, we have a contradiction.

PROPOSITION 5·5. *$\mathcal{H}$ is simple.*

Proof. Assume that $\mathcal{N} \trianglelefteq \mathcal{H}$ and $\mathcal{N}$ is supported on T. Then T is a strongly closed subgroup of $\mathcal{H}$. In particular, $T \trianglelefteq S$ and $Z(S) \le T$. Taking repeated normal closures of $Z(S)$ under the actions of $\mathrm{Aut}_{\mathcal{G}}(E_1)$ and $\mathrm{Aut}_{\mathcal{G}}(E_2)$, we apply the description of F_3 from [16, p. 100] to ascertain that $\Phi(E_1) \le T \not\le E_1$. Then $E_1 = \langle [T, E_1]^{\mathrm{Aut}_{\mathcal{G}}(E_1)} \rangle \le T$ and so $S = T$. Since $\mathrm{Aut}_{\mathcal{H}}(S)$ is generated by lifted morphisms from $O^{3'}(\mathrm{Aut}_{\mathcal{H}}(E_1))$ and $O^{3'}(\mathrm{Aut}_{\mathcal{H}}(E_2))$, in the language of Lemma 3·12 we have that $\mathrm{Aut}^0_{\mathcal{H}}(S) = \mathrm{Aut}_{\mathcal{H}}(S)$. Then [5, theorem II·9·8(d)] implies that $\mathcal{H}$ is simple.

PROPOSITION 5·6. *$\mathcal{H}$ is exotic.*

Proof. Aiming for a contradiction, suppose that $\mathcal{H} = \mathcal{F}_S(G)$ for some finite group G with $S \in \mathrm{Syl}_3(G)$. We may as well assume that $O_3(G) = O_{3'}(G) = \{1\}$ so that $F^*(G) = E(G)$ is a direct product of non-abelian simple groups, all of order divisible by 3. Since $\mathcal{F}_{S\cap F^*(G)}(F^*(G)) \trianglelefteq \mathcal{H}$, we have that $G = F^*(G)$. Furthermore, since $|\Omega_1(Z(S))| = 3$, we deduce that G is simple. We note that $m_3(\mathrm{F}_3) = 5$ by [21, Table 5·6·1]. In particular, we reduce to searching for simple groups with a Sylow 3-subgroup of order 3^{10} and 3-rank 5. Since E_3 is not normal in S, S does not have a unique elementary abelian subgroup of maximal rank.

If $G \cong \mathrm{Alt}(n)$ for some n then $m_3(\mathrm{Alt}(n)) = \lfloor \frac{n}{3} \rfloor$ by [21, proposition 5·2·10] and so $n < 18$. But a Sylow 3-subgroup of $\mathrm{Alt}(18)$ has order 3^8 and so $G \not\cong \mathrm{Alt}(n)$ for any n. If G is isomorphic to a group of Lie type in characteristic 3, then comparing with [21, Table 3·3·1], we see that the groups with a Sylow 3-subgroup which has 3-rank 5 are $\mathrm{PSL}_2(3^5)$, $\Omega_7(3)$, ${}^3\mathrm{D}_4(3)$ and $\mathrm{PSU}_5(3)$, and only $\mathrm{PSU}_5(3)$ has a Sylow 3-subgroup of order 3^{10} of these examples. Since the unipotent radicals of parabolic subgroups of $\mathrm{PSU}_5(3)$ are essential subgroups and since neither has index 3 in a Sylow 3-subgroup, we have shown that G is not a group of Lie type of characteristic 3.

Assume now that G is a group of Lie type in characteristic $r \neq 3$ with $m_3(G) = 5$. By [21, theorem 4·10·3], S has a unique elementary abelian subgroup of order 3^5 unless $G \cong \mathrm{G}_2(r^a)$, ${}^2\mathrm{F}_4(r^a)$, ${}^3\mathrm{D}_4(r^a)$, $\mathrm{PSU}_3(r^a)$ or $\mathrm{PSL}_3(r^a)$. Since S has more than one elementary abelian subgroup of order 3^5, we have that G is one of the listed exceptions. Then, applying [21, theorem 4·10·3(a)], none of the exceptions have 3-rank 5 and we conclude that G is not isomorphic to a group of Lie type in characteristic r.

Finally, checking the orders of the sporadic groups, we have that F_3 is the unique sporadic simple group with a Sylow 3-subgroup of order 3^{10}. Since the 3-fusion category of F_3 has 3 classes of essential subgroups, $G \not\cong \mathrm{F}_3$ and we have a final contradiction. Hence, $\mathcal{H}$ is exotic.

PROPOSITION 5·7. *$\mathcal{D}$ is a saturated fusion system with $\mathcal{D}^{frc} = \{E_1, E_3^{\mathcal{D}}, S\}$.*

Proof. In the statement of Proposition 3·9, letting $\mathcal{F}_0 = N_{\mathcal{G}}(E_1)$, $V = E_3$ and $\Delta = \mathrm{Aut}_{\mathcal{G}}(E_3)$ we have that $\mathcal{D}^\dagger = \langle \mathcal{F}_0, \mathrm{Aut}_{\mathcal{G}}(E_3) \rangle_S$ is a proper saturated subsystem of $\mathcal{G}$. But now, applying the Alperin–Goldschmidt theorem $\mathcal{F}_0 = \langle \mathrm{Aut}_{\mathcal{G}}(E_1), \mathrm{Aut}_{\mathcal{G}}(S) \rangle_S$ so that $\mathcal{D} = \langle \mathrm{Aut}_{\mathcal{G}}(E_1), \mathrm{Aut}_{\mathcal{G}}(E_3), \mathrm{Aut}_{\mathcal{G}}(S) \rangle_S = \langle \mathcal{F}_0, \mathrm{Aut}_{\mathcal{G}}(E_3) \rangle_S = \mathcal{D}^\dagger$. Therefore, $\mathcal{D}$ is saturated.

Let R be a fully $\mathcal{D}$-normalised, radical, centric subgroup of S not equal to E_1, S or a $\mathcal{D}$-conjugate of E_3. If any $\mathcal{D}$-conjugate of R is contained in a $\mathcal{D}$-conjugate of E_3, then since R is $\mathcal{D}$-centric and E_3 is elementary abelian, we have a contradiction. Hence R is not contained in a $\mathcal{D}$-conjugate of E_3 and by Proposition 5·2 and using that $E_2 \notin \mathcal{E}(\mathcal{D})$, R is contained in at most one $\mathcal{D}$-essential subgroup, namely E_1. Then, as E_1 is $\mathrm{Aut}_{\mathcal{D}}(S)$-invariant, Lemma 3·6 implies that $\mathrm{Out}_{E_1}(R) \trianglelefteq \mathrm{Out}_{\mathcal{D}}(R)$. Since R is $\mathcal{D}$-centric-radical we conclude that $E_1 \le R \le S$, a contradiction.

LEMMA 5·8. *E_1 is the unique proper non-trivial strongly closed subgroup of $\mathcal{D}$.*

Proof. Since every essential subgroup of $\mathcal{D}$ is contained in E_1, and since E_1 is characteristic in S, we deduce by the Alperin–Goldschmidt theorem that E_1 is strongly closed in $\mathcal{D}$. Assume that T is any proper non-trivial strongly closed subgroup of $\mathcal{D}$. Then $T \trianglelefteq S$ and so $Z(S) \leq T$ and $Z_2(S) = \langle Z(S)^{\mathrm{Aut}_{\mathcal{D}}(E_1)} \rangle \leq T$. Suppose first that $T \cap \Phi(E_1) = Z_2(S)$. Since $\Phi(E_1) \trianglelefteq S$ we have that $[\Phi(E_1), T] \leq Z_2(S)$. We calculate ([**44**, appendix A]) that $C_S(\Phi(E_1)/Z_2(S)) = E_1$ so that $T \leq E_1$. But then $[E_1, T] \leq \Phi(E_1) \cap T = Z_2(S) = Z(E_1)$ and $T \leq Z_2(E_1) = \Phi(E_1)$. We compute that $Z_2(E_1) = \Phi(E_1)$ so that $T = Z_2(S)$. However, then $T \leq E_3$ and by Lemma 5·3, $T < \langle T^{\mathrm{Aut}_{\mathcal{D}}(E_3)} \rangle$, a contradiction.

Thus, $T \cap \Phi(E_1) > Z_2(S)$ and from the description of F_3 given by [**16**, p. 100], we see that $\mathrm{Out}_{\mathcal{D}}(E_1)$ acts irreducibly on $\Phi(E_1)/Z_2(S)$. Therefore, we must have that $\Phi(E_1) \leq T$. But now, by Lemma 5·3, $E_3 = \langle (\Phi(E_1) \cap E_3)^{\mathrm{Aut}_{\mathcal{D}}(E_3)} \rangle \leq \langle (T \cap E_3)^{\mathrm{Aut}_{\mathcal{D}}(E_3)} \rangle \leq T$. Finally, again by Lemma 5·3, since $E_1 = \langle E_3^S \rangle \leq T$, we deduce that $T = E_1$, as desired.

PROPOSITION 5·9. *$\mathcal{D}$ is a saturated exotic simple fusion system.*

Proof. We note that $O^{3'}(\mathrm{Out}_{\mathcal{D}}(E_1)) \cong \mathrm{SL}_2(3)$ and Lemma 3·2 yields that $\mathrm{Aut}^0_{\mathcal{D}}(S)$ has index at most 2 in $\mathrm{Aut}_{\mathcal{D}}(S)$. Suppose $\mathrm{Aut}^0_{\mathcal{D}}(S)$ has index exactly 2 in $\mathrm{Aut}_{\mathcal{D}}(S)$. Then, since $\mathrm{Out}_{\mathcal{D}}(E_1) \cong \mathrm{GL}_2(3)$, an application of Lemma 3·2 yields that $\mathrm{Out}_{O^{3'}(\mathcal{D})}(E_1) \cong \mathrm{SL}_2(3)$. Observe that $O^{3'}(\mathrm{Aut}_{\mathcal{D}}(E_3)) \cong \mathrm{SL}_2(9)$. Let K be a Sylow 2-subgroup of $N_{O^{3'}(\mathrm{Aut}_{\mathcal{D}}(E_3))}(\mathrm{Aut}_S(E_3))$ which is cyclic of order 8 and contained in $O^{3'}(\mathcal{D})$. Then, as E_1 is $\mathrm{Aut}_{O^{3'}(\mathcal{D})}(S)$-invariant, applying Lemma 3·6, we deduce that the morphisms in K lift to morphisms in $\mathrm{Aut}_{O^{3'}(\mathcal{D})}(E_1)$. Hence, $\mathrm{Out}_{O^{3'}(\mathcal{D})}(E_1)$ contains a cyclic group of order 8. Since $\mathrm{Out}_{O^{3'}(\mathcal{D})}(E_1) \cong \mathrm{SL}_2(3)$, this is a contradiction. Thus $\mathrm{Aut}^0_{\mathcal{D}}(S) = \mathrm{Aut}_{\mathcal{D}}(S)$ and applying Lemma 3·12 we must have that $\mathcal{D} = O^{3'}(\mathcal{D})$.

Applying [**5**, theorem II·9·8(d)], if $\mathcal{D}$ is not simple with $\mathcal{N} \trianglelefteq \mathcal{D}$ then by Lemma 5·8 we have that $\mathcal{N}$ is supported on E_1. Then by [**5**, proposition I·6·4], $\mathrm{Aut}_{\mathcal{N}}(E_1) \trianglelefteq \mathrm{Aut}_{\mathcal{D}}(E_1)$ so that $\mathrm{Out}_{\mathcal{N}}(E_1)$ is isomorphic to a normal $3'$-subgroup of $\mathrm{Out}_{\mathcal{D}}(E_1) \cong \mathrm{GL}_2(3)$ and hence is a subgroup of the quaternion group of order 8. In particular, E_3 is not essential in $\mathcal{N}$ for otherwise, applying an argument similar to Lemma 5·3, we would have that $O^{3'}(\mathrm{Aut}_{\mathcal{N}}(E_3)) \cong \mathrm{SL}_2(9)$ and we could again lift a cyclic subgroup of order 8 to $\mathrm{Aut}_{\mathcal{N}}(E_1)$, using Lemma 3·6. Then, we apply Proposition 5·20 (or just perform the MAGMA calculation on which this relies) to deduce that $\mathcal{E}(\mathcal{N}) = \emptyset$ and $E_1 = O_3(\mathcal{N})$, and so $E_1 \trianglelefteq \mathcal{D}$, a contradiction by Proposition 3·10. Hence, $\mathcal{D}$ is simple.

Since $\mathcal{D}$ is a simple fusion system which contains a non-trivial proper strongly closed subgroup, we deduce by Theorem 3·14 that $\mathcal{D}$ is exotic.

It feels prudent at this point to draw comparisons with some of the other exotic fusion systems already documented in the literature. We remark that the set of essentials $\{E_3^{\mathcal{D}}\}$ in some ways behave similarly to *pearls* as defined in [**22**], or the extensions of pearls as found in [**31**]. In some ways, our class $\{E_3^{\mathcal{D}}\}$ motivates an examination of a generalisation of pearls to *q-pearls* P where $O^{p'}(G) \cong q^2{:}\mathrm{SL}_2(q)$ for G some model of $N_{\mathcal{F}}(P)$ and $q = p^n$, as in [**12**].

Perhaps one can investigate an even further generalisation where we need only stipulate that $O^{p'}(G)/Z(O^{p'}(G)) \cong q^2{:}\mathrm{SL}_2(q)$ and we allow for $Z(O^{p'}(G)) \neq \{1\}$. All of these cases are linked with *pushing up* problems more familiar in local group theory, and we speculate that all of these examples are special cases of a more general phenomenon in this setting.

We also record the following interesting observation. As shown in [22, theorem 3·6], p-pearls are *never* contained in any larger essential subgroups, in direct contrast to situation in the fusion system $\mathcal{D}$. Perhaps the fusion systems where there is a class of q-pearls contained in a strictly larger essential subgroup have a more rigid structure and so may be organized in some suitable fashion.

We now delve into the study of all saturated fusion systems on S and throughout the remainder of this section, we let $\mathcal{F}$ be a saturated fusion system on S. As in the study of Co_1, we first limit the possible combinations of essentials we have in a saturated fusion system supported on S, as well as the potential automisers.

LEMMA 5·10. *Suppose that $\mathcal{F}$ is a saturated fusion system on S with $E_1 \in \mathcal{E}(\mathcal{F})$. Then $Aut_{\mathcal{F}}(E_1)$ is $Aut(E_1)$-conjugate to a subgroup of $Aut_{\mathcal{G}}(E_1)$ and $O^{3'}(Out_{\mathcal{F}}(E_1)) \cong SL_2(3)$.*

Proof. Since $Z(E_1)$ has order 9, and from the actions present in F_3, we deduce that $\mathrm{Aut}(E_1)/C_{\mathrm{Aut}(E_1)}(Z(E_1)) \cong \mathrm{GL}_2(3)$. Indeed, we calculate (see [44, appendix A]) that $|\mathrm{Aut}(E_1)|_{3'} = 16$ so that $C_{\mathrm{Aut}(E_1)}(Z(E_1))$ is a normal 3-subgroup. It follows that $\mathrm{Out}_{\mathcal{F}}(E_1)$ is isomorphic to a subgroup of $\mathrm{GL}_2(3)$ which contains a strongly 3-embedded subgroup and so $\mathrm{Out}_{\mathcal{F}}(E_1) \cong \mathrm{SL}_2(3)$ or $\mathrm{GL}_2(3)$. Indeed, $\mathrm{Out}_{\mathcal{F}}(E_1)$ is normal in a subgroup isomorphic to $\mathrm{GL}_2(3)$. We calculate that there are two conjugacy classes of subgroups of $\mathrm{Aut}(E_1)$ containing $\mathrm{Inn}(E_1)$ whose quotient by $\mathrm{Inn}(E_1)$ is isomorphic to $\mathrm{SL}_2(3)$. Moreover, $\mathrm{Aut}_S(E_1)$ is a subgroup of a conjugate of exactly one of these classes (see [44, appendix A]). Since $\mathrm{Aut}_S(E_1) \leq \mathrm{Aut}_{\mathcal{F}}(E_1) \cap \mathrm{Aut}_{\mathcal{G}}(E_1)$, we conclude that $O^{3'}(\mathrm{Aut}_{\mathcal{F}}(E_1))$ is $\mathrm{Aut}(E_1)$-conjugate to $O^{3'}(\mathrm{Aut}_{\mathcal{G}}(E_1))$. Moreover, if $\mathrm{Out}_{\mathcal{F}}(E_1) \cong \mathrm{GL}_2(3)$ then $\mathrm{Aut}_{\mathcal{F}}(E_1)$ is the product of $O^{3'}(\mathrm{Aut}_{\mathcal{F}}(E_1))$ and a Sylow 2-subgroup of $N_{\mathrm{Aut}(E_1)}(O^{3'}(\mathrm{Aut}_{\mathcal{F}}(E_1)))$ and so is $\mathrm{Aut}(E_1)$-conjugate to $\mathrm{Aut}_{\mathcal{G}}(E_1)$.

The following lemma uses several facts about the group E_2. These may be gleaned from [16, section 13] ($E_2 = Q_\beta$, $Z(\Phi(E_2)) = V_\beta$, $C_2 = C_\beta$ and $W_2 = W_\beta$) but are also computed explicitly in [44, appendix A].

LEMMA 5·11. *Suppose that $\mathcal{F}$ is a saturated fusion system on S with $E_2 \in \mathcal{E}(\mathcal{F})$. Set $C_2 := C_{E_2}(Z_3(S))$ and $W_2 := C_{E_2}([E_2, C_2])$. Then $Z_3(S) = Z_2(E_2)$, $|W_2| = 3^6$, $|Z(W_2)| = 3^4$ and $O^{3'}(Out_{\mathcal{F}}(E_2)) \cong SL_2(3)$ acts irreducibly on E_2/C_2, $W_2/Z(W_2)$ and $Z_2(E_2)/Z(E_2)$.*

Proof. We calculate the following in MAGMA (see [44, appendix A]). We have that $Z(S) = Z(E_2)$ has order 3 and $Z_3(S) = Z_2(E_2)$ has order 3^3. Moreover, C_2 has order 3^7 and so has index 3^2 in E_2. We have $Z(W_2) = [E_2, C_2]$ has order 3^4, W_2 has order 3^6 and $Z_2(E_2) < Z(W_2) = C_{E_2}(W_2)$. Finally, we have that $C_2 = C_{E_2}(W_2/Z_2(E_2))$. It remains to prove that $O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_2)) \cong \mathrm{SL}_2(3)$ acts irreducibly on E_2/C_2, $W_2/Z(W_2)$ and $Z_2(E_2)/Z(E_2)$.

We observe that as $Z_2(E_2) \leq Z(W_2)$, we must have that $W_2 \leq C_2$. Then $|C_2/W_2| = |Z(W_2)/Z_2(E_2)| = |Z(E_2)| = 3$ and so $O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_2))$ centralises each of these chief factors. We note that $[E_2, W_2] \leq [E_2, C_2] = Z(W_2)$. Let $R := C_{O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_2))}(W_2/Z(W_2)) \trianglelefteq \mathrm{Out}_{\mathcal{F}}(E_2)$. Assume that R is non-trivial, and so as $O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_2))$ has a strongly 3-embedded subgroup, there is $r \in R$ of $3'$-order. Then $[r, W_2] \leq Z(W_2)$ and as $O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_2))$ centralises $Z(W_2)/Z_2(E_2)$, we have that $[r, Z(W_2)] \leq Z_2(E_2)$ and by coprime action we deduce that $[r, W_2] \leq Z_2(E_2)$. We have that $[E_2, W_2] \leq [E_2, C_2] = Z(W_2)$ and so $[E_2, W_2, r] \leq Z_2(E_2)$. By the three subgroups lemma, we have that $[r, E, W_2] \leq Z_2(E_2)$. But $C_{E_2}(W_2/Z_2(E_2)) = C_2$ and so we deduce that $[r, E_2] \leq C_2$.

Again, as $O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_2))$ centralises C_2/W_2, we have that $[r, C_2] \leq W_2$ and $[r, W_2] \leq Z_2(E_2)$, and by coprime action we deduce that $[r, E_2] \leq Z_2(E_2)$. Hence, $[r, E_2, C_2] = \{1\}$, $[C_2, r, E_2] \leq [Z_2(E_2), E_2] = Z(E_2)$ and by the three subgroups lemma we conclude that $[E_2, C_2, r] \leq Z(E_2)$. But $[E_2, C_2] = Z(W_2) \geq Z_2(E_2)$ and so we ascertain that $[Z_2(E_2), r] \leq Z(E_2)$ and as r centralises $Z(E_2)$, a final application of coprime action yields that $[r, E_2] = \{1\}$, a contradiction since r is non-trivial. Hence, $R = \{1\}$ and $O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_2))$ acts faithfully on $W_2/Z(W_2)$. As $|W_2/Z(W_2)| = 3^2$, we conclude that $W_2/Z(W_2)$ is a natural module for $O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_2)) \cong \mathrm{SL}_2(3)$.

Since $|E_2/C_2| = |Z_2(E_2)/Z(E_2)| = 3^2$, to complete the proof it remains to show that $1 \neq t \in Z(O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_2)))$ acts non-trivially on E_2/C_2 and $Z_2(E_2)/Z(E_2)$. Note that if $[t, Z_2(E_2)] \leq Z(E_2)$ then by coprime action, $[t, Z_2(E_2)] = \{1\}$. An application of the three subgroups lemma would then yield that $[t, E_2] \leq C_2$. Hence, it suffices to demonstrate that $[t, E_2] \not\leq C_2$. Assume otherwise. Since $[t, C_2] \leq W_2$, by coprime action we have that $[t, E_2] \leq W_2$. Then $[t, E_2, W_2] \leq [W_2, W_2] \leq [C_2, W_2] = Z_2(E_2)$. Moreover, $[E_2, W_2, t] = [Z(W_2), t] \leq Z_2(E_2)$. By the three subgroups lemma, we infer that $[t, W_2, E_2] \leq Z_2(E_2)$. But $W_2 = [t, W_2]Z(W_2)$ and $[E_2, Z(W_2)] \leq Z_2(E_2)$ so that $[W_2, E_2] \leq Z_2(E_2)$, a contradiction as $C_2 = C_{E_2}(W_2/Z_2(E_2))$. Hence, t acts non-trivially on E_2/C_2, which completes the proof.

LEMMA 5·12. *Suppose that $\mathcal{F}$ is a saturated fusion system on S such that $\{E_1, E_2\} \subseteq \mathcal{E}(\mathcal{F})$. Then $O_3(\mathcal{F}) = \{1\}$.*

Proof. Assume that $\mathcal{F}$ is a saturated fusion system on S such that $\{E_1, E_2\} \subseteq \mathcal{E}(\mathcal{F})$ and suppose that $\{1\} \neq Q \trianglelefteq \mathcal{F}$. By Proposition 3·10, we have that $Q \leq E_1 \cap E_2$. Then $Z(S) \leq Q$ and as $\mathrm{Out}_{\mathcal{F}}(E_1)$ acts irreducibly on $Z(E_1)$ by Lemma 5·10, we deduce that $Z(E_1) = Z_2(S) \leq Q$. By Lemma 5·11, we have that $O^{3'}(\mathrm{Aut}_{\mathcal{F}}(E_2))$ acts irreducibly on $Z_3(S)/Z(S)$ and so $Z_3(S) \leq Q$. Since $\Phi(E_1) \trianglelefteq S$ and $Z_2(S) < \Phi(E_1)$, we have that $Z_3(S) < \Phi(E_1)$. Then using the descriptions of F_3 in [**16**, p. 100], we have that $O^{3'}(\mathrm{Aut}_{\mathcal{G}}(E_1))$ acts irreducibly on $\Phi(E_1)/Z_2(S)$. By Lemma 5·10, $O^{3'}(\mathrm{Aut}_{\mathcal{F}}(E_1))$ is $\mathrm{Aut}(E_1)$-conjugate to $O^{3'}(\mathrm{Aut}_{\mathcal{G}}(E_1))$ and so we deduce that $\mathrm{Aut}_{\mathcal{F}}(E_1)$ acts irreducibly on $\Phi(E_1)/Z_2(S)$. Thus, $\Phi(E_1) \leq Q \leq E_1 \cap E_2$.

Now, if $\Phi(O_3(\mathcal{F}))$ is non-trivial then by the above argument we have that $\Phi(E_1) \leq \Phi(O_3(\mathcal{F})) \leq O_3(\mathcal{F}) \leq E_1 \cap E_2$. But $\Phi(O_3(\mathcal{F})) \leq \Phi(\Phi(E_1 \cap E_2)) \leq \Phi(E_1)$, and we conclude that $\Phi(E_1) \trianglelefteq \mathcal{F}$. If $\Phi(O_3(\mathcal{F})) = \{1\}$ and $O_3(\mathcal{F}) \neq \{1\}$ then $O_3(\mathcal{F})$ is elementary abelian and contains $\Phi(E_1)$, and since $\Phi(E_1)$ is elementary abelian of maximal order, the only possibility is that $O_3(\mathcal{F}) = \Phi(E_1)$. Either way $\Phi(E_1) \trianglelefteq \mathcal{F}$. But in the language of Lemma 5·11, we have by a calculation (see [**44**, appendix A]) that $Z(W_2) < \Phi(E_1) < W_2$. As $\mathrm{Out}_{\mathcal{F}}(E_2)$ acts irreducibly on $W_2/Z(W_2)$ by Lemma 5·11, this is a contradiction.

PROPOSITION 5·13. *Suppose that $\mathcal{F}$ is a saturated fusion system on S such that $O_3(\mathcal{F}) \neq \{1\}$. Then either*:

(i) $\mathcal{F} = N_{\mathcal{F}}(S)$; or
(ii) $\mathcal{F} = N_{\mathcal{F}}(E_i)$ where $O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_i)) \cong \mathrm{SL}_2(3)$ for $i \in \{1, 2\}$.

Proof. If $\mathcal{E}(\mathcal{F}) = \emptyset$, then outcome (i) is satisfied by the Alperin–Goldschmidt theorem. Thus, by Lemma 5·3 and Lemma 5·12, we may assume that E_i is the unique essential subgroup of $\mathcal{F}$ and apply Lemma 5·10 and Lemma 5·11.

LEMMA 5·14. *Suppose that $\mathcal{F}_1, \mathcal{F}_2$ are two saturated fusion systems supported on T where $E_1 \le T \le S$. If $E_3 \in \mathcal{E}(\mathcal{F}_1) \cap \mathcal{E}(\mathcal{F}_2)$ and $N_{\mathcal{F}_1}(E_1) = N_{\mathcal{F}_2}(E_1)$ then $Aut_{\mathcal{F}_1}(E_3) = Aut_{\mathcal{F}_2}(E_3)$.*

Proof. By Lemma 5·3, we have that $O^{3'}(\mathrm{Aut}_{\mathcal{F}_i}(E_3)) \cong \mathrm{SL}_2(9)$ for $i \in \{1, 2\}$. Write $X := O^{3'}(\mathrm{Aut}_{\mathcal{F}_1}(E_3))$ and $Y := O^{3'}(\mathrm{Aut}_{\mathcal{F}_2}(E_3))$. Set $K := N_{\mathrm{Aut}_{\mathcal{F}_1}(E_3)}(\mathrm{Aut}_T(E_3))$ so that, by Lemma 3·6, all morphisms in K lift to morphisms in $\mathrm{Aut}_{\mathcal{F}_1}(E_1) = \mathrm{Aut}_{\mathcal{F}_2}(E_1)$. In particular,

$$K = N_{\mathrm{Aut}_{N_{\mathcal{F}_1}(E_1)}(E_3)}(\mathrm{Aut}_T(E_3)) = N_{\mathrm{Aut}_{N_{\mathcal{F}_2}(E_1)}(E_3)}(\mathrm{Aut}_T(E_3)) = N_{\mathrm{Aut}_{\mathcal{F}_2}(E_3)}(\mathrm{Aut}_T(E_3)).$$

Let L be the unique cyclic subgroup of order 8 of a fixed Sylow 2-subgroup of K arranged such that $K_L := L\mathrm{Aut}_S(E_3) = N_{O^{3'}(\mathrm{Aut}_{\mathcal{F}_1}(E_3))}(\mathrm{Aut}_S(E_3))$. Then $K_L \le X \cap Y \le \mathrm{Aut}(E_1) \cong \mathrm{GL}_5(3)$. We record that there is a unique conjugacy class of subgroups isomorphic to $\mathrm{SL}_2(9)$ in $\mathrm{GL}_5(3)$ (see [**44**, appendix A]). Hence, there is $g \in \mathrm{Aut}(E_3)$ with $Y = X^g$.

Then $K_L, (K_L)^g \le Y$ and so, by Sylow's theorem, there is $y \in Y$ such that $(K_L)^g = (K_L)^y$. Thus, we have that $X^{gy^{-1}} = X^g$ and we calculate that $gy^{-1} \le N_{\mathrm{GL}_5(3)}(K_L) \le N_{\mathrm{GL}_5(3)}(X)$ (see [**44**, appendix A]). But then $X = X^g = Y$. By a Frattini argument, $\mathrm{Aut}_{\mathcal{F}_1}(E_3) = XK = YK = \mathrm{Aut}_{\mathcal{F}_2}(E_3)$.

THEOREM 5·15. *Suppose that $\mathcal{F}$ is saturated fusion system on S such that $O_3(\mathcal{F}) = \{1\}$. If $E_2 \notin \mathcal{E}(\mathcal{F})$ then $\mathcal{F} \cong \mathcal{D}$.*

Proof. Suppose that $E_2 \notin \mathcal{E}(\mathcal{F})$. Since $O_3(\mathcal{F}) = \{1\}$ and E_1 is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant, an application of Proposition 3·10 using Proposition 5·2 implies that some $E_3^{\mathcal{F}} \cap \mathcal{E}(\mathcal{F}) \ne \emptyset$. Hence by Lemma 5·3 we have that $E_1, E_3 \in \mathcal{E}(\mathcal{F})$ and $O^{3'}(\mathrm{Aut}_{\mathcal{F}}(E_3)) \cong \mathrm{SL}_2(9)$. Then for k an element of order 8 in $N_{O^{3'}(\mathrm{Aut}_{\mathcal{F}}(E_3))}(\mathrm{Aut}_S(E_3))$, by Lemma 3·6, and since E_1 is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant, k lifts to an element of order 8 in $\mathrm{Aut}(E_1)$. Now, by Lemma 5·10, we have that $O^{3'}(\mathrm{Out}_{\mathcal{F}}(E_1)) \cong \mathrm{SL}_2(3)$ and since $\mathrm{SL}_2(3)$ has no cyclic subgroups of order 8, the Sylow 2-subgroups of $\mathrm{Aut}_{\mathcal{F}}(E_1)$ have order at least 16. We calculate (see [**44**, appendix A]) that $|\mathrm{Aut}(E_1)|_{3'} = 16$ so that $\mathrm{Aut}_{\mathcal{F}}(E_1)$ contains a Sylow 2-subgroup of $\mathrm{Aut}(E_1)$, and $\mathrm{Out}_{\mathcal{F}}(E_1) \cong \mathrm{GL}_2(3)$.

Let t be an element of $\mathrm{Aut}(S)$ of order coprime to 3. Since E_1 is $\mathrm{Aut}(S)$-invariant, t normalises E_1. Since E_1 is self-centralising in S, an application of the three subgroups lemma and coprime action reveals that t acts non-trivially on E_1. Hence, a Hall $3'$-subgroup of $\mathrm{Aut}(S)$ restricts faithfully to $N_{\mathrm{Aut}(E_1)}(\mathrm{Aut}_S(E_1))$. As in Lemma 5·10, since $Z(E_1)$ has order 9 and from the actions present in F_3, we conclude that $\mathrm{Aut}(E_1)/C_{\mathrm{Aut}(E_1)}(Z(E_1)) \cong \mathrm{GL}_2(3)$ and $C_{\mathrm{Aut}(E_1)}(Z(E_1))$ is a normal 3-subgroup of $\mathrm{Aut}(E_1)$. Now, a Hall $3'$-subgroup of $\mathrm{Aut}(S)$ also normalises $Z(E_1)$ and so it restricts faithfully to $N_{\mathrm{Aut}(E_1)}(\mathrm{Aut}_S(E_1)C_{\mathrm{Aut}(E_1)}(Z(E_1)))$, and as $\mathrm{Aut}(E_1)/C_{\mathrm{Aut}(E_1)}(Z(E_1)) \cong \mathrm{GL}_2(3)$ and $[\mathrm{Aut}_S(E_1), Z(E_1)] \ne \{1\}$, we conclude that a Hall $3'$-subgroup of $\mathrm{Aut}(S)$ is elementary abelian of order at most 4. Since $N_{\mathrm{Aut}_{\mathcal{F}}(E_1)}(\mathrm{Aut}_S(E_1))$ contains an elementary abelian subgroup of order 4 which, by Lemma 3·2, lifts to $\mathrm{Aut}_{\mathcal{F}}(S) \le \mathrm{Aut}(S)$ we conclude that a Hall $3'$-subgroup of $\mathrm{Aut}(S)$ and of $\mathrm{Aut}_{\mathcal{F}}(S)$ is elementary abelian of order 4. In particular, $\mathrm{Out}_{\mathcal{F}}(S)$ in elementary abelian of order 4.

By the Alperin–Goldschmidt theorem and using that E_1 is characteristic in S, we have that $\mathcal{F} = \langle N_{\mathcal{F}}(E_1), \mathrm{Aut}_{\mathcal{F}}(E_3)\rangle_S$ and $\mathcal{D} = \langle N_{\mathcal{D}}(E_1), \mathrm{Aut}_{\mathcal{D}}(E_3)\rangle_S$. Hence, by Lemma 5·14 to show that $\mathcal{F} \cong \mathcal{D}$ it suffices to show that there is $\alpha \in \mathrm{Aut}(S)$ with $N_{\mathcal{F}^\alpha}(E_1) = N_{\mathcal{D}}(E_1)$.

Now, $\mathrm{Aut}_{\mathcal{F}}(S)$ contains a Hall $3'$-subgroup of $\mathrm{Aut}(S)$. By Hall's theorem, there is $\alpha_1 \in \mathrm{Aut}(S)$ such that $\mathrm{Aut}_{\mathcal{F}^{\alpha_1}}(S) = \mathrm{Aut}_{\mathcal{F}}(S)^{\alpha_1} = \mathrm{Aut}_{\mathcal{D}}(S)$. By the Alperin–Goldschmidt theorem, we see that $N_{\mathcal{F}^{\alpha_1}}(S) = N_{\mathcal{D}}(S)$. Then

$$\begin{aligned} K := N_{\mathrm{Aut}_{\mathcal{F}^{\alpha_1}}(E_1)}(\mathrm{Aut}_S(E_1)) &= N_{\mathrm{Aut}_{N_{\mathcal{F}^{\alpha_1}}(S)}(E_1)}(\mathrm{Aut}_S(E_1)) \\ &= N_{\mathrm{Aut}_{N_{\mathcal{D}}(S)}(E_1)}(\mathrm{Aut}_S(E_1)) \\ &= N_{\mathrm{Aut}_{\mathcal{D}}(E_1)}(\mathrm{Aut}_S(E_1)). \end{aligned}$$

We calculate that in $\mathrm{Aut}(E_1)$ there are three candidates for the group $\mathrm{Aut}_{\mathcal{F}}(E_1)$ which contain K appropriately and that there is an element which conjugates the three candidates and extends to an automorphism of S which preserves the class $\{E_3^S\}$ (see [**44**, appendix A]). In particular, there is $\alpha_2 \in \mathrm{Aut}(S)$ with $\mathrm{Aut}_{\mathcal{F}^{\alpha_1\alpha_2}}(E_1) = \mathrm{Aut}_{\mathcal{D}}(E_1)$. Hence, by Theorem 3·11 there is $\beta \in \mathrm{Aut}(S)$ with $\alpha := \alpha_1\alpha_2\beta$ and $N_{\mathcal{F}^{\alpha}}(E_1) = N_{\mathcal{D}}(E_1)$, as required.

We are now almost in a position to determine all saturated fusion systems on S. First, we require the notion of an amalgam of type F_3. We refer to [**16**, p. 100] for the notion of an amalgam of type F_3, noting that by a result of Delgado [**17**] such amalgams are unique up to *parabolic isomorphism*. We first record a short lemma recognising an amalgam of type F_3 from our hypothesis. For the following, as in [**16**], we conceal the relevant monomorphisms involved in the amalgam and instead work with identified subgroups.

LEMMA 5·16. *Let $\mathcal{A} := \mathcal{A}(G_1, G_2, G_{12})$ be an amalgam of finite groups. Write $Q_i := O_p(G_i)$ and $L_i = O^{p'}(G_i)$ for $i \in \{1, 2\}$. Suppose the following conditions hold*:

(i) there is $S \in \mathrm{Syl}_p(G_1) \cap \mathrm{Syl}_p(G_2)$ such that $G_{12} = N_{G_1}(S) = N_{G_2}(S)$;

(ii) $L_i/Q_i \cong \mathrm{SL}_2(3)$;

(iii) $C_{G_i}(Q_i) \leq Q_i$; and

(iv) S is isomorphic to a Sylow 3-subgroup of F_3.

Then $\mathcal{A}$ is an amalgam of type F_3.

Proof. Conditions (i), (ii) and (iii) promise that we satisfy [**16**, hypothesis A] so that $\mathcal{A}$ is a *weak BN-pair of rank 2*. We apply [**16**, theorem A]. Since S is isomorphic to a Sylow 3-subgroup of F_3, we have that $|S| = 3^{10}$. We appeal to [**21**, Table 2·2] for the structure of the rank two groups of Lie type in characteristic 3 (specifically the orders of their Sylow 3-subgroups), and it follows that that only possibilities are that $\mathcal{A}$ is isomorphic to an amalgam associated to $\mathrm{PSU}_5(3)$; or that $\mathcal{A}$ is amalgam of type F_3. However, in the first case we do not satisfy (ii) (the relevant parabolics in $\mathrm{PSU}_5(3)$ have quotient isomorphic to $\mathrm{PSU}_3(3)$ and $\mathrm{SL}_2(9)$). Hence, we have that $\mathcal{A}$ is an amalgam of type F_3.

In the setting above, we may freely use any of the structural results obtained in Section 13 of [**16**] pertaining to amalgams of type F_3. In particular, all the necessary conditions in Delgado's proof [**17**] that such amalgams are unique up to parabolic isomorphism follows from results there.

We provide the following result, which appears to have evaded the literature up until this point.

PROPOSITION 5·17. *Let $\mathcal{A}$ be an amalgam of type F_3. Then $\mathcal{A}$ is unique up to isomorphism.*

For this, we apply the computer implementation of Goldschmidt's lemma [19, (2·7)] found in Cano's PhD Thesis [11, p. 34] (mirrored in [44, appendix A]) in MAGMA. This takes as input four groups: P_1, B_1, P_2, B_2. It then outputs a 4-tuple, of which the first entry is the one we are interested in. We appeal to the online version of the Atlas of Finite Group Representations [1] for a matrix representation of the group F_3, namely its 248-dimensional representation over GF(2). We then use [1] to obtain the matrices which generate two distinct maximal subgroups of F_3 which contain a Sylow 3-subgroup. These groups represent P_1 and P_2 in our case.

By the main result of [17], the parabolic subgroups defining an amalgam of type F_3 are unique up to isomorphism and so, the groups P_1 and P_2 have the isomorphism type of the parabolic groups in any F_3-type amalgam. Hence, we are justified in our choice of subgroups to take. Then the groups B_i are defined as $N_{P_i}(S_i)$, where S_i is any Sylow 3-subgroup of P_i, for $i \in \{1,2\}$. The function then outputs 1 as its first entry, and so the amalgam is unique.

THEOREM 5·18. *Suppose that $\mathcal{F}$ is saturated fusion system on S such that $O_3(\mathcal{F}) = \{1\}$. Then $\mathcal{F} \cong \mathcal{D}, \mathcal{G}$ or $\mathcal{H}$.*

Proof. Observe that if $E_1 \notin \mathcal{E}(\mathcal{F})$ then by Proposition 5·2 and Lemma 5·3, we would have that $\mathcal{E}(\mathcal{F}) = \{E_2\}$. Since E_2 is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant, Proposition 3·10 would imply that $O_3(\mathcal{F}) \neq \{1\}$. Hence, as $O_3(\mathcal{F}) = \{1\}$, we must have that that $E_1 \in \mathcal{E}(\mathcal{F})$. By Theorem 5·15, we may assume that that $\{E_1, E_2\} \subseteq \mathcal{E}(\mathcal{F})$ and form $\mathcal{T} := \langle \mathrm{Aut}_{\mathcal{F}}(E_1), \mathrm{Aut}_{\mathcal{F}}(E_2), \mathrm{Aut}_{\mathcal{F}}(S)\rangle_S$. If $E_3 \in \mathcal{E}(\mathcal{F})$ then $\mathcal{T}$ is the $\mathcal{F}$-analogue of $\mathcal{H}$ and the proof that $\mathcal{T}$ is saturated is the same as the proof that $\mathcal{H}$ is saturated, relying on Lemma 3·8. If $E_3 \notin \mathcal{E}(\mathcal{F})$ then by the Alperin–Goldschmidt theorem we have that $\mathcal{F} = \mathcal{T}$. In either case, $\mathcal{E}(\mathcal{T}) = \{E_1, E_2\}$ and $O_3(\mathcal{T}) = \{1\}$ by Lemma 5·12.

For $i \in \{1,2\}$, let G_i be a model for $N_{\mathcal{F}}(E_i)$. Since E_i is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant, by the uniqueness of models provided by Theorem 3·11, we may embed the model for $N_{\mathcal{F}}(S)$, which we denote G_{12}, into G_i for $i \in \{1,2\}$. Applying [39, theorem 1], we ascertain that $\mathcal{T} = \langle \mathcal{F}_S(G_1), \mathcal{F}_S(G_2)\rangle_S = \mathcal{F}_S(G_1 *_{G_{12}} G_2)$. Furthermore, by Lemma 5·16, the tuple (G_1, G_2, G_{12}) (upon identifying subgroups in the free amalgamated product with the appropriate injective maps) forms an amalgam of type F_3. By Proposition 5·17, this amalgamated product is determined up to isomorphism, and so $\mathcal{T}$ is unique up to isomorphism. In particular, $\mathcal{T}$ is the unique (up to isomorphism) saturated fusion system on S with $O_3(\mathcal{T}) = \{1\}$ and $\mathcal{E}(\mathcal{T}) = \{E_1, E_2\}$. Since $\mathcal{H}$ satisfies these conditions, we must have that $\mathcal{T} \cong \mathcal{H}$.

We may as well assume now that $E_3 \in \mathcal{E}(\mathcal{F})$ and by the Alperin–Goldschmidt theorem, that $\mathcal{F} = \langle \mathcal{T}, \mathrm{Aut}_{\mathcal{F}}(E_3)\rangle_S$. By the proof of Lemma 5·3, utilising the MAGMA computations from [44, appendix A], we see that every elementary abelian subgroup A of E_1 of order 3^5 with $|N_{E_1}(A)| = 3^7$ is S-conjugate to E_3. Since E_1 is $\mathrm{Aut}(S)$-invariant, these conditions are maintained under the action of $\mathrm{Aut}(S)$ and so we conclude that $\{E_3^{\mathcal{F}}\} = \{E_3^S\} = \{E_3^{\mathrm{Aut}(S)}\}$. Hence, $\{E_3^{\mathcal{F}}\} = \{E_3^{\mathcal{F}^\alpha}\}$ for any $\alpha \in \mathrm{Aut}(S)$. Thus, replacing $\mathcal{F}$ by $\mathcal{F}^\alpha$ for some $\alpha \in \mathrm{Aut}(S)$, we have that $\mathcal{F} = \langle \mathcal{H}, \mathrm{Aut}_{\mathcal{F}}(E_3)\rangle_S$ and upon demonstrating that $\mathrm{Aut}_{\mathcal{F}}(E_3) = \mathrm{Aut}_{\mathcal{G}}(E_3)$ we will have shown that $\mathcal{F} = \mathcal{G}$. But $N_{\mathcal{F}}(E_1) = N_{\mathcal{H}}(E_1) = N_{\mathcal{G}}(E_1)$ and so Lemma 5·14 gives $\mathrm{Aut}_{\mathcal{F}}(E_3) = \mathrm{Aut}_{\mathcal{G}}(E_3)$, as desired.

We provide the following Table 2 summarizing the actions induced by the fusion systems described in Theorem 5·18 on their centric-radical subgroups. The entry "-" indicates that the subgroup is no longer centric-radical in the subsystem.

Table 2. *$\mathcal{G}$-conjugacy classes of radical-centric subgroups of S*

P	$\lvert P\rvert$	$\mathrm{Out}_{\mathcal{G}}(P)$	$\mathrm{Out}_{\mathcal{H}}(P)$	$\mathrm{Out}_{\mathcal{D}}(P)$
S	3^{10}	2×2	2×2	2×2
E_1	3^9	$\mathrm{GL}_2(3)$	$\mathrm{GL}_2(3)$	$\mathrm{GL}_2(3)$
E_2	3^9	$\mathrm{GL}_2(3)$	$\mathrm{GL}_2(3)$	–
E_3	3^5	$\mathrm{SL}_2(9).2$	–	$\mathrm{SL}_2(9).2$

We describe a pair of bonus exotic fusion systems related to the exotic system $\mathcal{D}$. Using that E_1 is characteristic in S, and applying Lemma 3·6, the morphisms in $N_{\mathrm{Aut}_{\mathcal{D}}(E_3)}(\mathrm{Aut}_S(E_3))$ extend to a group of morphisms in $\mathrm{Aut}_{\mathcal{D}}(E_1)$ which we denote by K. Then $|K|_{3'}=16$. Let G be a model for $N_{\mathcal{D}}(E_1)$ and let H be a subgroup of G chosen such that $\mathrm{Aut}_H(E_1)=K\mathrm{Inn}(E_1)$. In particular, H is the product of E_1 with some Sylow 2-subgroup of G. We define the subsystem

$$\mathcal{D}^* := \langle \mathrm{Aut}_{\mathcal{D}}(E_3), \mathcal{F}_{E_1}(H)\rangle_{E_1} \le \mathcal{D}.$$

Note that the conjugacy class of E_3 in S splits into three distinct classes upon restricting only to E_1. Indeed, in this way we have three choices for the construction of $\mathcal{D}^*$, corresponding to the three E_1-conjugacy classes of S-conjugates of E_3, which in turn correspond to the three choices of Sylow 2-subgroups of $\mathrm{Out}_{\mathcal{D}}(E_1)$. Since the choice is induced by an element of $\mathrm{Aut}(E_1)$, all choices give rise to isomorphic fusion systems.

PROPOSITION 5·19. *$\mathcal{D}^*$ is saturated fusion system on E_1 and $O^{3'}(\mathcal{D}^*)$ has index 2 in $\mathcal{D}^*$.*

Proof. We create H as in the construction of $\mathcal{D}^*$ and consider $\mathcal{F}_{E_1}(H)$. Since $\mathcal{F}_{E_1}(H)\subseteq\mathcal{D}$, and as E_3 is fully $\mathcal{D}$-normalised and $N_S(E_3)\le E_1$, E_3 is also fully $\mathcal{F}_{E_1}(H)$-normalised. Since $C_{E_1}(E_3)\le E_3$ we see that E_3 is also $\mathcal{F}_{E_1}(H)$-centric. Finally, since E_3 is abelian, it is minimal among S-centric subgroups with respect to inclusion and has the property that no proper subgroup of E_3 is essential in $\mathcal{F}_{E_1}(H)$. In the statement of Proposition 3·9, letting $\mathcal{F}_0=\mathcal{F}_{E_1}(H)$, $V=E_3$ and $\Delta=\mathrm{Aut}_{\mathcal{D}}(E_3)$, we have that $\widetilde{\Delta}:=\mathrm{Aut}_{\mathcal{F}_{E_1}(H)}(E_3)=N_{\mathrm{Aut}_{\mathcal{D}}(E_3)}(\mathrm{Aut}_S(E_3))$ is strongly 3-embedded in Δ. By that result, $\mathcal{D}^*=\langle\mathrm{Aut}_{\mathcal{D}}(E_3),\mathcal{F}_{E_1}(H)\rangle_{E_1}$ is a saturated fusion system.

In the construction of $\mathcal{D}^*$, we may have taken in place of K the group obtained by lifting the morphisms in $N_{\mathrm{Aut}_{O^{3'}(\mathcal{D})}(E_3)}(\mathrm{Aut}_S(E_3))$ to $\mathrm{Aut}_{\mathcal{D}}(E_1)$ and forming $\widehat{H}$ of index 2 in H with $\mathrm{Aut}_{\widehat{H}}(E_3)=N_{\mathrm{Aut}_{O^{3'}(\mathcal{D})}(E_3)}(\mathrm{Aut}_S(E_3))$. Letting $\mathcal{F}_0=\mathcal{F}_{E_1}(\widehat{H})$, $\widetilde{\Delta}=N_{\mathrm{Aut}_{O^{3'}(\mathcal{D})}(E_3)}(\mathrm{Aut}_S(E_3))$ $(\mathrm{Aut}_S(E_3))$ and $\Delta=\mathrm{Aut}_{O^{3'}(\mathcal{D})}(E_3)$ and applying Proposition 3·9, the fusion system $\widehat{\mathcal{D}^*}=\langle O^{3'}(\mathrm{Aut}_{\mathcal{D}}(E_3)),\mathcal{F}_{E_1}(\widehat{H})\rangle_{E_1}$ is a saturated fusion subsystem of $\mathcal{D}$.

By construction, $\mathcal{D}^*=\langle\widehat{\mathcal{D}^*},\mathrm{Aut}_{\mathcal{D}^*}(E_1)\rangle_{E_1}$ and it is clear that for all $\alpha\in\mathrm{Aut}_{\mathcal{D}^*}(E_1)$, $\widehat{\mathcal{D}^*}^{\alpha}=\widehat{\mathcal{D}^*}$. Hence, applying [5, proposition I·6·4], we have that $\widehat{\mathcal{D}^*}$ is weakly normal in $\mathcal{D}^*$ in the sense of [5, definition I·6·1] and [15, theorem A] yields that $O^{3'}(\widehat{\mathcal{D}^*})\trianglelefteq\mathcal{D}^*$. Then $O^{3'}(\mathrm{Aut}_{\mathcal{D}^*}(T))\le\mathrm{Aut}_{O^{3'}(\widehat{\mathcal{D}^*})}(T)\trianglelefteq\mathrm{Aut}_{\mathcal{D}^*}(T)$ by [5, proposition I·6·4] for all $T\le E_1$, and we deduce that $O^{3'}(\widehat{\mathcal{D}^*})$ has index prime to 3 in $\mathcal{D}^*$. It quickly follows that $\widehat{\mathcal{D}^*}$ has index prime to 3 in $\mathcal{D}^*$ and as $\mathrm{Aut}_{\widehat{\mathcal{D}^*}}(E_1)\le\mathrm{Aut}^0_{\mathcal{D}^*}(E_1)$, we see that $\mathrm{Aut}_{\widehat{\mathcal{D}^*}}(E_1)=\mathrm{Aut}^0_{\mathcal{D}^*}(E_1)$ has index 2 in $\mathrm{Aut}_{\mathcal{D}^*}(E_1)$. A final application of Lemma 3·12 gives that $O^{3'}(\mathcal{D}^*)=\widehat{\mathcal{D}^*}$ has index 2 in $\mathcal{D}^*$, as desired.

We provide some more generic results regarding all possible saturated fusion systems supported on E_1. Although we do not formally prove the following proposition, its conclusion merits some explanation. Let $\mathcal{F}$ be a saturated fusion system on E_1. It is fairly easy to show that $\mathcal{E}(\mathcal{F}) \subseteq \{E_3^S\}$ so we take this as a starting point.

For E_3^s some S-conjugate of E_3 with $s \notin E_1$, if $E_3, E_3^s \in \mathcal{E}(\mathcal{F})$ then it quickly follows that $O^{3'}(\mathrm{Aut}_{\mathcal{F}}(E_3)) \cong O^{3'}(\mathrm{Aut}_{\mathcal{F}}(E_3^s)) \cong \mathrm{SL}_2(9)$ (as witnessed in Lemma 5·23). Applying Lemma 3·6 to E_3 and E_3^s, we have that for T a cyclic subgroup of $O^{3'}(\mathrm{Aut}_{\mathcal{F}}(E_3))$ of order 8 which normalises $N_{E_1}(E_3)$ the morphisms in T lift to morphisms in $\mathrm{Aut}_{\mathcal{F}}(E_1)$. Similarly, T^s also lifts. Note that no element of T centralises $Z_2(S) = Z(E_1)$ and so both T and T^s project to cyclic subgroups of order 8 in $\mathrm{Aut}(E_1)/C_{\mathrm{Aut}(E_1)}(Z(E_1)) \cong \mathrm{GL}_2(3)$. But then the projection of $\langle T, T^s\rangle$ is divisible by 3, a contradiction since $\mathrm{Inn}(E_1) \in \mathrm{Syl}_3(\mathrm{Aut}_{\mathcal{F}}(E_1))$ and $\mathrm{Inn}(E_1) \le C_{\mathrm{Aut}(E_1)}(Z(E_1))$.

We conclude that if $E_3 \in \mathcal{E}(\mathcal{F})$ then the only S-conjugates of E_3 in $\mathcal{E}(\mathcal{F})$ are the E_1 conjugates of E_3. Moreover, for $s \in S \setminus E_1$ and α_s the automorphism of E_1 induced by conjugation by s, $\mathcal{F}^{\alpha_s} \cong \mathcal{F}$ and if $\{E_3^{E_1}\} \subseteq \mathcal{E}(\mathcal{F})$ then $\{(E_3^s)^{E_1}\} \in \mathcal{E}(\mathcal{F}^{\alpha_s})$. Since we only care about classifying fusion systems up to isomorphism, we may as well assume that $E_3 \in \mathcal{E}(\mathcal{F})$, leading to the following result (which is verified computationally [**44**, appendix A]).

PROPOSITION 5·20. *Let $\mathcal{F}$ be a saturated fusion system supported on E_1. Then $\mathcal{E}(\mathcal{F}) \subseteq \{E_3^{E_1}\}$.*

We return to some properties of the systems $\mathcal{D}$ and $O^{3'}(\mathcal{D})$.

PROPOSITION 5·21. *$N_{E_1}(E_3)$ is the unique, proper, non-trivial, strongly closed subgroup in both $O^{3'}(\mathcal{D}^*)$ and $\mathcal{D}^*$, and $\mathcal{D}^{*frc} = O^{3'}(\mathcal{D}^*)^{frc} = \{E_3^{\mathcal{D}^*}, E_1\}$.*

Proof. Since $N_{E_1}(E_3)$ is normalised by $\mathrm{Aut}_{\mathcal{D}^*}(E_1)$ and contains all $\mathcal{D}^*$-conjugates of E_3, we have by the Alperin–Goldschmidt theorem that $N_{E_1}(E_3)$ is strongly closed in $\mathcal{D}^*$ and $O^{3'}(\mathcal{D}^*)$. Assume that T is a proper non-trivial strongly closed subgroup of $\mathcal{D}^*$. Then $\langle (T \cap E_3)^{\mathrm{Aut}_{\mathcal{D}}(E_3)}\rangle \le T$ and since $T \trianglelefteq E_1$, we must have by Lemma 5·3 that $[E_3, \mathrm{Aut}_{\mathcal{D}^*}(E_3)] \le T$. But then $\langle [E_3, \mathrm{Aut}_{\mathcal{D}^*}(E_3)]^{E_1}\rangle \le T$ and one can calculate (see [**44**, appendix A]) that this implies that $N_{E_1}(E_3) \le T$.

Let τ be a non-trivial involution in $Z(O^{3'}(\mathrm{Aut}_{\mathcal{D}^*}(E_3)))$. By Lemma 3·6, τ lifts to $\widetilde{\tau} \in \mathrm{Aut}_{O^{3'}(\mathcal{D})}(E_1)$. Suppose that $[\widetilde{\tau}, E_1] \le N_{E_1}(E_3)$. Since $\widetilde{\tau}$ is the extension of τ to $\mathrm{Aut}_{\mathcal{D}^*}(N_{E_1}(E_3))$ and τ centralises $\mathrm{Aut}_{E_1}(E_3)$, we conclude that $[\widetilde{\tau}, N_{E_1}(E_3)] \le E_3$ and so by coprime action, we have that $[\widetilde{\tau}, E_1] \le E_3$. Since E_3 is abelian and $[E_1, E_3, \widetilde{\tau}] \le [\Phi(E_1), \widetilde{\tau}] = Z(E_1)$, the three subgroups lemma implies that $[E_3, \widetilde{\tau}, E_1] \le Z(E_1)$. But then, as $E_3 = [E_3, \tau]Z(N_{E_1}(E_3))$ and $[E_1, Z(N_{E_1}(E_3))] \le [E_1, \Phi(E_1)] = Z(E_1)$, we have that $E_3 \trianglelefteq E_1$, a contradiction. Hence, $\widetilde{\tau}$ acts non-trivially on $E_1/N_{E_1}(E_3)$ and since the Sylow 2-subgroups of $\mathrm{Aut}_{O^{5'}(\mathcal{D}^*)}(E_1)$ are cyclic of order 8, we deduce that a Sylow 2-subgroup of $\mathrm{Aut}_{O^{5'}(\mathcal{D}^*)}(E_1)$ acts faithfully and irreducibly on $E_1/N_{E_1}(E_3)$. We conclude that $T = N_{E_1}(E_3)$ is the unique proper non-trivial strongly closed subgroup of both $O^{3'}(\mathcal{D}^*)$ and $\mathcal{D}^*$.

Let $\mathcal{F} \in \{\mathcal{D}, O^{3'}(\mathcal{D})\}$ and assume that $R \in \mathcal{F}^{frc}$ but R is not equal to E_1. Since R is $\mathcal{F}$-radical, Lemma 3·6 implies that some $\mathcal{F}$-conjugate of R is contained in at least one $\mathcal{F}$-essential subgroup. But Proposition 5·20 then implies that some $\mathcal{F}$-conjugate of R is contained in an E_1-conjugate of E_3. Since E_3 is elementary abelian and R is $\mathcal{F}$-centric, we must have that R is E_1-conjugate to E_3, as required.

PROPOSITION 5·22. *$O^{3'}(\mathcal{D}^*)$ is simple and both $O^{3'}(\mathcal{D}^*)$ and $\mathcal{D}^*$ are exotic fusion systems.*

Proof. Let $\mathcal{N} \trianglelefteq O^{3'}(\mathcal{D}^*)$ supported on $\{1\} \le P \le E_1$. By [5, theorem II·9·8(d)] we may assume that $P < E_1$, and P is strongly closed in $O^{3'}(\mathcal{D}^*)$. Hence, $\mathcal{N}$ is supported on $N_{E_1}(E_3)$ and we have that $\mathrm{Aut}_{\mathcal{N}}(E_3) \trianglelefteq \mathrm{Aut}_{O^{3'}(\mathcal{D})}(E_3)$ by [5, proposition I·6·4(c)] so that $\mathrm{Aut}_{\mathcal{N}}(E_3) = \mathrm{Aut}_{O^{3'}(\mathcal{D})}(E_3) \cong \mathrm{SL}_2(9)$.

Let τ be a non-trivial involution in $Z(\mathrm{Aut}_{\mathcal{N}}(E_3))$. By Lemma 3·6, τ lifts to $\widetilde{\tau} \in \mathrm{Aut}_{O^{3'}(\mathcal{D})}(E_1)$ and restricts to $\widehat{\tau} \in \mathrm{Aut}_{O^{3'}(\mathcal{D})}(N_{E_1}(E_3))$. Indeed, $\widehat{\tau} \in \mathrm{Aut}_{\mathcal{N}}(N_{E_1}(E_3)) \trianglelefteq \mathrm{Aut}_{O^{3'}(\mathcal{D})}(N_{E_1}(E_3))$ and we ascertain that $[\widehat{\tau}, \mathrm{Aut}_{E_1}(N_{E_1}(E_3))] \le \mathrm{Inn}(N_{E_1}(E_3))$. Since $\widetilde{\tau}$ is the lift of τ, we infer that $[\widetilde{\tau}, E_1] \le N_{E_1}(E_3)$. But as in witnessed in the proof of Proposition 5·21, the Sylow 2-subgroups of $\mathrm{Aut}_{O^{3'}(\mathcal{D})}(E_1)$ act faithfully on $E_1/N_{E_1}(E_3)$, a contradiction. Hence, $O^{3'}(\mathcal{D}^*)$ is simple.

Assume that there is $\mathcal{N}$ is a proper non-trivial normal subsystem of $\mathcal{D}^*$. Applying [5, Theorem II·9·1] and using that $O^{3'}(\mathcal{D}^*)$ is simple, we deduce that $O^{3'}(\mathcal{D}^*) \le \mathcal{N}$ and it quickly follows that $O^{3'}(\mathcal{D}^*) = \mathcal{N}$. Since $N_{E_1}(E_3)$ is a strongly closed subgroup of both $\mathcal{D}$ and $O^{3'}(\mathcal{D}^*)$, by Theorem 3·14, we conclude that both $\mathcal{D}$ and $O^{3'}(\mathcal{D}^*)$ are exotic.

We now classify all saturated fusion systems supported on E_1. We preface this classification with the following lemma.

LEMMA 5·23. *Suppose that $\mathcal{F}$ is saturated fusion system on E_1 with $E_3 \in \mathcal{E}(\mathcal{F})$. Then $O_3(\mathcal{F}) = \{1\}$ and $O^{3'}(\mathrm{Aut}_{\mathcal{F}}(E_3)) \cong SL_2(9)$.*

Proof. As in Lemma 5·3, since $\Phi(E_1)$ induces an FF-action on E_3, an application of Theorem 3·7 implies that $O^{3'}(\mathrm{Aut}_{\mathcal{F}}(E_3)) \cong \mathrm{SL}_2(9)$ and $E_3 = [E_3, O^{3'}(\mathrm{Aut}_{\mathcal{F}}(E_3))] \times C_{E_3}(O^{3'}(\mathrm{Aut}_{\mathcal{F}}(E_3)))$. Moreover, $Z(E_1) \le [E_3, O^{3'}(\mathrm{Aut}_{\mathcal{F}}(E_3))] \ntrianglelefteq E_1$ and $|C_{E_3}(O^{3'}(\mathrm{Aut}_{\mathcal{F}}(E_3)))| = 3$. Since $E_3 \in \mathcal{E}(\mathcal{F})$, by Proposition 3·10, $O_3(\mathcal{F})$ is an $\mathrm{Aut}_{\mathcal{F}}(E_3)$-invariant subgroup of E_3 which is also normal in E_1, so that $O_3(\mathcal{F}) = \{1\}$.

THEOREM D. *Suppose that $\mathcal{F}$ is saturated fusion system on E_1 such that $E_1 \ntrianglelefteq \mathcal{F}$. Then $\mathcal{F} \cong O^{3'}(\mathcal{D}^*)$ or $\mathcal{D}^*$.*

Proof. Since $E_1 \ntrianglelefteq \mathcal{F}$, we must have that E_3 is essential in $\mathcal{F}$ by Proposition 5·20. By Lemma 5·23, we have that $O_3(\mathcal{F}) = \{1\}$ and $O^{3'}(\mathrm{Aut}_{\mathcal{F}}(E_3)) \cong \mathrm{SL}_2(9)$. Let K be a Hall $3'$-subgroup of $N_{\mathrm{Aut}_{\mathcal{F}}(E_3)}(\mathrm{Aut}_S(E_3))$ so that by Lemma 3·6, K lifts to a group of automorphisms of E_1, which we denote by $\widehat{K}$. As in Lemma 5·10, we calculate that $|\mathrm{Aut}(E_1)|_{3'} = 16$ and so $\mathrm{Out}_{\mathcal{D}^*}(E_1)$ is a Sylow 2-subgroup of $\mathrm{Out}(E_1)$. Set $L := K \cap O^{3'}(\mathrm{Aut}_{\mathcal{F}}(E_3))$ and $\widehat{L}$ the lift of L to $\mathrm{Aut}_{\mathcal{F}}(E_1)$. Then $\widehat{L}$ is the unique cyclic subgroup of $\widehat{K}$ of order 8 and has index at most 2 in $\widehat{K}$. We may choose $\alpha \in \mathrm{Aut}(E_1)$ so that $\widehat{K}^{\alpha}\mathrm{Inn}(E_1) \le \mathrm{Aut}_{\mathcal{F}^{\alpha}}(E_1) \le \mathrm{Aut}_{\mathcal{D}^*}(E_1)$. Indeed, $\widehat{L}^{\alpha}\mathrm{Inn}(E_1) = \mathrm{Aut}_{O^{3'}(\mathcal{D}^*)}(E_1)$. Applying Theorem 3·11, we deduce that there is $\beta \in \mathrm{Aut}(E_1)$ with

$$N_{O^{3'}(\mathcal{D}^*)}(E_1) \le N_{\mathcal{F}^{\alpha\beta}}(E_1) \le N_{\mathcal{D}^*}(E_1).$$

In either case, we invoke Lemma 5·14 so that $\mathrm{Aut}_{\mathcal{F}^{\alpha\beta}}(E_3) = \mathrm{Aut}_{O^{3'}(\mathcal{D}^*)}(E_3)$ if $N_{\mathcal{F}^{\alpha\beta}}(E_1) = N_{O^{3'}(\mathcal{D}^*)}(E_1)$, while $\mathrm{Aut}_{\mathcal{F}^{\alpha\beta}}(E_3) = \mathrm{Aut}_{\mathcal{D}^*}(E_3)$ if $N_{\mathcal{F}^{\alpha\beta}}(E_1) = N_{\mathcal{D}^*}(E_1)$. Then the Alperin–Goldschmidt theorem implies that $\mathcal{F}^{\alpha\beta} = O^{3'}(\mathcal{D}^*)$ or $\mathcal{D}^*$ and the theorem holds.

The following Table 3 summarises the actions induced by the fusion systems described in Theorem D on their centric-radical subgroups.

Table 3. *$\mathcal{D}$-conjugacy classes of radical-centric subgroups of E_1*

P	$\|P\|$	$\mathrm{Out}_{\mathcal{D}^*}(P)$	$\mathrm{Out}_{O^{3'}(\mathcal{D}^*)}(P)$
E_1	3^9	SD_{16}	C_8
E_3	3^5	$\mathrm{SL}_2(9).2$	$\mathrm{SL}_2(9)$

6. *Fusion systems related to a Sylow 5-subgroup of M*

In this final section, we investigate saturated fusion systems on a 5-group S which is isomorphic to a Sylow 5-subgroup of the Monster sporadic simple group M. As in the previous section, we document some exotic fusion systems supported on S and some exotic fusion systems supported on a particular index 5 subgroup of S. Once again, the Atlas [13] is an invaluable tool in illustrating the structure of M and its actions. As a starting point, we consider the following maximal 5-local subgroups of M:

$$M_1 \cong 5^2.5^2.5^4{:}(\mathrm{Sym}(3) \times \mathrm{GL}_2(5))$$

$$M_2 \cong 5_+^{1+6}{:}4.\mathrm{J}_2.2$$

$$M_3 \cong 5^4{:}(3 \times \mathrm{SL}_2(25)).2$$

$$M_4 \cong 5^{3+3}.(2 \times \mathrm{PSL}_3(5))$$

remarking that $|S| = 5^9$, and for a given $S \in \mathrm{Syl}_5(\mathrm{M})$ each M_i be chosen such that $S \cap M_i \in \mathrm{Syl}_5(M_i)$. Choose M_i such that this holds.

Let $E_1 := O_5(M_1) = C_S(Z_2(S))$ of order 5^8, and $E_3 := O_5(M_3)$ elementary abelian of order 5^4. Furthermore, note that $\mathbf{Q} := O_5(M_2)$ is the unique extraspecial subgroup of S of order 5^7 and so is characteristic in S.

We appeal to the online version of the Atlas of Finite Group Representations [1] for representations of M_i for $i \in \{1, 2, 3, 4\}$. These are accessible without the need to construct the Monster computationally. We consider M_1 as a permutation group on 750 points, M_2 in its 8-dimensional matrix representation over GF(5), and M_4 as a permutation group on 7750 points. Naturally, we access S and E_1 computationally via M_1.

We note some important structural properties of M_1 which will be used later. Namely, we have that $\Phi(E_1)$ is of order 5^4 and $Z(E_1) = Z_2(S)$ is of order 5^2. Moreover, we can choose a subgroup isomorphic to Sym(3) in M_1/E_1 which acts trivially on $Z(E_1)$. We shall denote this subgroup A_1 and refer to A_1 as the "pure" Sym(3) in M_1/E_1. We record that the unique normal subgroup of M_1/E_1 isomorphic to $\mathrm{GL}_2(5)$ acts faithfully on $Z(E_1)$ and centralises A_1. In this way, we have that $M_1/E_1 = A_1 \times B_1 \cong \mathrm{Sym}(3) \times \mathrm{GL}_2(5)$. Moreover, $O^{5'}(M_1) = C_{M_1}(\Phi(E_1)/Z(E_1))$, $O^{5'}(M_1/E_1) = O^{5'}(M_1)/E_1 \cong \mathrm{SL}_2(5)$ and $O^{5'}(M_1/E_1) \leq B_1$.

We desire more candidates for essentials subgroups of the 5-fusion category of M and for this we examine the structure of M_2. Let $X \trianglelefteq M_2$ with $M_2/X \cong \mathrm{J}_2.2$ and consider the maximal subgroup $H \cong (\mathrm{Alt}(5) \times \mathrm{Dih}(10)).2$ of M_2/X. Define E_2 to be the largest normal 5-subgroup of the preimage of H in M_2 so that

$$N_{\mathrm{M}}(E_2) = N_{M_2}(E_2) \cong 5_+^{1+6}.5{:}(2 \times \mathrm{GL}_2(5)).$$

Then E_2 is an essential subgroup of $\mathcal{F}_S(\mathrm{M})$ of order 5^8, $\mathbf{Q}$ is characteristic in E_2 and $[N_{\mathrm{M}}(S){:}N_{N_{\mathrm{M}}(S)}(E_2)] = 3$.

We remark that $M_2 = \langle N_{\mathrm{M}}(S), N_{\mathrm{M}}(E_2)\rangle$ and we can arrange, up to conjugacy, that $M_4 = \langle O^{5'}(N_{\mathrm{M}}(E_1)), N_{\mathrm{M}}(E_2)\rangle$. In particular, setting $\mathbf{R} := O_5(M_4)$, we have that $[N_{\mathrm{M}}(S){:}N_{N_{\mathrm{M}}(S)}(\mathbf{R})] = 3$. For ease of notation, we fix $\mathcal{G} := \mathcal{F}_S(\mathrm{M})$ for the remainder of this section.

PROPOSITION 6·1. *$\mathcal{E}(\mathcal{G}) = \{E_1, E_2^{\mathcal{G}}, E_3^{\mathcal{G}}\}$ and $\mathcal{G}^{frc} = \{E_1, E_2^{\mathcal{G}}, E_3^{\mathcal{G}}, \boldsymbol{Q}, \boldsymbol{R}^{\mathcal{G}}, S\}$.*

Proof. See [**48**, theorem 5].

As in Section 5, before describing any exotic subsystems of $\mathcal{G}$, we require an observation regarding the containment of some essentials in others and lean on MAGMA for the determination of all possible essential subgroups of a saturated fusion system $\mathcal{F}$ supported on S. The following proposition is verified computationally (see [**44**, appendix A]).

PROPOSITION 6·2. *Suppose that $\mathcal{F}$ is saturated fusion system on S. Then $\mathcal{E}(\mathcal{F}) \subseteq \{E_1, E_2^{\mathcal{G}}, E_3^{\mathcal{G}}\}$.*

We remark that each of the three $\mathcal{G}$-conjugates of E_2 is normal in S. We record that upon restricting to S, the $\mathcal{G}$-conjugates of E_3 split into four distinct classes, fused by elements of $N_{\mathrm{Aut}_{\mathcal{G}}(S)}(E_2)$. We provide some generic results regarding all saturated fusion systems on S which also elucidate some of the structure of $\mathcal{G}$.

LEMMA 6·3. *Every $\mathcal{G}$-conjugate of E_3 is contained in E_1 and not contained in any $\mathcal{G}$-conjugate of E_2. Moreover, $\{E_3^{\mathcal{G}}\} = \{E_3^{N_{Aut_{\mathcal{G}}(E_2)}(Aut_S(E_2))}\}$.*

Proof. We verify that $\{E_3^{\mathcal{G}}\} = \{E_3^{N_{\mathrm{Aut}_{\mathcal{G}}(E_2)}(\mathrm{Aut}_S(E_2))}\}$ computationally (see [**44**, appendix A]). Since E_1 and $E_2\alpha$ are normalised by $N_{\mathrm{Aut}_{\mathcal{G}}(S)}(E_2)$ for all $\alpha \in \mathrm{Aut}_{\mathcal{G}}(S)$, for the first statement of the lemma it suffices to prove that $E_3 \le E_1$ and $E_3 \not\le E_2\alpha$ for all $\alpha \in \mathrm{Aut}_{\mathcal{G}}(S)$. To this end, we note that $[Z_2(S), E_3] = \{1\}$ so that $E_3 \le E_1$. One can see this in $\mathcal{G}$, for otherwise since E_3 is elementary abelian we would have that $Z_2(S) \not\le E_3$ and $[Z_2(S), E_3] \le Z(S)$, a contradiction since $O^{5'}(\mathrm{Out}_{\mathcal{G}}(E_3)) \cong \mathrm{SL}_2(25)$ has no non-trivial modules exhibiting this behaviour. If $E_3 \le E_2\alpha$ for some $\alpha \in \mathrm{Aut}_{\mathcal{G}}(S)$, then as $E_2\alpha \trianglelefteq S$, we have that $E_1 = \langle E_3^S\rangle \le E_2\alpha$, an obvious contradiction.

LEMMA 6·4. *Suppose that $\mathcal{F}$ is a saturated fusion system on S with $E_2 \in \mathcal{E}(\mathcal{F})$. Then $|\Phi(E_2)| = 5^5$, $O^{5'}(Aut_{\mathcal{F}}(E_2))$ acts trivially on $E_2/\boldsymbol{Q}$, and both $\boldsymbol{Q}/\Phi(E_2)$ and $\Phi(E_2)/\Phi(\boldsymbol{R})$ are natural modules for $O^{5'}(Out_{\mathcal{F}}(E_2)) \cong SL_2(5)$.*

Proof. We compute (see [**44**, appendix A]) that $\Phi(E_2)$ is of order 5^5 and so has index 5^3 in E_2. Then $\mathbf{Q}$ has index 5 in E_2 and $\Phi(E_2)$ has index 5^2 in $\mathbf{Q}$. Thus, $O^{5'}(\mathrm{Aut}_{\mathcal{F}}(E_2))$ acts trivially on $E_2/\mathbf{Q}$ and since $O^{5'}(\mathrm{Aut}_{\mathcal{F}}(E_2))$ acts faithfully on $E_2/\Phi(E_2)$, we deduce that $O^{5'}(\mathrm{Out}_{\mathcal{F}}(E_2)) \cong \mathrm{SL}_2(5)$ and $\mathbf{Q}/\Phi(E_2)$ is a natural module.

We have that $\Phi(E_2) < \mathbf{R} \le E_2$ so that $\Phi(\mathbf{R}) < \Phi(E_2)$. We calculate (see [**44**, appendix A]) that $\Phi(\mathbf{R}) = [E_2, \Phi(E_2)]$ is characteristic in E_2 and so $\mathbf{R} = C_{E_2}(\Phi(\mathbf{R}))$ is also characteristic in E_2. Since $|\mathbf{R}/\Phi(E_2)| = 5$, S centralises $\mathbf{R}/\Phi(E_2)$ and we either have that $O^{5'}(\mathrm{Out}_{\mathcal{F}}(E_2))$ acts faithfully on $\Phi(E_2)/\Phi(\mathbf{R})$ of order 5^2; or $O^{5'}(\mathrm{Out}_{\mathcal{F}}(E_2))$ acts trivially on $\mathbf{R}$. Since $\mathbf{R}$

is self-centralising in S the latter case clearly gives a contradiction. Hence, the former case holds and $\Phi(E_2)/\Phi(\mathbf{R})$ is a natural module for $O^{5'}(\mathrm{Out}_{\mathcal{F}}(E_2)) \cong \mathrm{SL}_2(5)$.

The above lemma also holds for any $\mathcal{G}$-conjugate of E_2 which is essential in $\mathcal{F}$, with $\mathbf{R}$ replaced by $\mathbf{R}\alpha$ for some appropriate $\alpha \in \mathrm{Aut}_{\mathcal{G}}(S)$.

LEMMA 6·5. *Let $\mathcal{F}$ be a saturated fusion system on S. Let P be some $\mathcal{G}$-conjugate of E_3. If $P \in \mathcal{E}(\mathcal{F})$, then $O^{5'}(Aut_{\mathcal{F}}(P)) \cong SL_2(25)$, P is a natural module for $O^{5'}(Aut_{\mathcal{F}}(P))$, $E_1 \in \mathcal{E}(\mathcal{F})$ and $O_5(\mathcal{F}) = \{1\}$.*

Proof. Let P be some $\mathcal{G}$-conjugate of E_3 and suppose that $P \in \mathcal{E}(\mathcal{F})$. Then $\Phi(E_1)$ is elementary abelian of order 5^4 and is not contained in P. Furthermore, by Lemma 6·3 $[P, \Phi(E_1)] \le [E_1, \Phi(E_1)] = Z_2(S) \le P$ so that $\Phi(E_1) \le N_S(P)$. Since P is $\mathcal{G}$-essential, and $|N_S(P)/P| = 5^2$, applying Theorem 3·7 we see that $N_S(P) = P\Phi(E_1)$, $P \cap \Phi(E_1) = Z_2(S)$ and $\Phi(E_1)$ induces an FF-action on P. Then for $L := O^{5'}(\mathrm{Aut}_{\mathcal{F}}(P))$, Theorem 3·7 implies that $L \cong \mathrm{SL}_2(25)$ and $P = [P, L]$ is a natural module.

Let K be a Hall $5'$-subgroup of $N_L(\mathrm{Aut}_S(P))$ so that K is cyclic of order 24 and acts irreducibly on $Z_2(S)$. If E_1 is not essential then applying Lemma 3·6, Proposition 6·2 and Lemma 6·3, the morphisms in K must lift to automorphisms of S. But then, upon restriction, the morphisms in K would normalise $Z(S)$, contradicting the irreducibility of $Z_2(S)$ under the action of K. Hence, $E_1 \in \mathcal{E}(\mathcal{F})$. Since $O_5(\mathcal{F}) \trianglelefteq S$, P is irreducible under the action of $\mathrm{Aut}_{\mathcal{F}}(P)$ and, by Proposition 3·10, $O_5(\mathcal{F}) \le P$, we conclude that $O_5(\mathcal{F}) = \{1\}$.

LEMMA 6·6. *Suppose that $\mathcal{F}$ is a saturated fusion system on S with $E_1 \in \mathcal{E}(\mathcal{F})$. Then $|\Phi(E_1)| = 5^4$, $O^{5'}(Aut_{\mathcal{F}}(E_1))$ acts trivially on $\Phi(E_1)/Z(E_1)$, $Z(E_1)$ is a natural module for $O^{5'}(Out_{\mathcal{F}}(E_1)) \cong SL_2(5)$, and $O^{5'}(Aut_{\mathcal{F}}(E_1))$ normalises every $Aut_{\mathcal{G}}(S)$-conjugate of* $\mathbf{R}$.

Proof. We compute (see [**44**, appendix A]) that $\Phi(E_1)$ is elementary abelian of order 5^4 and that S centralises $\Phi(E_1)/Z(E_1)$. In particular, $O^{5'}(\mathrm{Aut}_{\mathcal{F}}(E_1))$ acts trivially on $\Phi(E_1)/Z(E_1)$. Set $L := O^{5'}(\mathrm{Out}_{\mathcal{F}}(E_1))$ and notice that for $r \in L$ of $5'$-order, if r acts trivially on $\Phi(E_1)$ then, by the three subgroups lemma, r centralises $E_1/C_{E_1}(\Phi(E_1))$. Since $\Phi(E_1)$ is self-centralising in E_1, we deduce that L acts faithfully on $\Phi(E_1)$. In particular, $C_L(Z(E_1)) = \{1\}$. Since $Z(E_1)$ has order 5^2, we conclude that $Z(E_1)$ is natural module for $L \cong \mathrm{SL}_2(5)$.

We note that $Z(E_1)$, $\Phi(E_1)$ and E_1 are all invariant under $\mathrm{Aut}_{\mathcal{G}}(S)$. Hence, for $\alpha \in \mathrm{Aut}_{\mathcal{G}}(S)$, $\mathbf{R}\alpha \le E_1$ so that $\Phi(\mathbf{R}\alpha) \le \Phi(E_1)$. Since $Z(E_1)$ centralises $\mathbf{R}\alpha$, we deduce that $Z(E_1) \le Z(\mathbf{R}\alpha) = \Phi(\mathbf{R}\alpha)$ and as $O^{5'}(\mathrm{Aut}_{\mathcal{F}}(E_1))$ centralises $\Phi(E_1)/Z(E_1)$, $O^{5'}(\mathrm{Aut}_{\mathcal{F}}(E_1))$ normalises $\Phi(\mathbf{R}\alpha)$ and so normalises $C_{E_1}(\Phi(\mathbf{R}\alpha)) = \mathbf{R}\alpha$ (where the last equality follows from a MAGMA computation [**44**, appendix A]).

LEMMA 6·7. *Suppose that $\mathcal{F}$ is a saturated fusion system on S with $E_1 \in \mathcal{E}(\mathcal{F})$. Then there is $\gamma \in Aut(E_1)$ with $Aut_{\mathcal{F}}(E_1)^{\gamma} \le Aut_{\mathcal{G}}(E_1)$ and we may choose $A, B \le Out(E_1)$ such that $A = A_1^{\gamma} \cong Sym(3)$ with $[A, Z(E_1)] = \{1\}$, $B = B_1^{\gamma} \cong GL_2(5)$ with $[B, A] = \{1\}$), and $Out_{\mathcal{F}}(E_1) \le A \times B$ with $O^{5'}(Out_{\mathcal{F}}(E_1)) \le B$.*

Proof. Let T be a Sylow 2-subgroup of $O^{5'}(\mathrm{Aut}_{\mathcal{F}}(E_1))$ so that $|T| = 2^3$. Then $N_T(\mathrm{Aut}_S(E_1))$ is cyclic of order 4 and T centralises $\Phi(E_1)/Z(E_1)$ and so centralises $\mathrm{Aut}_{\Phi(E_1)}(E_1)$. We calculate ([**44**, appendix A]) that $N_T(\mathrm{Aut}_S(E_1))$ is a Sylow 2-subgroup of $C_{N_{\mathrm{Aut}(E_1)}(\mathrm{Aut}_S(E_1))}(\mathrm{Aut}_{\Phi(E_1)}(E_1))$ and $N_T(\mathrm{Aut}_S(E_1))$ is conjugate by an element of $N_{\mathrm{Aut}(E_1)}(\mathrm{Aut}_S(E_1))$ to a Sylow 2-subgroup $N_{O^{5'}(\mathrm{Aut}_{\mathcal{G}}(E_1))}(\mathrm{Aut}_S(E_1))$.

We have that there is a unique $\mathrm{Aut}(E_1)$-conjugacy class of subgroups which contain $N_{O^{5'}(\mathrm{Aut}_{\mathcal{G}}(E_1))}(\mathrm{Aut}_S(E_1))$ with quotient by $\mathrm{Inn}(E_1)$ isomorphic to $\mathrm{SL}_2(5)$. Indeed, $O^{5'}(\mathrm{Aut}_{\mathcal{G}}(E_1))$ satisfies these conditions and tracing backwards, we ascertain that $O^{5'}(\mathrm{Aut}_{\mathcal{F}}(E_1))$ is $\mathrm{Aut}(E_1)$-conjugate to $O^{5'}(\mathrm{Aut}_{\mathcal{G}}(E_1))$. Finally, the normaliser in $\mathrm{Out}(E_1)$ of $O^{5'}(\mathrm{Out}_{\mathcal{G}}(E_1))$ is $\mathrm{Out}_{\mathcal{G}}(E_1)$ so that $\mathrm{Out}_{\mathcal{F}}(E_1) \le N_{\mathrm{Out}(E_1)}(O^{5'}(\mathrm{Out}_{\mathcal{F}}(E_1)))$ and $N_{\mathrm{Out}(E_1)}(O^{5'}(\mathrm{Out}_{\mathcal{F}}(E_1)))$ is $\mathrm{Out}(E_1)$ conjugate to $\mathrm{Out}_{\mathcal{G}}(E_1)$. Hence, $\mathrm{Aut}_{\mathcal{F}}(E_1)$ is $\mathrm{Aut}(E_1)$-conjugate to a subgroup of $\mathrm{Aut}_{\mathcal{G}}(E_1)$ and the rest of the result follows from the description of $\mathrm{Out}_{\mathcal{G}}(E_1) \cong M_1/E_1$.

LEMMA 6·8. *Let $\mathcal{F}$ be a saturated fusion system on S. Let P be some $\mathcal{G}$-conjugate of E_3. If $P \in \mathcal{E}(\mathcal{F})$, then $Out_{\mathcal{F}}(E_1) = C_{Out_{\mathcal{F}}(E_1)}(Z(E_1)) \times B$, where $B \cong GL_2(5)$, $[B, C_{Out_{\mathcal{F}}(E_1)}(Z(E_1))] = \{1\}$ and $|C_{Out_{\mathcal{F}}(E_1)}(Z(E_1))| \in \{3, 6\}$.*

Proof. Let $P \in \mathcal{E}(\mathcal{F})$ where P is a $\mathcal{G}$-conjugate of E_3. Applying Lemma 6·5, $E_1 \in \mathcal{E}(\mathcal{F})$ and $O^{5'}(\mathrm{Aut}_{\mathcal{F}}(P)) \cong \mathrm{SL}_2(25)$, and following the notation from the proof of that result, we set K to be a Hall $5'$-subgroup of $N_{O^{5'}(\mathrm{Aut}_{\mathcal{F}}(P))}(\mathrm{Aut}_S(P))$. Then K is cyclic of order 24 and using that E_1 is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant and applying Lemma 3·6 and Proposition 6·2, we see that K lifts to a group of automorphisms of E_1 which we denote $\widehat{K}$. Then $\widehat{K}$ acts on $Z(E_1)$ as K does. In particular, $\widehat{K}$ acts faithfully on $Z(E_1)$. By Lemma 6·7, $\mathrm{Out}_{\mathcal{F}}(E_1)$ is $\mathrm{Out}(E_1)$-conjugate to a subgroup of $\mathrm{Out}_{\mathcal{G}}(E_1)$ so that $\mathrm{Out}_{\mathcal{F}}(E_1)/C_{\mathrm{Out}_{\mathcal{F}}(E_1)}(Z(E_1))$ is isomorphic to a subgroup of $\mathrm{GL}_2(5)$. But $O^{5'}(\mathrm{Out}_{\mathcal{F}}(E_1)) \cap C_{\mathrm{Out}_{\mathcal{F}}(E_1)}(Z(E_1)) = (\widehat{K}\mathrm{Inn}(E_1)/\mathrm{Inn}(E_1)) \cap C_{\mathrm{Out}_{\mathcal{F}}(E_1)}(Z(E_1)) = \{1\}$ and we deduce that $\mathrm{Out}_{\mathcal{F}}(E_1)/C_{\mathrm{Out}_{\mathcal{F}}(E_1)}(Z(E_1)) \cong \mathrm{GL}_2(5)$.

Furthermore, again using that $\mathrm{Out}_{\mathcal{F}}(E_1)$ is $\mathrm{Out}(E_1)$-conjugate to a subgroup of $\mathrm{Out}_{\mathcal{G}}(E_1)$, we deduce that $|C_{\mathrm{Out}_{\mathcal{F}}(E_1)}(Z(E_1))| \le 6$. Now, a Sylow 3-subgroup of $O^{5'}(\mathrm{Out}_{\mathcal{F}}(E_1))$ centralises $\Phi(E_1)/Z(E_1)$. Since a Sylow 3-subgroup of $\widehat{K}$ acts on $\Phi(E_1)/Z(E_1)$ as K acts on $\mathrm{Aut}_S(P) \cong \Phi(E_1)P/P \cong \Phi(E_1)/Z(E_1)$, we have that a Sylow 3-subgroup of $\widehat{K}$ acts non-trivially on $\Phi(E_1)/\mathrm{Inn}(E_1)$. Hence, $9 \mid |\mathrm{Out}_{\mathcal{F}}(E_1)|$ and it follows that $|C_{\mathrm{Out}_{\mathcal{F}}(E_1)}(Z(E_1))| \in \{3, 6\}$.

Since $\mathrm{Out}_{\mathcal{F}}(E_1)$ is $\mathrm{Out}(E_1)$-conjugate to a subgroup of $\mathrm{Out}_{\mathcal{G}}(E_1)$, if $|C_{\mathrm{Out}_{\mathcal{F}}(E_1)}(Z(E_1))| = 6$ then $\mathrm{Out}_{\mathcal{F}}(E_1)$ is $\mathrm{Out}(E_1)$-conjugate to $\mathrm{Out}_{\mathcal{G}}(E_1)$ and the result is easily seen to hold. Hence, we assume that $|C_{\mathrm{Out}_{\mathcal{F}}(E_1)}(Z(E_1))| = 3$ so that $\mathrm{Out}_{\mathcal{F}}(E_1) = \langle \widehat{K}\mathrm{Inn}(E_1)/\mathrm{Inn}(E_1), O^{5'}(\mathrm{Out}_{\mathcal{F}}(E_1))\rangle$. We note that $\{E_3^{\mathcal{G}}\}$ is the unique class of elementary abelian subgroups H of E_1 of order 5^4 with $|N_{E_1}(H)| = 5^6$ and $[N_{E_1}(H), E_1] = Z(E_1)$ (see [**44**, appendix A]). In particular, this class is invariant under $\mathrm{Aut}(E_1)$. Since $\mathrm{Aut}_{\mathcal{F}}(E_1)$ is $\mathrm{Out}(E_1)$-conjugate to a subgroup of $\mathrm{Aut}_{\mathcal{G}}(E_1)$, and we can choose a Sylow 3-subgroup of $\mathrm{Aut}_{\mathcal{G}}(E_1)$ to normalise P, we can also choose a Sylow 3-subgroup of $\mathrm{Aut}_{\mathcal{F}}(E_1)$ to normalise P. In particular, there is a 3-element t of $\mathrm{Aut}_{\mathcal{F}}(E_1)$ which normalises P and centralises $\Phi(E_1)/Z(E_1) \cong \mathrm{Aut}_S(P)$. Since $O^{5'}(\mathrm{Aut}_{\mathcal{F}}(P)) \cong \mathrm{SL}_2(25)$, we must have that $t|_P$ centralises $O^{5'}(\mathrm{Aut}_{\mathcal{F}}(P))$. Hence, $\widehat{K}$ centralises a Sylow 3-subgroup of $\mathrm{Aut}_{\mathcal{F}}(E_1)$ and so $\widehat{K}\mathrm{Inn}(E_1)/\mathrm{Inn}(E_1)$ centralises $C_{\mathrm{Out}_{\mathcal{F}}(E_1)}(Z(E_1))$. But $O^{5'}(\mathrm{Out}_{\mathcal{F}}(E_1))$ centralises $C_{\mathrm{Out}_{\mathcal{F}}(E_1)}(Z(E_1))$ and so we see that $\mathrm{Out}_{\mathcal{F}}(E_1)$ centralises $C_{\mathrm{Out}_{\mathcal{F}}(E_1)}(Z(E_1))$. Finally, since $\mathrm{Out}_{\mathcal{F}}(E_1)$ is $\mathrm{Out}(E_1)$-conjugate to a subgroup of $\mathrm{Out}_{\mathcal{G}}(E_1)$, the lemma holds.

LEMMA 6·9. *Let $\mathcal{F}$ be a saturated fusion system on S. Let P be some $\mathcal{G}$-conjugate of E_3 and set $Aut^*_{\mathcal{F}}(S)$ the subgroup of $Aut_{\mathcal{F}}(S)$ generated by all $\mathcal{F}$-automorphisms of S that restrict to elements of $O^{5'}(Aut_{\mathcal{F}}(R))$, where $R \in \{E_1, P^{\mathcal{F}}, S\}$. If $P \in \mathcal{E}(\mathcal{F})$, then* *(i)* $|\mathrm{Aut}_{\mathcal{F}}(S)/\mathrm{Aut}^*_{\mathcal{F}}(S)| = |C_{\mathrm{Out}_{\mathcal{F}}(E_1)}(Z(E_1))|/3$;

(ii) $\{P^{\mathcal{F}}\} = \{E_3^{\mathcal{G}}\}$; *and*

(iii) $[\mathrm{Aut}_{\mathcal{F}}(P){:}O^{5'}(\mathrm{Aut}_{\mathcal{F}}(P))] = |C_{\mathrm{Out}_{\mathcal{F}}(E_1)}(Z(E_1))|$.

Moreover, if $\{E_2^{\mathcal{G}}\} \cap \mathcal{E}(\mathcal{F}) = \emptyset$ *then* $\mathrm{Aut}^*_{\mathcal{F}}(S) = \mathrm{Aut}^0_{\mathcal{F}}(S)$, $\mathrm{Out}_{O^{5'}(\mathcal{F})}(E_1) \cong 3 \times \mathrm{GL}_2(5)$ *and* $\mathrm{Aut}_{O^{5'}(\mathcal{F})}(E_3) \cong 3 \times \mathrm{SL}_2(25)$.

Proof. Since E_1 is self-centralising and characteristic in S, an application of the three subgroups lemma implies that any morphism in $O^{5'}(\mathrm{Aut}_{\mathcal{F}}(R))$ which extends to automorphisms of S restricts faithfully to a morphism in $\mathrm{Aut}_{\mathcal{F}}(E_1)$. Indeed, it follows that $|\mathrm{Aut}_{\mathcal{F}}(S)|/|\mathrm{Aut}^*_{\mathcal{F}}(S)| = |\mathrm{Aut}_{\mathcal{F}}(E_1)|/|\langle \mathrm{Aut}^*_{\mathcal{F}}(S)|_{E_1}, O^{5'}(\mathrm{Aut}_{\mathcal{F}}(E_1))\rangle|$. By the proof of Lemma 6·8, and using that $\mathrm{Out}_{\mathcal{F}}(E_1) \le A \times B$ in the language of Lemma 6·7, we have that

$$\langle \mathrm{Aut}^*_{\mathcal{F}}(S)|_{E_1}, O^{5'}(\mathrm{Aut}_{\mathcal{F}}(E_1))\rangle \mathrm{Inn}(E_1)/\mathrm{Inn}(E_1) = O_3(A) \times B \cong 3 \times \mathrm{GL}_2(5).$$

Hence, (i) holds.

We observe that the subgroup of $\mathrm{Aut}_{\mathcal{G}}(E_1)$ with quotient by $\mathrm{Inn}(E_1)$ isomorphic to $3 \times \mathrm{GL}_2(5)$ acts transitively on the set $\{E_3^{\mathcal{G}}\}$ (see [**44**, appendix A]), and is conjugate by $\mathrm{Aut}(E_1)$ to a subgroup of $\mathrm{Aut}_{\mathcal{F}}(E_1)$ by Lemma 6·8. Since $\{E_3^{\mathcal{G}}\}$ is preserved by $\mathrm{Aut}(E_1)$ (as in Lemma 6·8) and P is $\mathcal{G}$-conjugate to E_3, we have that $\{P^{\mathcal{F}}\} = \{E_3^{\mathcal{G}}\}$ and so (ii) holds. We may take $P = E_3$ to prove the remainder of the claims.

Now, it follows by a Frattini argument that $|N_{\mathrm{Aut}_{\mathcal{F}}(E_3)}(\mathrm{Aut}_S(E_3))| = [\mathrm{Aut}_{\mathcal{F}}(E_3){:}O^{5'}(\mathrm{Aut}_{\mathcal{F}}(E_3))]|N_{O^{5'}(\mathrm{Aut}_{\mathcal{F}}(E_3))}(\mathrm{Aut}_S(E_3))|$. By Lemma 3·6, and using that E_1 is characteristic in S, we see that all morphisms in $N_{\mathrm{Aut}_{\mathcal{F}}(E_3)}(\mathrm{Aut}_S(E_3))$ lift to morphisms in $\mathrm{Aut}_{\mathcal{F}}(E_1)$ which normalise E_3. But $\mathrm{Aut}_{\mathcal{F}}(E_1)$ is $\mathrm{Aut}(E_1)$-conjugate to a subgroup of $\mathrm{Aut}_{\mathcal{G}}(E_1)$ and as $\mathrm{Inn}(E_1)$ preserves the class $\{E_3^{\mathcal{G}}\}$, we may calculate $|N_{\mathrm{Aut}_{\mathcal{F}}(E_3)}(\mathrm{Aut}_S(E_3))|$ from $N_{\mathrm{Aut}_{\mathcal{G}}(E_1)}(E_3)$. Writing H for the subgroup of $\mathrm{Aut}_{\mathcal{G}}(E_1)$ with $H/\mathrm{Inn}(E_1) \cong 3 \times \mathrm{GL}_2(5)$, we calculate (see [**44**, appendix A]) that $N_H(E_3)$ has index 2 in $N_{\mathrm{Aut}_{\mathcal{G}}(E_1)}(E_3)$ and so (iii) holds.

Finally, assume that $\{E_2^{\mathcal{G}}\} \cap \mathcal{E}(\mathcal{F}) = \emptyset$. We clearly have that $\mathrm{Aut}^*_{\mathcal{F}}(S) \le \mathrm{Aut}^0_{\mathcal{F}}(S) \le \mathrm{Aut}_{\mathcal{F}}(S)$. Aiming for a contradiction, suppose that $\mathrm{Aut}^*_{\mathcal{F}}(S) < \mathrm{Aut}^0_{\mathcal{F}}(S)$ so that $\mathrm{Aut}^0_{\mathcal{F}}(S) = \mathrm{Aut}_{\mathcal{F}}(S)$. Then we see that $\mathrm{Out}_{\mathcal{F}}(E_1) = A \times B \cong \mathrm{Sym}(3) \times \mathrm{GL}_2(5)$. By Theorem 3·11, we let H be a model for $N_{\mathcal{F}}(E_1)$ and let $H^* \trianglelefteq H$ such that $H^*/E_1 \cong 3 \times \mathrm{GL}_2(5)$. Indeed, H^* is unique with respect to this property. Form the fusion system $\mathcal{F}^* := \langle O^{5'}(\mathrm{Aut}_{\mathcal{F}}(P)), \mathcal{F}_S(H^*)\rangle_S$. Applying Proposition 3·9 and by the definition of $\mathrm{Aut}^*_{\mathcal{F}}(S)$, we have that $\mathcal{F}^*$ is saturated and $\mathcal{F} = \langle \mathcal{F}^*, \mathrm{Aut}_{\mathcal{F}}(S)\rangle_S$. Moreover, for all $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ we have that $\mathcal{F}^{*\alpha} = \mathcal{F}^*$. Hence, applying [**5**, proposition I·6·4], we have that $\mathcal{F}^*$ is weakly normal in $\mathcal{F}$ in the sense of [**5**, definition I·6·1] and [**15**, theorem A] yields that $O^{5'}(\mathcal{F}^*) \trianglelefteq \mathcal{F}$. Then $O^{5'}(\mathrm{Aut}_{\mathcal{F}}(T)) \le \mathrm{Aut}_{O^{5'}(\mathcal{F}^*)}(T) \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(T)$ by [**5**, proposition I·6·4] for all $T \le S$, and we deduce that $O^{5'}(\mathcal{F}^*)$ has index prime to 5 in a $\mathcal{F}$, a contradiction by Lemma 3·12 since $\mathrm{Aut}^0_{\mathcal{F}}(S) = \mathrm{Aut}_{\mathcal{F}}(S)$. Hence, $\mathrm{Aut}^*_{\mathcal{F}}(S) = \mathrm{Aut}^0_{\mathcal{F}}(S)$.

Then $\mathrm{Out}_{O^{5'}(\mathcal{F})}(E_1) = \langle \mathrm{Aut}^0_{\mathcal{F}}(S)|_{E_1}, O^{5'}(\mathrm{Aut}_{\mathcal{F}}(E_1))\rangle \mathrm{Inn}(E_1)/\mathrm{Inn}(E_1) = O_3(A) \times B \cong 3 \times \mathrm{GL}_2(5)$. As in Lemma 6·8, we see that we may choose $t \in \mathrm{Aut}_{\mathcal{F}}(E_1)$ to normalises E_3 with $[t, \Phi(E_1)] \le Z(E_1)$ so that $t|_{E_3}$ centralises $O^{5'}(\mathrm{Aut}_{\mathcal{F}}(E_3))$. Then part (iii) implies that $\mathrm{Aut}_{O^{5'}(\mathcal{F})}(E_3) \cong 3 \times \mathrm{SL}_2(25)$.

As a consequence of Lemma 6·5 and Lemma 6·9, if any $\mathcal{G}$-conjugate of E_3 is essential in $\mathcal{F}$, then $\{E_1, \{E_3^{\mathcal{G}}\}\} \subseteq \mathcal{E}(\mathcal{F})$.

We now construct some exotic fusion subsystems of $\mathcal{G}$ in a similar manner to Section 5, and persist with the same notations. That is, we set

$$\mathcal{H} = \langle \mathrm{Aut}_{\mathcal{G}}(E_1), \mathrm{Aut}_{\mathcal{G}}(E_2), \mathrm{Aut}_{\mathcal{G}}(S)\rangle_S$$

and

$$\mathcal{D} = \langle \mathrm{Aut}_{\mathcal{G}}(E_1), \mathrm{Aut}_{\mathcal{G}}(E_3), \mathrm{Aut}_{\mathcal{G}}(S)\rangle_S.$$

PROPOSITION 6·10. *$\mathcal{H}$ is a saturated fusion system with with $\mathcal{E}(\mathcal{H}) = \{E_1, E_2^{\mathcal{H}}\}$ and $\mathcal{H}^{frc} = \{E_1, E_2^{\mathcal{H}}, \boldsymbol{Q}, \boldsymbol{R}^{\mathcal{H}}, S\}$.*

Proof. By applying Lemma 3·8 to $\mathcal{G}$ with $P = E_3$ we deduce that $\mathcal{H}$ is saturated. Moreover, by Lemma 5·3 we have that $\{E_3^{\mathcal{F}}\} = \{E_3^{N_{\mathrm{Aut}_{\mathcal{G}}(E_2)}(\mathrm{Aut}_S(E_2))}\}$. Then Lemma 3·8 and Proposition 6·2 reveal that $\mathcal{E}(\mathcal{H}) = \{E_1, E_2^{\mathcal{G}}\}$.

Let R be a fully $\mathcal{H}$-normalised, radical, centric subgroup of S not equal to one described in the conclusion of the proposition. Then an $\mathcal{H}$-conjugate of R must be contained in an $\mathcal{H}$-essential subgroup for otherwise, by Lemma 3·6, we infer that $\mathrm{Out}_S(R) \trianglelefteq \mathrm{Out}_{\mathcal{H}}(R)$ and R is not $\mathcal{H}$-radical. If an $\mathcal{H}$-conjugate of R is contained in a $\mathcal{G}$-conjugate of E_3 then since R is $\mathcal{H}$-centric, R is $\mathcal{G}$-conjugate to E_3. Then $\mathrm{Out}_S(R) \le O^{5'}(\mathrm{Out}_{\mathcal{H}}(R)) \le O^{5'}(\mathrm{Out}_{\mathcal{G}}(R)) \cong \mathrm{SL}_2(25)$. Since R is not $\mathcal{H}$-essential, it follows that $O^{5'}(\mathrm{Out}_{\mathcal{H}}(R))$ is contained in the unique maximal subgroup of $O^{5'}(\mathrm{Out}_{\mathcal{G}}(R))$ which contains $\mathrm{Out}_S(R)$ and so $\mathrm{Out}_S(R) \trianglelefteq O^{5'}(\mathrm{Out}_{\mathcal{H}}(R))$. Then the Frattini argument implies that $\mathrm{Out}_S(R) \trianglelefteq \mathrm{Out}_{\mathcal{H}}(R)$, a contradiction.

Thus, no $\mathcal{H}$-conjugate of R is contained in an $\mathcal{G}$-conjugate of E_3. Hence, by the Alperin–Goldschmidt theorem and using Proposition 6·2, since $\mathcal{H} = \langle \mathrm{Aut}_{\mathcal{G}}(E_1), \mathrm{Aut}_{\mathcal{G}}(E_2), \mathrm{Aut}_{\mathcal{G}}(S)\rangle_S$ and R is fully $\mathcal{H}$-normalised, R is fully $\mathcal{G}$-normalised and so is $\mathcal{G}$-centric. Finally, since $O_5(\mathrm{Out}_{\mathcal{G}}(R)) \le O_5(\mathrm{Out}_{\mathcal{H}}(R)) = \{1\}$, we conclude that R is $\mathcal{G}$-centric-radical and comparing with Proposition 6·1, we have a contradiction.

PROPOSITION 6·11. *$\mathcal{H}$ is simple.*

Proof. Assume that $\mathcal{N} \trianglelefteq \mathcal{H}$ and $\mathcal{N}$ is supported on T. Then T is a strongly closed subgroup of $\mathcal{H}$. In particular, $T \trianglelefteq S$ and $Z(S) \le T$. Observe that since $N_{\mathcal{G}}(\mathbf{Q}) = \langle N_{\mathcal{G}}(S), N_{\mathcal{G}}(E_2)\rangle_S \le \mathcal{H}$, we have that $N_{\mathcal{H}}(\mathbf{Q}) = N_{\mathcal{G}}(\mathbf{Q})$. In particular, $\mathrm{Aut}_{\mathcal{H}}(\mathbf{Q})$ is irreducible on $\mathbf{Q}/Z(S)$. Since $\mathrm{Aut}_{\mathcal{H}}(E_1) = \mathrm{Aut}_{\mathcal{G}}(E_1)$ is irreducible on $Z_2(S)$, we have that $\mathbf{Q} \le T$. Then $E_1 = \langle (E_1 \cap \mathbf{Q})^{\mathrm{Aut}_{\mathcal{G}}(E_1)}\rangle \le T$ and so $S = E_1\mathbf{Q} = T$. Since $\mathrm{Aut}_{\mathcal{H}}(S)$ is generated by lifted morphisms from $O^{5'}(\mathrm{Aut}_{\mathcal{H}}(E_1))$ and $O^{5'}(\mathrm{Aut}_{\mathcal{H}}(\mathbf{Q}))$, in the language of Lemma 3·12 we have that $\mathrm{Aut}^0_{\mathcal{H}}(S) = \mathrm{Aut}_{\mathcal{H}}(S)$. Then [**5**, theorem II·9·8(d)] implies that $\mathcal{H}$ is simple.

PROPOSITION 6·12. *$\mathcal{H}$ is exotic.*

Proof. Aiming for a contradiction, suppose that $\mathcal{H} = \mathcal{F}_S(G)$ for some finite group G with $S \in \mathrm{Syl}_5(G)$. We may as well assume that $O_5(G) = O_{5'}(G) = \{1\}$ so that $F^*(G) = E(G)$ is a direct product of non-abelian simple groups, all divisible by 5. Then, as $\mathcal{H}$ is simple and $\mathcal{F}_{S\cap F^*(G)}(F^*(G)) \trianglelefteq \mathcal{F}_S(G)$, we may assume that $G = F^*(G)$. Hence, every component in G is normal and is divisible by 5 and as $|\Omega(Z(S))| = 5$, we have that G is simple. We note that $m_5(\mathrm{M}) = 5$ by [**21**, Table 5·6·1].

If $G \cong \mathrm{Alt}(n)$ for some n then $m_5(\mathrm{Alt}(n)) = \lfloor \frac{n}{5} \rfloor$ by [21, proposition 5·2·10] and so $n < 25$. But a Sylow 5-subgroup of Alt(25) has order 5^6 and so $G \not\cong \mathrm{Alt}(n)$ for any n. If G is isomorphic to a group of Lie type in characteristic 5, then comparing with [21, Table 3·3·1], we see that the groups with a Sylow 5-subgroup which has 5-rank 4 are $\mathrm{PSL}_2(5^4)$, $\mathrm{PSL}_3(25)$, $\mathrm{PSU}_3(25)$, $\mathrm{PSL}_4(5)$ or $\mathrm{PSU}_4(5)$ and none of these examples have a Sylow 5-subgroup of order 5^9.

Assume now that G is a group of Lie type in characteristic $r \neq 5$ with $m_5(G) = 4$. By [21, theorem 4·10·3], S has a unique elementary abelian subgroup of order 5^4 unless $G \cong \mathrm{G}_2(r^a)$, ${}^2\mathrm{F}_4(r^a)$, ${}^3\mathrm{D}_4(r^a)$, $\mathrm{PSU}_3(r^a)$ or $\mathrm{PSL}_3(r^a)$. Since S has more than one elementary abelian subgroup of order 5^4, we have that G is one of the listed exceptions. Then, applying [21, theorem 4·10·3(a)], none of the exceptions have 5-rank 4 and we conclude that G is not isomorphic to a group of Lie type in characteristic r.

Finally, checking the orders of the sporadic groups, we have that M is the unique sporadic simple group with a Sylow 5-subgroup of order 5^9. Since M has 3 classes of essential subgroups, $G \not\cong \mathrm{M}$ and $\mathcal{H}$ is exotic.

PROPOSITION 6·13. *$\mathcal{D}$ is a saturated fusion system and $O^{5'}(\mathcal{D})$ has index 2 in $\mathcal{D}$.*

Proof. In the statement of Proposition 3·9, letting $\mathcal{F}_0 = N_{\mathcal{G}}(E_1)$, $V = E_3$ and $\Delta = \mathrm{Aut}_{\mathcal{G}}(E_3)$ we have that $\mathcal{D}^\dagger = \langle \mathcal{F}_0, \mathrm{Aut}_{\mathcal{G}}(E_3)\rangle_S$ is a proper saturated subsystem of $\mathcal{G}$. But now, applying the Alperin–Goldschmidt theorem $\mathcal{F}_0 = \langle \mathrm{Aut}_{\mathcal{G}}(E_1), \mathrm{Aut}_{\mathcal{G}}(S)\rangle_S$ so that $\mathcal{D} = \langle \mathrm{Aut}_{\mathcal{G}}(E_1), \mathrm{Aut}_{\mathcal{G}}(E_3), \mathrm{Aut}_{\mathcal{G}}(S)\rangle_S = \langle \mathcal{F}_0, \mathrm{Aut}_{\mathcal{G}}(E_3)\rangle_S = \mathcal{D}^\dagger$. Therefore, $\mathcal{D}$ is saturated.

We note that as $\mathcal{D} < \mathcal{G}$, no $\mathcal{G}$-conjugate of E_2 is essential in $\mathcal{D}$. Applying Lemma 6·9, we have that $\mathrm{Aut}^*_{\mathcal{D}}(S) = \mathrm{Aut}^0_{\mathcal{D}}(S)$ and $|\mathrm{Aut}_{\mathcal{D}}(S)/\mathrm{Aut}^0_{\mathcal{D}}(S)| = |C_{\mathrm{Aut}_{\mathcal{D}}(E_1)}(Z(E_1))|/3 = 2$. Hence, Lemma 3·12 implies that $O^{5'}(\mathcal{D})$ is the unique proper subsystem of $\mathcal{D}$ of p'-index and has index 2 in $\mathcal{D}$.

PROPOSITION 6·14. $\mathcal{D}^{frc} = O^{5'}(\mathcal{D})^{frc} = \{E_1, E_3^{\mathcal{G}}, S\}$.

Proof. Let $\mathcal{F}$ be one of $\mathcal{D}$ or $O^{5'}(\mathcal{D})$ and R be a fully $\mathcal{F}$-normalised, radical, centric subgroup of S not equal to E_1, S or a $\mathcal{D}$-conjugate of E_3. If an $\mathcal{F}$-conjugate of R is contained in a $\mathcal{G}$-conjugate of E_3, then since R is $\mathcal{F}$-centric and E_3 is elementary abelian, we have that R is $\mathcal{G}$-conjugate to E_3. Since no $\mathcal{G}$-conjugate of E_3 is contained in E_2, the Alperin–Goldschmidt theorem implies that $\{E_3^{\mathcal{D}}\} = \{E_3^{\mathcal{G}}\}$ and so R is $\mathcal{D}$-conjugate to E_3, a contradiction. Hence R is not contained in a $\mathcal{G}$-conjugate of E_3 and by Proposition 6·2 and using that $E_2 \notin \mathcal{E}(\mathcal{F})$, every $\mathcal{D}$-conjugate of R is contained in at most one $\mathcal{F}$-essential, namely E_1. Then, as E_1 is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant, Lemma 3·6 implies that $\mathrm{Out}_{E_1}(R) \trianglelefteq \mathrm{Out}_{\mathcal{F}}(R)$. Since R is $\mathcal{F}$-centric-radical we see that $E_1 \le R \le S$, a contradiction.

LEMMA 6·15. *E_1 is the unique proper non-trivial strongly closed subgroup of $\mathcal{D}$ and $O^{5'}(\mathcal{D})$*

Proof. Assume that T is a proper non-trivial strongly closed subgroup of $\mathcal{F}$, where $\mathcal{F}$ is one of $\mathcal{D}$ or $O^{5'}(\mathcal{D})$. Then $T \trianglelefteq S$ and so $Z(S) \le T$. Then applying Lemma 6·5, the irreducibility of $O^{5'}(\mathrm{Aut}_{\mathcal{D}}(E_3)) \le \mathrm{Aut}_{\mathcal{F}}(E_3)$ on E_3 implies that $E_3 \le T$. We calculate (see [44, appendix A]) that $E_1 = \langle E_3^S \rangle$ from which we deduce that $E_1 \le T$. Since E_1 is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant and every essential subgroup of $\mathcal{F}$ is contained in E_1 by Proposition 6·2, it follows from the Alperin–Goldschmidt theorem that E_1 is strongly closed in $\mathcal{F}$.

PROPOSITION 6·16. *$O^{5'}(\mathcal{D})$ is a simple saturated fusion system on S and both $\mathcal{D}$ and $O^{5'}(\mathcal{D})$ are exotic.*

Proof. If $O^{5'}(\mathcal{D})$ is not simple with $\mathcal{N} \trianglelefteq O^{5'}(\mathcal{D})$ and $\mathcal{N}$ supported on $T < S$ then by Lemma 6·15, $\mathcal{N}$ is supported on E_1. By [5, proposition I·6·4], $\mathrm{Aut}_{\mathcal{N}}(E_1) \trianglelefteq \mathrm{Aut}_{O^{5'}(\mathcal{D})}(E_1)$ so that $\mathrm{Out}_{\mathcal{N}}(E_1)$ is isomorphic to a normal $5'$-subgroup of $\mathrm{Out}_{O^{5'}(\mathcal{D})}(E_1) \cong 3 \times \mathrm{GL}_2(5)$. In particular, no $\mathcal{D}$-conjugate of E_3 is essential in $\mathcal{N}$ for otherwise we could again lift a cyclic subgroup of order 24 to $\mathrm{Aut}_{\mathcal{N}}(E_1)$, using Lemma 3·6. Thus, applying Proposition 6·36 (or performing the MAGMA calculation on which this relies), we deduce that $\mathcal{E}(\mathcal{N}) = \emptyset$ and $E_1 = O_5(\mathcal{N})$ so that $E_1 \trianglelefteq O^{5'}(\mathcal{D})$, a contradiction by Proposition 3·10.

Hence, if $O^{5'}(\mathcal{D})$ is not simple then $\mathcal{N}$ is supported on S. But then by [5, theorem II·9·8(d)], we have that $O^{5'}(O^{5'}(\mathcal{D})) < O^{5'}(\mathcal{D})$, a contradiction. Thus $O^{5'}(\mathcal{D})$ is simple.

Assume that there is $\mathcal{N}$, a proper non-trivial normal subsystem of $\mathcal{D}$. Applying [5, theorem II·9·1] and using that $O^{5'}(\mathcal{D})$ is simple, we deduce that $O^{5'}(\mathcal{D}) \leq \mathcal{N}$ and it quickly follows that $O^{5'}(\mathcal{D}) = \mathcal{N}$. Since E_1 is strongly closed in $\mathcal{D}$ and $O^{5'}(\mathcal{D})$, by Theorem 3·14, we conclude that $\mathcal{D}$ and $O^{5'}(\mathcal{D})$ are exotic.

We now begin the task of determining all saturated fusion systems supported on S. We first record a lemma limiting the possible combinations of essential subgroups in $\mathcal{F}$.

LEMMA 6·17. *Let $\mathcal{F}$ be a saturated fusion system on S with $E_3^{\mathcal{G}} \cap \mathcal{E}(\mathcal{F}) \neq \emptyset$. If $T \in \{E_2^{\mathcal{G}}\} \cap \mathcal{E}(\mathcal{F})$ then T is not $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant and $\{E_2^{\mathcal{G}}\} \subseteq \mathcal{E}(\mathcal{F})$.*

Proof. Assume that $T \in \{E_2^{\mathcal{G}}\} \cap \mathcal{E}(\mathcal{F})$ and $E_3^{\mathcal{G}} \cap \mathcal{E}(\mathcal{F}) \neq \emptyset$. Then by Lemma 6·9(ii), we may assume that $E_3 \in \mathcal{E}(\mathcal{F})$. Moreover, there is a 3-element in $\mathrm{Aut}_{\mathcal{F}}(E_1)$ which centralises S/E_1 and $Z(E_1)$, normalises E_3 and lifts to some $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ by Lemma 3·2. Then α normalises $S/\mathbf{Q}$ by Lemma 3·6. Thus, if T is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant, as $|S/T| = |T/\mathbf{Q}| = 5$ and by coprime action, α centralises $S/\mathbf{Q}$ and so centralises $E_3\mathbf{Q}/\mathbf{Q}$. But as an $\langle\alpha\rangle$-module, $E_3\mathbf{Q}/\mathbf{Q} \cong E_3/Z(E_1)$ and coprime action yields that α centralises E_3, a contradiction. Thus, T is not $\mathrm{Aut}_{\mathcal{F}}(S)$ invariant and we deduce that all $\mathcal{G}$-conjugates of T are essential in $\mathcal{F}$.

PROPOSITION 6·18. *Suppose that $\mathcal{F}$ is a saturated fusion system on S such that $\mathcal{E}(\mathcal{F}) \subseteq \{E_i\alpha\}$ for some $i \in \{1, 2\}$ and $\alpha \in \mathrm{Aut}_{\mathcal{G}}(S)$. Then either*:

(i) $\mathcal{F} = N_{\mathcal{F}}(S)$; or

(ii) $\mathcal{F} = N_{\mathcal{F}}(E_i\alpha)$ where $O^{5'}(\mathrm{Out}_{\mathcal{F}}(E_i\alpha)) \cong \mathrm{SL}_2(5)$ for $i \in \{1, 2\}$.

Proof. If $\mathcal{E}(\mathcal{F}) = \emptyset$, then (i) holds by the Alperin–Goldschmidt theorem. If $\mathcal{E}(\mathcal{F}) = \{E_1\}$ or $\{E_2\alpha\}$ for some $\alpha \in \mathrm{Aut}_{\mathcal{G}}(S)$, then (ii) holds by Lemma 6·4 and Lemma 6·7.

PROPOSITION 6·19. *Suppose that $\mathcal{F}$ is a saturated fusion system on S with $\{E_2^{\mathcal{G}}\} \subseteq \mathcal{E}(\mathcal{F})$. Then $O^{5'}(\mathrm{Out}_{\mathcal{F}}(\mathbf{Q})) \cong 2.\mathrm{J}_2$, $\mathcal{E}(N_{\mathcal{F}}(\mathbf{Q})) = \{E_2^{\mathcal{G}}\}$ and if $\{E_2^{\mathcal{G}}\} = \mathcal{E}(\mathcal{F})$ then $\mathcal{F} = N_{\mathcal{F}}(\mathbf{Q})$.*

Proof. Assume that $\{E_2^{\mathcal{G}}\} \subseteq \mathcal{E}(\mathcal{F})$. Note that $\mathbf{Q} \trianglelefteq N_{\mathcal{F}}(E_2\alpha) \leq N_{\mathcal{F}}(\mathbf{Q})$ for all $\alpha \in \mathrm{Aut}_{\mathcal{G}}(S)$. Then Proposition 3·10 implies that $\{E_2^{\mathcal{G}}\} \subseteq \mathcal{E}(N_{\mathcal{F}}(\mathbf{Q}))$, $O_5(N_{\mathcal{F}}(\mathbf{Q})) = \mathbf{Q}$ and $\mathcal{F} = N_{\mathcal{F}}(\mathbf{Q})$ whenever $\{E_2^{\mathcal{G}}\} = \mathcal{E}(\mathcal{F})$. Furthermore, any essential subgroup of $N_{\mathcal{F}}(\mathbf{Q})$ contains $\mathbf{Q}$, and as $\mathbf{Q} \not\leq E_1$, an appeal to Proposition 6·2 gives $\mathcal{E}(N_{\mathcal{F}}(\mathbf{Q})) = \{E_2^{\mathcal{G}}\}$. Finally, $O^{5'}(\mathrm{Out}_{\mathcal{F}}(\mathbf{Q}))$ satisfies the hypothesis of Lemma 2·9 so that $O^{5'}(\mathrm{Out}_{\mathcal{F}}(\mathbf{Q})) \cong 2.\mathrm{J}_2$.

PROPOSITION 6·20. *Suppose that $\mathcal{F}$ is a saturated fusion system on S with $\{E_1, E_2\alpha\} \subseteq \mathcal{E}(\mathcal{F})$ for some $\alpha \in \mathcal{G}$. Then $O^{5'}(Out_{\mathcal{F}}(\boldsymbol{R}\alpha)) \cong PSL_3(5)$, $\mathcal{E}(N_{\mathcal{F}}(\boldsymbol{R}\alpha)) = \{E_1, E_2\alpha\}$ and if $\{E_1, E_2\alpha\} = \mathcal{E}(\mathcal{F})$ then $\mathcal{F} = N_{\mathcal{F}}(\boldsymbol{R}\alpha)$.*

Proof. Assume that $\mathcal{E}(\mathcal{F}) = \{E_1, E_2\alpha\}$ for some $\alpha \in \mathrm{Aut}_{\mathcal{G}}(S)$. Adjusting by an automorphism of S if necessary, we may as well assume that $\mathcal{E}(\mathcal{F}) = \{E_1, E_2\}$. Since any $\mathrm{Aut}_{\mathcal{F}}(S)$-conjugate of E_2 is also essential in $\mathcal{F}$, we infer from this that E_2 is $\mathrm{Aut}_{\mathcal{F}}(S)$-invariant. In particular, since $\mathbf{R} \trianglelefteq N_{\mathcal{F}}(E_2)$ by Lemma 6·4, $\mathbf{R}$ is normalised by $\mathrm{Aut}_{\mathcal{F}}(S)$. By Lemma 6·7 $O^{5'}(\mathrm{Aut}_{\mathcal{F}}(E_1))$ normalises $\mathbf{R}$ and so applying Lemma 3·2 and a Frattini argument to $\mathrm{Aut}_{\mathcal{F}}(E_1)$, we deduce that $\mathbf{R}$ is normalised by $\mathrm{Aut}_{\mathcal{F}}(E_1)$. In particular, $\{E_1, E_2\} \subseteq \mathcal{E}(N_{\mathcal{F}}(\mathbf{R}))$.

Note that if $\mathbf{R} \leq E_2\alpha \neq E_2$ for some $\alpha \in \mathrm{Aut}_{\mathcal{G}}(S)$, we have that $\mathbf{R} \leq E_2 \cap E_2\alpha = \mathbf{Q}$, a contradiction. Hence, by Proposition 3·10, we see that $\mathcal{E}(N_{\mathcal{F}}(\mathbf{R})) = \{E_1, E_2\}$. Since $\mathrm{Aut}_{\mathcal{F}}(E_2)$ acts irreducibly on $\mathbf{Q}/\Phi(E_2) = \mathbf{Q}/\mathbf{Q} \cap \mathbf{R}$, we have that $\mathrm{Aut}_{\mathcal{F}}(E_2)$ acts irreducibly on $E_2/\mathbf{R}$ and we conclude that $\mathbf{R} = O_5(N_{\mathcal{F}}(\mathbf{R}))$. If $\{E_1, E_2\} = \mathcal{E}(\mathcal{F})$ then $\mathcal{F} = N_{\mathcal{F}}(\mathbf{R})$. Moreover, the actions described in Lemma 6·4 and Lemma 6·7 imply that the only non-trivial normal subgroups of $\mathcal{F}$ are $\mathbf{R}$ and $\Phi(\mathbf{R})$. Since $\mathrm{M} = \langle N_{\mathrm{M}}(S), M_4 \rangle$, where M is the Monster, we see that $\Phi(\mathbf{R})$ is not characteristic in S. In particular, no non-trivial characteristic subgroup of S is normal in $\mathcal{F}$.

By Theorem 3·11, there is a finite group G with $S \in \mathrm{Syl}_5(G)$, $N_{\mathcal{F}}(\mathbf{R}) = \mathcal{F}_S(G)$ and $F^*(G) = \mathbf{R}$. Moreover, by the uniqueness of models provided in Theorem 3·11 we can embed the models of $N_{\mathcal{F}}(S)$, $N_{\mathcal{F}}(E_1)$ and $N_{\mathcal{F}}(E_2)$, which we write as G_{12}, G_1 and G_2 respectively, into G. Indeed, by the Alperin–Goldschmidt theorem, we may as well assume that $G = \langle G_1, G_2 \rangle$ and $G_{12} = G_1 \cap G_2$. Then the triple $(G_1/\mathbf{R}, G_2/\mathbf{R}, G_{12}/\mathbf{R})$ along with the appropriate induced injective maps forms a weak BN-pair of rank 2, and since $S/\mathbf{R} \cong 5_+^{1+2}$, applying [**16**, theorem A] and using the terminology there, we deduce that $O^{5'}(G)/\mathbf{R}$ is locally isomorphic to $\mathrm{PSL}_3(5)$. By [**28**, theorem 1], $O^{5'}(G)/\mathbf{R} \cong \mathrm{PSL}_3(5)$, and $\mathbf{R}/Z(\mathbf{R})$ and $Z(\mathbf{R})$ are dual natural modules for $O^{5'}(G)/\mathbf{R}$. Hence, we have that $O^{5'}(\mathrm{Out}_{\mathcal{F}}(\mathbf{R})) \cong \mathrm{PSL}_3(5)$, as desired.

Remark. In the above, the groups of shape $5^{3+3}.\mathrm{PSL}_3(5)$ come from a situation where a weak BN-pair of rank 2 of type $\mathrm{PSL}_3(5)$ is pushed up. Indeed, this case occurs as outcome (12) of [**28**, theorem 1] with the stipulation that $q = 5$. There, this phenomena could also occur for $q = 3^n$ for all $n \in \mathbb{N}$. We speculate that these cases could result in a class of interesting fusion systems. In particular, when $q = 3$, a similar Sylow subgroup already supports the 3-fusion categories of $\Omega_7(3)$, Fi_{22} and ${}^2\mathrm{E}_6(2)$. We note however that in our case S is *not* isomorphic to a Sylow 5-subgroup of $\Omega_7(5)$.

PROPOSITION 6·21. *Suppose that $\mathcal{F}$ is a saturated fusion system on S. Then $O_5(\mathcal{F}) = \{1\}$ if and only if $\mathcal{E}(\mathcal{F}) = \{E_1, E_2^{\mathcal{G}}\}$ or $\{E_3^{\mathcal{G}}\} \cap \mathcal{E}(\mathcal{F}) \neq \emptyset$.*

Proof. Suppose first that $\mathcal{E}(\mathcal{F}) = \{E_1, E_2^{\mathcal{G}}\}$. By Proposition 6·19, we have that $\mathrm{Out}_{\mathcal{F}}(\mathbf{Q}) \cong 2.\mathrm{J}_2$ acts irreducibly on $\mathbf{Q}/Z(S)$. As $Q_2 \not\leq E_1$, by Proposition 3·10, we conclude that $O_5(\mathcal{F}) \leq Z(S)$. But by Lemma 6·7, $\mathrm{Out}_{\mathcal{F}}(E_1)$ acts irreducibly on $Z(E_1)$ and we conclude that $O_5(\mathcal{F}) = \{1\}$. If $\{E_3^{\mathcal{G}}\} \cap \mathcal{E}(\mathcal{F}) \neq \emptyset$ then $O_5(\mathcal{F}) = \{1\}$ by Lemma 6·5.

Suppose that $O_5(\mathcal{F}) = \{1\}$. By Lemma 6·9, if $E_3 \notin \mathcal{E}(\mathcal{F})$ then no $\mathcal{G}$-conjugate of E_3 is contained in $\mathcal{E}(\mathcal{F})$. Then Proposition 6·2 and Proposition 6·18-Proposition 6·20 imply that $\mathcal{E}(\mathcal{F}) = \{E_1, E_2^{\mathcal{G}}\}$ as desired.

As a consequence of this result, if $O_5(\mathcal{F}) \neq \{1\}$ then $\mathcal{F}$ is described in Proposition 6·18-Proposition 6·20. We additionally note that if $E_3\alpha \in \mathcal{E}(\mathcal{F})$ then Lemma 6·5 implies that $E_1 \in \mathcal{E}(\mathcal{F})$ and Lemma 6·17 implies that either $\{E_2^{\mathcal{G}}\} \cap \mathcal{E}(\mathcal{F}) = \emptyset$ or $\{E_2^{\mathcal{G}}\} \subset \mathcal{E}(\mathcal{F})$.

LEMMA 6·22. *Suppose that $\mathcal{F}_1, \mathcal{F}_2$ are two saturated fusion systems supported on S. If $E_1 \in \mathcal{E}(\mathcal{F}_1) \cap \mathcal{E}(\mathcal{F}_2)$ and $N_{\mathcal{F}_1}(S) = N_{\mathcal{F}_2}(S)$ then $N_{\mathcal{F}_1}(E_1) = N_{\mathcal{F}_2}(E_1)$.*

Proof. We know that E_1 is characteristic in S so that $N_{\mathcal{F}_1}(E_1) \geq N_{\mathcal{F}_1}(S) \leq N_{\mathcal{F}_2}(E_1)$. By [**33**, proposition 2·11], it suffices to show that $\mathrm{Aut}_{\mathcal{F}_1}(E_1) = \mathrm{Aut}_{\mathcal{F}_2}(E_1)$ and that the homomorphism $H^1(\mathrm{Out}_{\mathcal{F}_1}(E_1); Z(E_1)) \to H^1(\mathrm{Out}_{N_{\mathcal{F}_1}(S)}(E_1); Z(E_1))$ induced by restriction is surjective. We observe by Lemma 6·7 and Lemma 6·8 that $\mathrm{Out}_{\mathcal{F}_i}(E_1)$ contains a subgroup isomorphic to $3 \times \mathrm{GL}_2(5)$ of index at most 2 for $i \in \{1, 2\}$. Moreover, since E_1 is characteristic in S, all morphisms in $\mathrm{Aut}_{\mathcal{F}_i}(S)$ restrict faithfully to morphisms in $\mathrm{Aut}_{\mathcal{F}_i}(E_1)$ for $i \in \{1, 2\}$. In particular, $\mathrm{Aut}_{\mathcal{F}_i}(S)$ is generated by lifted morphisms in $N_{\mathrm{Aut}_{\mathcal{F}_i}(E_1)}(\mathrm{Aut}_S(E_1))$.

Let K be a Hall $5'$-subgroup of $N_{\mathrm{Aut}_{\mathcal{F}_1}(E_1)}(\mathrm{Aut}_S(E_1)) = \mathrm{Aut}_{N_{\mathcal{F}_1}(S)}(E_1) = \mathrm{Aut}_{N_{\mathcal{F}_2}(S)}(E_1)$ so that $K \cong 3 \times C_4 \times C_4$ or $\mathrm{Sym}(3) \times C_4 \times C_4$. Then K lifts to a group of automorphisms $\widehat{K}$ of $\mathrm{Aut}(S)$ with $\widehat{K}\mathrm{Inn}(S) = \mathrm{Aut}_{\mathcal{F}_1}(S) = \mathrm{Aut}_{\mathcal{F}_2}(S)$. We calculate that $|\mathrm{Aut}(S)|_{5'} = |\mathrm{Out}_{\mathcal{G}}(S)|$ in [**44**, appendix A] (but Theorem 6·24 also provides a genuine proof). In particular, $\widehat{K}$ is either a Hall $5'$-subgroup itself or is the centraliser of the unique Sylow 3-subgroup of a Hall $5'$-subgroup of $\mathrm{Aut}(S)$. Either way set $L := C_K(O_3(K))$ so that $L \cong 3 \times C_4 \times C_4$.

Since $C_{\mathrm{Out}_{\mathcal{F}_i}(E_1)}(O_3(\mathrm{Out}_{\mathcal{F}_i}(E_1))) \cong 3 \times \mathrm{GL}_2(5)$, $Z(C_{\mathrm{Out}_{\mathcal{F}_i}(E_1)}(O_3(\mathrm{Out}_{\mathcal{F}_i}(E_1))))$ is cyclic of order 12 for $i \in \{1, 2\}$. Indeed, there is a unique subgroup L^* of L cyclic of order 12 such that $[L^*|_{E_1}\mathrm{Inn}(E_1)/\mathrm{Inn}(E_1), C_{\mathrm{Out}_{\mathcal{F}_i}(E_1)}(O_3(\mathrm{Out}_{\mathcal{F}_i}(E_1)))] = \{1\}$. In particular, by a Frattini argument, we see that $C_{\mathrm{Aut}_{\mathcal{F}_i}(E_1)}(L^*|_{E_1})\mathrm{Inn}(E_1)$ is the preimage in $\mathrm{Aut}_{\mathcal{F}_i}(E_1)$ of $C_{\mathrm{Out}_{\mathcal{F}_i}(E_1)}(O_3(\mathrm{Out}_{\mathcal{F}_i}(E_1)))$. We verify using MAGMA (See [**44**, appendix A]) that $C_{\mathrm{Aut}(E_1)}(L^*|_{E_1}) \cong 3 \times \mathrm{GL}_2(5)$ and so we have that $C_{\mathrm{Aut}_{\mathcal{F}_1}(E_1)}(L^*|_{E_1})\mathrm{Inn}(E_1) = C_{\mathrm{Aut}_{\mathcal{F}_2}(E_1)}(L^*|_{E_1})\mathrm{Inn}(E_1)$. Finally, since $N_{\mathrm{Aut}_{\mathcal{F}_1}(E_1)}(\mathrm{Aut}_S(E_1)) = \mathrm{Aut}_{N_{\mathcal{F}_1}(S)}(E_1) = \mathrm{Aut}_{N_{\mathcal{F}_2}(S)}(E_1)$, a Frattini argument implies that $\mathrm{Aut}_{\mathcal{F}_1}(E_1) = \mathrm{Aut}_{\mathcal{F}_2}(E_1)$.

It remains to prove that the homomorphism $H^1(\mathrm{Out}_{\mathcal{F}_1}(E_1); Z(E_1)) \to H^1(\mathrm{Out}_{N_{\mathcal{F}_1}(S)}(E_1); Z(E_1))$ induced by restriction is surjective. We observe by Lemma 6·7 and Lemma 6·8 that $\mathrm{Out}_{\mathcal{F}_1}(E_1) \cong 3 \times \mathrm{GL}_2(5)$ or $\mathrm{Sym}(3) \times \mathrm{GL}_2(5)$. One can compute (e.g. in MAGMA as in [**44**, appendix A]) that $H^1(\mathrm{Out}_{N_{\mathcal{F}_1}(S)}(E_1); Z(E_1)) = \{1\}$. Hence, the result.

LEMMA 6·23. *Suppose that $\mathcal{F}_1, \mathcal{F}_2$ are two saturated fusion systems supported on T where $E_1 \leq T \leq S$. If $E_3 \in \mathcal{E}(\mathcal{F}_1) \cap \mathcal{E}(\mathcal{F}_2)$ and $N_{\mathcal{F}_1}(E_1) = N_{\mathcal{F}_2}(E_1)$ then $\mathrm{Aut}_{\mathcal{F}_1}(E_3) = \mathrm{Aut}_{\mathcal{F}_2}(E_3)$.*

Proof. By Lemma 6·5, we have that $O^{5'}(\mathrm{Aut}_{\mathcal{F}_i}(E_3)) \cong \mathrm{SL}_2(25)$ for $i \in \{1, 2\}$. Write $X := O^{5'}(\mathrm{Aut}_{\mathcal{F}_1}(E_3))$ and $Y := O^{5'}(\mathrm{Aut}_{\mathcal{F}_2}(E_3))$. Set $K := N_{\mathrm{Aut}_{\mathcal{F}_1}(E_3)}(\mathrm{Aut}_T(E_3))$ so that, by Lemma 6·6, all morphisms in K lift to morphisms in $\mathrm{Aut}_{\mathcal{F}_1}(E_1) = \mathrm{Aut}_{\mathcal{F}_2}(E_1)$. In particular, by Lemma 6·8

$$K = N_{\mathrm{Aut}_{N_{\mathcal{F}_1}(E_1)}(E_3)}(\mathrm{Aut}_T(E_3)) = N_{\mathrm{Aut}_{N_{\mathcal{F}_2}(E_1)}(E_3)}(\mathrm{Aut}_T(E_3)) = N_{\mathrm{Aut}_{\mathcal{F}_2}(E_3)}(\mathrm{Aut}_T(E_3)).$$

Let L be a cyclic subgroup of order 24 in a Hall $5'$-subgroup of K arranged such that $K_L := L\mathrm{Aut}_S(E_3) = N_{O^{5'}(\mathrm{Aut}_{\mathcal{F}_1}(E_3))}(\mathrm{Aut}_S(E_3))$. Then $K_L \leq X \cap Y \leq \mathrm{Aut}(E_1) \cong \mathrm{GL}_4(5)$.

We record that there is a unique conjugacy class of subgroups isomorphic to $\mathrm{SL}_2(25)$ in $\mathrm{GL}_4(5)$ (see [44, appendix A]). Hence, there is $g \in \mathrm{Aut}(E_3)$ with $Y = X^g$.

Then $K_L, (K_L)^g \leq Y$ and so there is $y \in Y$ such that $(K_L)^g = (K_L)^y$. Thus, we have that $X^{gy^{-1}} = X^g$ and we calculate that $gy^{-1} \leq N_{\mathrm{GL}_4(5)}(K_L) \leq N_{\mathrm{GL}_4(5)}(X)$ (see [44, appendix A]). But then $X = X^g = Y$. By a Frattini argument, $\mathrm{Aut}_{\mathcal{F}_1}(E_3) = XK = YK = \mathrm{Aut}_{\mathcal{F}_2}(E_3)$.

THEOREM 6·24. *Suppose that $\mathcal{F}$ is saturated fusion system on S. If $\{E_1, E_2^{\mathcal{G}}\} \subseteq \mathcal{E}(\mathcal{F})$ then $\mathcal{F} \cong \mathcal{G}$ or $\mathcal{H}$.*

Proof. We observe by the Alperin–Goldschmidt theorem, Proposition 6·2 and Lemma 6·9 that either $\mathcal{E}(\mathcal{F}) = \{E_1, E_2^{\mathcal{G}}\}$ or $\mathcal{E}(\mathcal{F}) = \{E_1, E_2^{\mathcal{G}}, E_3^{\mathcal{G}}\}$. Moreover, applying Lemma 6·22 and Lemma 6·23, if $N_{\mathcal{F}}(S) = N_{\mathcal{G}}(S)$ and $N_{\mathcal{F}}(\mathbf{Q}) = N_{\mathcal{G}}(\mathbf{Q})$ then the Alperin–Goldschmidt theorem and Proposition 6·2 yields that $\mathcal{F} = \mathcal{G}$ or $\mathcal{F} = \mathcal{H}$ depending on whether or not $E_3 \in \mathcal{E}(\mathcal{F})$.

Since $\{E_2^{\mathcal{G}}\} \subseteq \mathcal{E}(\mathcal{F})$, we have by Proposition 6·19 that $\mathbf{Q} = O_5(N_{\mathcal{F}}(\mathbf{Q}))$ and $O^{5'}(\mathrm{Out}_{\mathcal{F}}(\mathbf{Q})) \cong 2.\mathrm{J}_2$ acts trivially on $Z(S)$. Let L be a complement to $\mathrm{Aut}_S(E_1)$ in $N_{O^{5'}(\mathrm{Aut}_{\mathcal{F}}(E_1))}(\mathrm{Aut}_S(E_1))$, recalling that $O^{5'}(\mathrm{Out}_{\mathcal{F}}(E_1)) \cong \mathrm{SL}_2(5)$ by Lemma 6·7. Then L acts faithfully on $Z(S)$ and lifts by Lemma 3·2 to $\widehat{L} \leq \mathrm{Aut}_{\mathcal{F}}(S)$ which also acts faithfully on $Z(S)$. Since $\mathbf{Q}$ is characteristic in S, $\widehat{L}|_{\mathbf{Q}}$ induces a cyclic subgroup of $\mathrm{Aut}_{\mathcal{F}}(\mathbf{Q})$ of order 4 which acts faithfully on $Z(S)$. Indeed, we have that $|\mathrm{Out}_{\mathcal{F}}(\mathbf{Q})/O^{5'}(\mathrm{Out}_{\mathcal{F}}(\mathbf{Q}))| \geq 4$.

Since a maximal subgroup of $\mathrm{Out}(\mathbf{Q}) \cong \mathrm{Sp}_6(5){:}4$ containing $O^{5'}(\mathrm{Out}_{\mathcal{F}}(\mathbf{Q}))$ has shape $4.\mathrm{J}_2 : 2$ by [47] and [9, Table 8·28], we deduce that $\mathrm{Out}_{\mathcal{F}}(\mathbf{Q}) \cong 4.\mathrm{J}_2 : 2$ is maximal in $\mathrm{Out}(\mathbf{Q})$. We calculate in MAGMA that $|\mathrm{Aut}(S)|_{5'} = 2^5.3 = |N_{\mathrm{Aut}_{\mathcal{F}}(\mathbf{Q})}(\mathrm{Aut}_S(Q))|$ so that a Hall $5'$-subgroup of $N_{\mathrm{Aut}(\mathbf{Q})}(\mathrm{Aut}_S(Q))$ has the same order as a Hall $5'$-subgroup of $\mathrm{Aut}(S)$. By Lemma 3·2, every automorphism in $N_{\mathrm{Aut}_{\mathcal{F}}(\mathbf{Q})}(\mathrm{Aut}_S(Q))$ of $5'$-order extends to an element of $\mathrm{Aut}_{\mathcal{F}}(S)$ of $5'$-order. We deduce that the subgroup of $\mathrm{Aut}_{\mathcal{F}}(S)$ generated by lifts of elements of a fixed Hall $5'$-subgroup of $N_{\mathrm{Aut}_{\mathcal{F}}(\mathbf{Q})}(\mathrm{Aut}_S(Q))$ has the same order as a Hall $5'$-subgroup of $\mathrm{Aut}(S)$, and so is a Hall $5'$-subgroup of $\mathrm{Aut}(S)$. Indeed, $\mathrm{Aut}_{\mathcal{F}}(S)$ contains a Hall $5'$-subgroup of $\mathrm{Aut}(S)$, and by a similar reasoning, $\mathrm{Aut}_{\mathcal{G}}(S)$ contains a Hall $5'$-subgroup of $\mathrm{Aut}(S)$. Therefore, there is $\alpha \in \mathrm{Aut}(S)$ such that $\mathrm{Aut}_{\mathcal{F}^\alpha}(S) = \mathrm{Aut}_{\mathcal{F}}(S)^\alpha = \mathrm{Aut}_{\mathcal{G}}(S)$ and by the Alperin–Goldschmidt theorem, we have $N_{\mathcal{F}^\alpha}(S) = N_{\mathcal{F}}(S)^\alpha = N_{\mathcal{G}}(S)$.

Let K be the embedding of the restriction of $\mathrm{Aut}_{\mathcal{F}^\alpha}(S)$ to $\mathbf{Q}$ into $\mathrm{Aut}(\mathbf{Q}) \cong 5^6{:}\mathrm{Sp}_6(5).4$. Set $X = \mathrm{Aut}_{\mathcal{F}^\alpha}(\mathbf{Q})$ and $Y = \mathrm{Aut}_{\mathcal{G}}(\mathbf{Q})$ so that $K \leq X \cap Y$. We observe that there is one conjugacy class of subgroups isomorphic to $\mathrm{Aut}_{\mathcal{F}^\alpha}(\mathbf{Q})$ in $\mathrm{Aut}(\mathbf{Q})$ and so there is $g \in \mathrm{Aut}(\mathbf{Q})$ with $Y = X^g$. Then $K, K^g \leq Y$ and K, K^g are both Sylow 5-subgroup normalisers in Y. Thus, there is $m \in Y$ with $K^m = K^g$ so that $gm^{-1} \in N_{\mathrm{Aut}(\mathbf{Q})}(K)$ and $X^{gm^{-1}} = Y$. We calculate in MAGMA that $N_{\mathrm{Aut}(\mathbf{Q})}(K) \leq N_{\mathrm{Aut}(\mathbf{Q})}(X)$ ([44, appendix A]) so that $X = Y$ and $\mathrm{Aut}_{\mathcal{F}^\alpha}(\mathbf{Q}) = \mathrm{Aut}_{\mathcal{G}}(\mathbf{Q})$. Hence, by Theorem 3·11 there is $\beta \in \mathrm{Aut}(S)$ such that $N_{\mathcal{F}^{\alpha\beta}}(\mathbf{Q}) = N_{\mathcal{F}^\alpha}(\mathbf{Q})^\beta = N_{\mathcal{G}}(\mathbf{Q})$. Then for $\gamma := \alpha\beta \in \mathrm{Aut}(S)$, we deduce that $N_{\mathcal{F}^\gamma}(S) = N_{N_{\mathcal{F}^\gamma}(\mathbf{Q})}(S) = N_{N_{\mathcal{G}}(\mathbf{Q})}(S) = N_{\mathcal{G}}(S)$ and we conclude that $\mathcal{F}^\gamma = \mathcal{H}$ or $\mathcal{G}$. Hence, $\mathcal{F} \cong \mathcal{H}$ or $\mathcal{G}$, as desired.

Remark. The techniques in the above proof can be used to show that the symplectic amalgam $\mathcal{A}_{53}$ found in [38] is determined up to isomorphism.

THEOREM 6·25. *Suppose that $\mathcal{F}$ is saturated fusion system on S such that $O_5(\mathcal{F}) = \{1\}$ and $\{E_1, E_2^{\mathcal{G}}\} \not\subseteq \mathcal{E}(\mathcal{F})$. Then $\mathcal{F} \cong \mathcal{D}$ or $O^{5'}(\mathcal{D})$.*

Proof. Since $O_5(\mathcal{F}) = \{1\}$, Proposition 6·21 implies that $\{E_3^{\mathcal{G}}\} \cap \mathcal{E}(\mathcal{F}) \neq \emptyset$. Then Lemma 6·9(ii) implies that $\{E_3^{\mathcal{G}}\} \subseteq \mathcal{E}(\mathcal{F})$. Moreover, Lemma 6·5 implies that $E_1 \in \mathcal{E}(\mathcal{F})$ and Lemma 6·17 implies that $\{E_2^{\mathcal{G}}\} \cap \mathcal{E}(\mathcal{F}) = \emptyset$. Hence, $\mathcal{E}(\mathcal{F}) = \{E_1, E_3^{\mathcal{G}}\}$.

By Lemma 6·7 and Lemma 6·8, we have that $C_{\mathrm{Out}_{\mathcal{F}}(E_1)}(O_3(\mathrm{Out}_{\mathcal{F}}(E_1))) \cong 3 \times \mathrm{GL}_2(5)$ has index at most 2 in $\mathrm{Out}_{\mathcal{F}}(E_1)$ and $\{E_3^{\mathcal{F}}\} = \{E_3^{\mathcal{G}}\}$. Let A be the preimage in $\mathrm{Aut}_{\mathcal{F}}(E_1)$ of this subgroup and consider the group $K := N_A(\mathrm{Aut}_S(E_1))$. Then, by Lemma 3·2 K lifts to a subgroup $\widehat{K} \leq \mathrm{Aut}_{\mathcal{F}}(S)$ such that $\widehat{K}\mathrm{Inn}(S)/\mathrm{Inn}(S) \cong 3 \times C_4 \times C_4$. In particular, $|\widehat{K}\mathrm{Inn}(S)/\mathrm{Inn}(S)| = |\mathrm{Out}_{\mathcal{G}}(S)|/2$. As observed in Theorem 6·24, $\mathrm{Out}_{\mathcal{G}}(S)$ is a Hall $5'$-subgroup of $\mathrm{Out}(S)$ and so $\widehat{K}\mathrm{Inn}(S)/\mathrm{Inn}(S)$ has index 2 in some Hall $5'$-subgroup Y of $\mathrm{Out}(S)$ which is conjugate in $\mathrm{Out}(S)$ to $\mathrm{Out}_{\mathcal{G}}(S)$. Indeed, $\widehat{K}\mathrm{Inn}(S)/\mathrm{Inn}(S) = C_Y(O_3(Y))$. Hence, we have that $\widehat{K}$ is conjugate in $\mathrm{Aut}(S)$ to $\mathrm{Aut}_{O^{5'}(\mathcal{D})}(S) = C_{\mathrm{Aut}_{\mathcal{D}}(S)}(O_3(\mathrm{Aut}_{\mathcal{D}}(S)))$. Since $N_{\mathrm{Aut}(S)}(\mathrm{Aut}_{O^{5'}(\mathcal{D})}(S)) = \mathrm{Aut}_{\mathcal{G}}(S)$ by [44, appendix A], we see that either $\mathrm{Aut}_{\mathcal{F}}(S)$ is conjugate to $\mathrm{Aut}_{O^{5'}(\mathcal{D})}(S)$ or $\mathrm{Aut}_{\mathcal{G}}(S)$. In particular, applying Theorem 3·11, $N_{\mathcal{F}}(S)$ is $\mathrm{Aut}(S)$-conjugate to $N_{O^{5'}(\mathcal{D})}(S)$ or $N_{\mathcal{D}}(S)$.

Suppose that there is $\alpha \in \mathrm{Aut}(S)$ with $N_{\mathcal{F}^\alpha}(S) = N_{\mathcal{F}}(S)^\alpha = N_{\mathcal{D}}(S)$. Applying Lemma 6·22 we have that $N_{\mathcal{F}^\alpha}(E_1) = N_{\mathcal{D}}(E_1)$. We note that by Lemma 6·9 that $\{E_3^{\mathcal{G}}\} = \{E_3^{\mathcal{F}^\alpha}\}$. Hence, $\mathcal{F}^\alpha$ and $\mathcal{D}$ have the same essential subgroups and we deduce by the Alperin–Goldschmidt theorem that $\mathcal{F}^\alpha = \langle N_{\mathcal{F}^\alpha}(E_1), \mathrm{Aut}_{\mathcal{F}^\alpha}(E_3)\rangle_S$. Then Lemma 6·23 implies that $\mathcal{F}^\alpha = \mathcal{D}$ and $\mathcal{F} \cong \mathcal{D}$.

Suppose now that there is $\alpha \in \mathrm{Aut}(S)$ with $N_{\mathcal{F}^\alpha}(S) = N_{\mathcal{F}}(S)^\alpha = N_{O^{5'}(\mathcal{D})}(S)$ so that by Lemma 6·22, we have $N_{\mathcal{F}^\alpha}(E_1) = N_{O^{5'}(\mathcal{D})}(E_1)$. Then by the Alperin–Goldschmidt theorem, Proposition 6·2 and Lemma 6·9, we have that $\mathcal{F}^\alpha = \langle N_{\mathcal{F}^\alpha}(E_1), \mathrm{Aut}_{\mathcal{F}^\alpha}(E_3)\rangle_S$ and Lemma 6·23 implies that $\mathcal{F}^\alpha = O^{5'}(\mathcal{D})$ and $\mathcal{F} \cong O^{5'}(\mathcal{D})$, completing the proof.

We provide the following Tables 4 and 5 summarising the actions induced by the fusion systems described in Theorem 6·24 and Theorem 6·25 on their centric-radical subgroups. The entry "-" indicates that the subgroup is no longer centric-radical in the subsystem.

Table 4. *$\mathcal{G}$-conjugacy classes of radical-centric subgroups of S*

P	$\|P\|$	$\mathrm{Out}_{\mathcal{G}}(P)$	$\mathrm{Out}_{\mathcal{H}}(P)$
S	5^9	$\mathrm{Sym}(3) \times 4 \times 4$	$\mathrm{Sym}(3) \times 4 \times 4$
E_1	5^8	$\mathrm{Sym}(3) \times \mathrm{GL}_2(5)$	$\mathrm{Sym}(3) \times \mathrm{GL}_2(5)$
E_2	5^8	$2 \times \mathrm{GL}_2(5)$	$2 \times \mathrm{GL}_2(5)$
E_3	5^4	$(3 \times \mathrm{SL}_2(25)).2$	–
$\mathbf{Q}$	5^7	$4.\mathrm{J}_2 : 2$	$4.\mathrm{J}_2 : 2$
$\mathbf{R}$	5^6	$2 \times \mathrm{PSL}_3(5)$	$2 \times \mathrm{PSL}_3(5)$

Table 5. *$\mathcal{G}$-conjugacy classes of radical-centric subgroups of S*

P	$\|P\|$	$\mathrm{Out}_{\mathcal{D}}(P)$	$\mathrm{Out}_{O^{5'}(\mathcal{D})}(P)$
S	5^9	$\mathrm{Sym}(3) \times 4 \times 4$	$3 \times 4 \times 4$
E_1	5^8	$\mathrm{Sym}(3) \times \mathrm{GL}_2(5)$	$3 \times \mathrm{GL}_2(5)$
E_2	5^8	–	–
E_3	5^4	$(3 \times \mathrm{SL}_2(25)).2$	$3 \times \mathrm{SL}_2(25)$
$\mathbf{Q}$	5^7	–	–
$\mathbf{R}$	5^6	–	–

In a similar manner to Section 5, we now construct some additional exotic fusion systems related to the system $\mathcal{D}$ and supported on E_1. We note that the lift to $\mathrm{Aut}_{\mathcal{D}}(E_1)$ of a cyclic subgroup of order 24 in $N_{O^{5'}(\mathrm{Aut}_{\mathcal{D}}(E_3))}(\mathrm{Aut}_{E_1}(E_3))$ projects as a group of order 24 in the unique normal subgroup of $\mathrm{Out}_{\mathcal{D}}(E_1)\cong \mathrm{Sym}(3)\times \mathrm{GL}_2(5)$ which is isomorphic to $\mathrm{GL}_2(5)$. Indeed, there is a unique up to conjugacy cyclic subgroup of $\mathrm{GL}_2(5)$ of order 24 which is contained in a unique $5'$-order overgroup, in which it has index 2. We set $K^*\cong \mathrm{Sym}(3)\times (\mathrm{C}_{24}:2)$ to be the unique $5'$-order overgroup of a chosen cyclic subgroup of order 24 in $\mathrm{Out}_{\mathcal{D}}(E_1)$ and denote by K its preimage in $\mathrm{Aut}_{\mathcal{D}}(E_1)$. Indeed, $N_{\mathrm{Aut}_{\mathcal{D}(E_1)}}(E_3)$ has index 50 in $K\mathrm{Inn}(E_1)$.

Let G be a model for $N_{\mathcal{D}}(E_1)$ and let H be a subgroup of G chosen such that $\mathrm{Aut}_H(E_1)=K\mathrm{Inn}(E_1)$. We define the subsystem

$$\mathcal{D}^*=\langle \mathcal{F}_{E_1}(H),\mathrm{Aut}_{\mathcal{D}}(E_3)\rangle_{E_1}\leq \mathcal{D}.$$

We observe that we could have chosen any of the 10 $\mathrm{Aut}_{\mathcal{D}}(E_1)$-conjugates of K to form a saturated fusion system. By definition, all of the created fusion systems are isomorphic. Moreover, the $\mathcal{D}$-conjugacy class of E_3 splits into 10 classes upon restricting to $\mathcal{D}^*$, which in turn correspond the possible choices of a cyclic subgroup of order 24.

PROPOSITION 6·26. *$\mathcal{D}^*$ is saturated fusion system on E_1 and $\mathcal{E}(\mathcal{D}^*)=\{E_3^{\mathcal{D}^*}\}$.*

Proof. We create H as in the construction of $\mathcal{D}^*$ and consider $\mathcal{F}_{E_1}(H)$. Since $\mathcal{F}_{E_1}(H)\leq\mathcal{D}$, and as E_3 is fully $\mathcal{D}$-normalised and $N_S(E_3)\leq E_1$, E_3 is also fully $\mathcal{F}_{E_1}(H)$-normalised. Since $C_{E_1}(E_3)\leq E_3$ we see that E_3 is also $\mathcal{F}_{E_1}(H)$-centric. Finally, since E_3 is abelian, it is minimal among S-centric subgroups with respect to inclusion and has the property that no proper subgroup of E_3 is essential in $\mathcal{F}_{E_1}(H)$. In the statement of Proposition 3·9, letting $\mathcal{F}_0=\mathcal{F}_{E_1}(H)$, $V=E_3$ and $\Delta=\mathrm{Aut}_{\mathcal{D}}(E_3)$, we have that $\widetilde{\Delta}:=\mathrm{Aut}_{\mathcal{F}_{E_1}(H)}(E_3)=N_{\mathrm{Aut}_{\mathcal{D}}(E_3)}(\mathrm{Aut}_S(E_3))$ is strongly 5-embedded in Δ. By that result, $\mathcal{D}^*=\langle\mathcal{F}_{E_1}(H),\mathrm{Aut}_{\mathcal{D}}(E_3)\rangle_{E_1}$ is a saturated fusion system.

Since each morphism in $\mathcal{D}^*$ is a composite of morphisms in $\mathcal{F}_{E_1}(H)$ and $\mathrm{Aut}_{\mathcal{D}}(E_3)$, we must have that an essential subgroup of $\mathcal{D}^*$ is contained in some H-conjugate of E_3 and so $\mathcal{E}(\mathcal{D}^*)=\{E_3^{\mathcal{D}^*}\}$.

PROPOSITION 6·27. *$O^{5'}(\mathcal{D}^*)$ has index 4 in $\mathcal{D}^*$.*

Proof. Let K be a Hall $5'$-subgroup of $N_{O^{5'}(\mathrm{Aut}_{\mathcal{D}^*}(E_3))}(\mathrm{Aut}_{E_1}(E_3))$ so that K is cyclic of order 24. Then K centralises a Sylow 3-subgroup of $\mathrm{Aut}_{\mathcal{D}^*}(E_3)$ and, by Lemma 3·6, lifts to a group of morphisms in $\mathrm{Aut}_{\mathcal{D}^*}(E_1)$ which we denote by $\widehat{K}$. Indeed, it follows that $\widehat{K}$ centralises a Sylow 3-subgroup of $\mathrm{Aut}_{\mathcal{D}^*}(E_1)$, and this holds for all $\mathcal{D}^*$ conjugates of E_3. Now, by the definition of $\mathcal{D}^*$, if R is a $\mathcal{D}^*$-centric subgroup which is not equal to a $\mathcal{D}^*$-conjugate of E_3 then by Lemma 3·6, it follows that $\mathrm{Aut}_{E_1}(R)\trianglelefteq\mathrm{Aut}_{\mathcal{D}^*}(R)$ and so $O^{5'}(\mathrm{Aut}_{\mathcal{D}^*}(R))$ is a 5-group. Hence, we have by definition that $\mathrm{Out}^0_{\mathcal{D}^*}(E_1)$ centralises a Sylow 3-subgroup of $\mathrm{Out}_{\mathcal{D}^*}(E_1)$. We observe that the centraliser in $\mathrm{Out}_{\mathcal{D}^*}(E_1)\cong\mathrm{Sym}(3)\times(\mathrm{C}_{24}:2)$ of a Sylow 3-subgroup is isomorphic to $3\times\mathrm{C}_{24}$ and so $O^{5'}(\mathcal{D}^*)$ has index at least 4 in $\mathcal{D}^*$ by Lemma 3·12.

Since $\widehat{K}$ is cyclic of order 24, we have that $\mathrm{Out}^0_{\mathcal{D}^*}(E_1)$ is of order at least 24 and $O^{5'}(\mathcal{D}^*)$ has index at most 12 in $\mathcal{D}^*$ by Lemma 3·12. Aiming for a contradiction, assume that $\mathrm{Out}^0_{\mathcal{D}^*}(E_1)=\widehat{K}\mathrm{Inn}(E_1)/\mathrm{Inn}(E_1)$ is cyclic of order 24. Then $\widehat{K}\mathrm{Inn}(E_1)/\mathrm{Inn}(E_1)\trianglelefteq\mathrm{Out}_{\mathcal{D}^*}(E_1)$. But then for $T\in\mathrm{Syl}_3(\widehat{K}\mathrm{Inn}(E_1)/\mathrm{Inn}(E_1))$, we see that $T\trianglelefteq\mathrm{Out}_{\mathcal{D}^*}(E_1)$ and in the language of Lemma 6·7 we have that $T \in \mathrm{Syl}_3(A)$ or $T \in \mathrm{Syl}_3(B)$. That is, either T centralises $Z(E_1)$ or T centralises $\Phi(E_1)/Z(E_1)$. Since T is induced by the lift of a morphism in K, this is a contradiction. Hence, $\mathrm{Out}^0_{\mathcal{D}^*}(E_1) \cong 3 \times \mathrm{C}_{24}$ and by Lemma 3·12 we have that $O^{5'}(\mathcal{D}^*)$ has index 4 in $\mathcal{D}^*$.

PROPOSITION 6·28. *$O^{5'}(\mathcal{D}^*)$ is simple and there is $\alpha \in \mathrm{Aut}_{\mathcal{D}^*}(E_1)$ such that $\mathcal{E}(O^{5'}(\mathcal{D}^*)) = \{E_3^{\mathcal{D}^*}\} = \{E_3^{O^{5'}(\mathcal{D}^*)}, (E_3\alpha)^{O^{5'}(\mathcal{D}^*)}\} \neq \{E_3^{O^{5'}(\mathcal{D}^*)}\}$.*

Proof. Let $\mathcal{N} \trianglelefteq O^{5'}(\mathcal{D}^*)$ supported on $P \leq E_1$. By Lemma 3·12 we may assume that $P < E_1$, and P is strongly closed in $\mathcal{D}^*$. By the irreducible action of $O^{5'}(\mathrm{Aut}_{\mathcal{D}^*}(E_3))$ on E_3, we deduce that $E_3 \leq P$ and since $P \trianglelefteq E_1$, we have (as calculated in [**44**, appendix A]) that $N_{E_1}(E_3) = \langle E_3^{E_1} \rangle \leq P$. Indeed, as $\mathrm{Out}_{O^{5'}(\mathcal{D}^*)}(E_1)$ acts irreducibly on $E_1/N_{E_1}(E_3)$ we see that $P = N_{E_1}(E_3)$. By [**5**, proposition I·6·4(c)] we have that $O^{5'}(\mathrm{Aut}_{\mathcal{N}}(E_3)) = O^{5'}(\mathrm{Aut}_{O^{5'}(\mathcal{D})}(E_3)) \cong \mathrm{SL}_2(25)$.

Let τ be a non-trivial involution in $Z(O^{5'}(\mathrm{Aut}_{\mathcal{N}}(E_3)))$. By Lemma 3·6, τ lifts to $\widetilde{\tau} \in \mathrm{Aut}_{O^{5'}(\mathcal{D}^*)}(E_1)$ and restricts to $\widehat{\tau} \in \mathrm{Aut}_{O^{5'}(\mathcal{D}^*)}(P)$. Indeed, $\widehat{\tau} \in \mathrm{Aut}_{\mathcal{N}}(P) \trianglelefteq \mathrm{Aut}_{O^{5'}(\mathcal{D}^*)}(P)$ and we ascertain that $[\widehat{\tau}, \mathrm{Aut}_{E_1}(P)] \leq \mathrm{Inn}(P)$. Since $\widehat{\tau}$ is the extension of $\tau \in \mathrm{Aut}_{\mathcal{N}}(E_3)$ to P, we have that $[\widehat{\tau}, \mathrm{Aut}_{E_1}(P)] \leq \mathrm{Aut}_{E_3}(P)$. Since $\widetilde{\tau}$ is the lift of τ to $\mathrm{Aut}_{\mathcal{D}^*}(E_1)$, we infer that $[\widetilde{\tau}, E_1] \leq E_3$. But then, as E_3 is abelian and $[E_1, E_3, \widetilde{\tau}] \leq [\Phi(E_1), \widetilde{\tau}] \leq Z(E_1)$, the three subgroups lemma implies that $[E_3, \widetilde{\tau}, E_1] \leq Z(E_1)$ and as $E_3 = [E_3, \tau]$ and $Z(E_1) \leq E_3$, we have that $E_3 \trianglelefteq E_1$, a contradiction. Hence, $O^{5'}(\mathcal{D}^*)$ is simple.

By Lemma 3·13, we see that $\mathcal{E}(O^{5'}(\mathcal{D}^*)) = \mathcal{E}(\mathcal{D}^*) = \{E_3^{\mathcal{D}^*}\}$. We note that $\mathrm{Aut}_{O^{5'}(\mathcal{D}^*)}(E_1) \leq N_{\mathrm{Aut}_{\mathcal{D}^*}(E_1)}(E_3)\mathrm{Inn}(E_1)$. Since $N_{\mathrm{Aut}_{\mathcal{D}^*}(E_1)}(E_3)$ has index 50 in $\mathrm{Aut}_{\mathcal{D}^*}(E_1)$ and $|\mathrm{Inn}(E_1)/N_{\mathrm{Inn}(E_1)}(E_3)| = 25$, it follows that $\{E_3^{\mathcal{D}^*}\}$ splits into two conjugacy classes upon restricting to the action of $N_{\mathrm{Aut}_{\mathcal{D}^*}(E_1)}(E_3)\mathrm{Inn}(E_1)$, each of size 25. Since $\mathrm{Inn}(E_1) \leq \mathrm{Aut}_{O^{5'}(\mathcal{D}^*)}(E_1)$, we deduce that

$$\left\{E_3^{\mathrm{Aut}_{O^{5'}(\mathcal{D}^*)}(E_1)}\right\} = \left\{E_3^{N_{\mathrm{Aut}_{\mathcal{D}^*}(E_1)}(E_3)\mathrm{Inn}(E_1)}\right\}.$$

Finally, there is $\alpha \in \mathrm{Aut}_{\mathcal{D}^*}(E_1)$ with $E_3\alpha \notin \{E_3^{\mathrm{Aut}_{O^{5'}(\mathcal{D}^*)}(E_1)}\}$ and it follows that for such an α, $\mathcal{E}(O^{5'}(\mathcal{D}^*)) = \{E_3^{O^{5'}(\mathcal{D}^*)}, E_3\alpha^{O^{5'}(\mathcal{D}^*)}\}$.

PROPOSITION 6·29. *There are three proper saturated subsystems of $\mathcal{D}^*$ which properly contain $O^{5'}(\mathcal{D}^*)$. Moreover, every saturated subsystem $\mathcal{F}$ of $\mathcal{D}^*$ of index prime to 5 satisfies $\mathcal{F}^{frc} = \{E_3^{\mathcal{D}^*}, E_1\}$.*

Proof. Applying Lemma 3·12, we simply enumerate the proper subgroups of $\mathrm{Out}_{\mathcal{D}^*}(E_1)$ which properly contain $\mathrm{Out}_{O^{5'}(\mathcal{D}^*)}(E_1)$, which gives three non-isomorphic subgroups of shapes $\mathrm{Sym}(3) \times \mathrm{C}_{24}$, $3 \times (\mathrm{C}_{24} : 2)$ and $(3 \times \mathrm{C}_{24}) : 2$.

Let $\mathcal{F}$ be a fusion subsystem of $\mathcal{D}^*$ of index prime to 5 and assume that $R \in \mathcal{F}^{frc}$ with $R \neq E_1$. Applying Lemma 3·6, since R is $\mathcal{F}$-radical, some $\mathcal{F}$-conjugate of R is contained in at least one $\mathcal{F}$-essential subgroup. But Proposition 6·28 then implies that R is contained in a $\mathcal{D}^*$-conjugate of E_3. Since E_3 is elementary abelian and R is $\mathcal{F}$-centric, we must have that R is $\mathcal{D}^*$-conjugate to E_3, as required.

PROPOSITION 6·30. *Every saturated subsystem $\mathcal{F}$ of $\mathcal{D}^*$ of index prime to 5 is an exotic fusion system.*

Proof. Assume that there is $\mathcal{N}$ is a non-trivial normal subsystem of $\mathcal{F}$. Applying [**5**, theorem II·9·1] and using that $O^{5'}(\mathcal{D}^*)$ is simple and normal in $\mathcal{F}$, we deduce that $O^{5'}(\mathcal{D}^*) \leq \mathcal{N}$. Hence, every normal subsystem of $\mathcal{F}$ is supported on E_1.

Suppose that there is a finite group G containing E_1 as a Sylow 5-subgroup with $\mathcal{F} = \mathcal{F}_{E_1}(G)$. We may as well assume that $O_5(G) = O_{5'}(G) = \{1\}$, and since $\mathcal{F}_{F^*(G)\cap E_1}(F^*(G)) \trianglelefteq \mathcal{F}$, we have that $E_1 \in \mathrm{Syl}_5(F^*(G))$. Since $|\Omega_1(Z(E_1))| = 25$, we conclude that $F^*(G) = E(G)$ is a direct product of at most two non-abelian simple groups.

If $F^*(G)$ is a direct product of exactly two simple groups, K_1 and K_2 say, then $N_{N_G(E_1)}(K_i \cap \Omega(Z(E_1)))$ has index at most 2 in $N_G(E_1)$. But a 3-element of $\mathrm{Aut}_{\mathcal{F}}(E_1)$ acts irreducibly on $\Omega(Z(E_1))$ and we have a contradiction. Thus, $F^*(G)$ is simple.

If $F^*(G) \cong \mathrm{Alt}(n)$ for some n then $m_5(\mathrm{Alt}(n)) = \lfloor \frac{n}{5} \rfloor$ by [**21**, proposition 5·2·10] and so $n < 25$. But a Sylow 5-subgroup of Alt(25) has order 5^6 and so $F^*(G) \not\cong \mathrm{Alt}(n)$ for any n. If $F^*(G)$ is isomorphic to a group of Lie type in characteristic 5, then comparing with [**21**, Table 3·3·1], we see that the groups with a Sylow 5-subgroup which has 5-rank 4 are $\mathrm{PSL}_2(5^4)$, $\mathrm{PSL}_3(25)$, $\mathrm{PSU}_3(25)$, $\mathrm{PSL}_4(5)$ or $\mathrm{PSU}_4(5)$ and none of these examples have a Sylow 5-subgroup of order 5^8.

Assume now that $F^*(G)$ is a group of Lie type in characteristic $r \neq 5$. Since E_1 has multiple elementary abelian subgroups of order 5^4, we arrive at the same contradiction as in Proposition 6·12.

Finally, no sporadic groups have Sylow 5-subgroup of order 5^8 and we conclude that $\mathcal{F}$ is exotic.

As observed in Proposition 6·28, the $\mathcal{D}^*$-classes of E_3 split into two distinct classes upon restriction to $O^{5'}(\mathcal{D}^*)$ (in fact, this holds restricting to $\mathcal{F}_{E_1}(E_1)$). Indeed, there is a system of index 2 in $\mathcal{D}^*$ in which this happens and this is the largest subsystem of $\mathcal{D}^*$ in which this happens. This subsystem, which we denote by $\mathcal{L}$, contains $O^{5'}(\mathcal{D}^*)$ with index 2 and has $\mathrm{Out}_{\mathcal{L}}(E_1) = N_{\mathrm{Out}_{\mathcal{D}^*}(E_1)}(E_3) \cong (3 \times \mathrm{C}_{24}) : 2$.

We may apply Lemma 3·8 to $\mathcal{L}$ and $O^{5'}(\mathcal{D}^*)$, and as the two classes of essential subgroups are fused by an element of $\mathrm{Aut}(E_1)$, regardless of the choice of class we obtain a saturated subsystem defined up to isomorphism. We denote the subsystems obtained by $\mathcal{L}_{\mathcal{P}}$ and $O^{5'}(\mathcal{D}^*)_{\mathcal{P}}$ and the convention we adopt is that $E_3 \in \mathcal{E}(\mathcal{L}_{\mathcal{P}}) \cap \mathcal{E}(O^{5'}(\mathcal{D}^*)_{\mathcal{P}})$. It is clear from Lemma 3·8 that $\mathcal{E}(\mathcal{L}_{\mathcal{P}}) = \mathcal{E}(O^{5'}(\mathcal{D}^*)_{\mathcal{P}}) = \{E_3^{\mathcal{L}}\}$.

PROPOSITION 6·31. *$O^{5'}(\mathcal{L}_{\mathcal{P}})$ has index 6 in $\mathcal{L}_{\mathcal{P}}$ and is simple. Moreover, $N_{E_1}(E_3)$ is the unique proper non-trivial strongly closed subgroup in every saturated subsystem $\mathcal{F}$ of $\mathcal{L}_{\mathcal{P}}$ which contains $O^{5'}(\mathcal{L}_{\mathcal{P}})$.*

Proof. It is immediate from Lemma 3·12 and Proposition 6·27 that $O^{5'}(\mathcal{D}^*)_{\mathcal{P}}$ has index 2 in $\mathcal{L}_{\mathcal{P}}$ and $O^{5'}(\mathcal{L}_{\mathcal{P}})$ has index prime to 5 in $O^{5'}(\mathcal{D}^*)_{\mathcal{P}}$. Hence, $O^{5'}(\mathcal{L}_{\mathcal{P}}) = O^{5'}(O^{5'}(\mathcal{D}^*)_{\mathcal{P}})$ and for the first part of the lemma, it suffices to prove that $O^{5'}(\mathcal{L}_{\mathcal{P}})$ has index 3 in $O^{5'}(\mathcal{D}^*)_{\mathcal{P}}$. Note that $O^{5'}(\mathrm{Aut}_{O^{5'}(\mathcal{D}^*)_{\mathcal{P}}}(E_3)) \cong \mathrm{SL}_2(25)$ and that, as in Proposition 6·27 we can select a cyclic subgroup of order 24 labeled K which lifts to a subgroup $\widehat{K}$ of $\mathrm{Aut}_{O^{5'}(\mathcal{D}^*)_{\mathcal{P}}}(E_1)$ and $\widehat{K}\mathrm{Inn}(E_1) \trianglelefteq \mathrm{Aut}_{O^{5'}(\mathcal{D}^*)_{\mathcal{P}}}(E_1)$. For R a $O^{5'}(\mathcal{D}^*)_{\mathcal{P}}$-centric subgroup, we have that either $\mathrm{Aut}_{E_1}(R) \trianglelefteq \mathrm{Aut}_{O^{5'}(\mathcal{D}^*)_{\mathcal{P}}}(R)$ or that R is $O^{5'}(\mathcal{D}^*)_{\mathcal{P}}$-conjugate to E_3. Then as $\widehat{K}/\mathrm{Inn}(E_1) \trianglelefteq$

$\mathrm{Aut}_{O^{5'}(\mathcal{D}^*)_{\mathcal{P}}}(E_1)$, it follows that $\widehat{K}\mathrm{Inn}(E_1) = \mathrm{Aut}^0_{O^{5'}(\mathcal{D}^*)_{\mathcal{P}}}(E_1)$ and $O^{5'}(\mathcal{L}_{\mathcal{P}})$ has index 6 in $\mathcal{L}_{\mathcal{P}}$.

Let $\mathcal{F}$ be a saturated subsystem $\mathcal{F}$ of $\mathcal{L}_{\mathcal{P}}$ which contains $O^{5'}(\mathcal{L}_{\mathcal{P}})$. Then $O^{5'}(\mathrm{Aut}_{\mathcal{F}}(E_3)) \cong \mathrm{SL}_2(25)$ acts irreducibly on E_3. Hence, if P is a non-trivial strongly closed subgroup of $\mathcal{F}$, then since $P \cap Z(E_1) \neq \{1\}$, we infer that $E_3 \leq P$. Indeed, $N_{E_1}(E_3) = \langle E_3^{E_1} \rangle \leq P$. Note that $\mathrm{Aut}_{\mathcal{F}}(E_1) = N_{\mathrm{Aut}_{\mathcal{F}}(E_1)}(E_3)\mathrm{Inn}(E_1)$ and so $N_{E_1}(E_3)$ contains all essential subgroups of $\mathcal{F}$ and is normalised by $\mathrm{Aut}_{\mathcal{F}}(E_1)$. Hence, $N_{E_1}(E_3)$ is strongly closed in $\mathcal{F}$ and as $\mathrm{Aut}_{\mathcal{F}}(E_1)$ acts irreducibly on $E_1/N_{E_1}(E_3)$, $N_{E_1}(E_3)$ is the unique proper non-trivial strongly closed subgroup of $\mathcal{F}$.

Let $\mathcal{N}$ be a proper non-trivial normal subsystem of $O^{5'}(\mathcal{L}_{\mathcal{P}})$. Then by Lemma 3·12, we may assume that $\mathcal{N}$ is supported on $N_{E_1}(E_3)$. We then repeat parts of the proof of Proposition 6·28 with $O^{5'}(\mathcal{L}_{\mathcal{P}})$ in place of $O^{5'}(\mathcal{D}^*)$ to see that $E_3 \trianglelefteq E_1$, a contradiction. Hence, $O^{5'}(\mathcal{L}_{\mathcal{P}})$ is simple, completing the proof.

PROPOSITION 6·32. *Up to isomorphism, there are two proper saturated subsystems of $\mathcal{L}_{\mathcal{P}}$ which properly contain $O^{5'}(\mathcal{L}_{\mathcal{P}})$, one of which has index 3 while the other, $O^{5'}(\mathcal{D}^*)_{\mathcal{P}}$, has index 2. Furthermore, every saturated subsystem $\mathcal{F}$ of $\mathcal{L}_{\mathcal{P}}$ which contains $O^{5'}(\mathcal{L}_{\mathcal{P}})$ is an exotic fusion system, and satisfies $\mathcal{F}^{frc} = \{E_3^{O^{5'}(\mathcal{D}^*)}, E_1\}$.*

Proof. As in Proposition 6·29, applying [**5**, theorem I·7·7], we enumerate proper subgroups of $\mathrm{Out}_{\mathcal{L}_{\mathcal{P}}}(E_1)$ which properly contain $\mathrm{Out}_{O^{5'}(\mathcal{L}_{\mathcal{P}})}(E_1)$, noting that this corresponds to calculating subgroups of Sym(3). Thus, there is a unique subsystem of index 2 and three systems of index 3 which, since they are all conjugate under an automorphism of E_1, are pairwise isomorphic. Since $O^{5'}(\mathcal{D}^*)_{\mathcal{P}}$ has index 2 in $\mathcal{L}_{\mathcal{P}}$, we have verified the first part of the proposition. We now let $\mathcal{F}$ be a fusion subsystem of $\mathcal{L}_{\mathcal{P}}$ of index prime to 5.

Assume that R in $\mathcal{F}^{frc}$ but not equal to E_1. Applying Lemma 3·6, since R is $\mathcal{F}$-radical, an $\mathcal{F}$-conjugate of R is contained in at least one $\mathcal{F}$-essential subgroup. But then R is contained in a $\mathcal{L}$-conjugate of E_3. Since E_3 is elementary abelian and R is $\mathcal{F}$-centric, we must have that R is $\mathcal{L}$-conjugate to E_3, as required.

Assume that there is $\mathcal{N}$, a proper non-trivial normal subsystem of $\mathcal{F}$. Applying [**5**, theorem II·9·1] and using that $O^{5'}(\mathcal{F}) = O^{5'}(\mathcal{L}_{\mathcal{P}})$ is simple, we deduce that $O^{5'}(\mathcal{L}_{\mathcal{P}}) \leq \mathcal{N}$ and so $N_{E_1}(E_3)$ supports no normal subsystem of $\mathcal{F}$. Hence, applying Theorem 3·14, we see that $\mathcal{F}$ is exotic.

We now determine all fusion systems supported on E_1 up to isomorphism. We begin with the following general lemmas.

LEMMA 6·33. *Suppose that $\mathcal{F}$ is saturated fusion system on E_1 with $P \in \{E_3^{\mathcal{G}}\} \cap \mathcal{E}(\mathcal{F}) \neq \emptyset$. Then $O_5(\mathcal{F}) = \{1\}$, $O^{5'}(\mathrm{Aut}_{\mathcal{F}}(P)) \cong \mathrm{SL}_2(25)$ and P is natural module for $O^{5'}(\mathrm{Aut}_{\mathcal{F}}(P))$.*

Proof. Let $P \in \{E_3^{\mathcal{G}}\} \cap \mathcal{E}(\mathcal{F})$. The proof that $O^{5'}(\mathrm{Aut}_{\mathcal{F}}(P)) \cong \mathrm{SL}_2(25)$ and the verification of the action on P is the same as Lemma 6·5. Then by Proposition 3·10, $O_5(\mathcal{F})$ is an $\mathrm{Aut}_{\mathcal{F}}(P)$-invariant subgroup of P which is also normal in E_1, so that $O_5(\mathcal{F}) = \{1\}$.

LEMMA 6·34. *Suppose that $\mathcal{F}$ is saturated fusion system on E_1 with $E_3^{\mathcal{G}} \cap \mathcal{E}(\mathcal{F}) \neq \emptyset$. Then $\mathrm{Out}_{\mathcal{F}}(E_1)$ is $\mathrm{Aut}(E_1)$-conjugate to a subgroup of $\mathrm{Out}_{\mathcal{D}^*}(E_1)$.*

Proof. Let $P \in \{E_3^{\mathcal{G}}\} \cap \mathcal{E}(\mathcal{F})$. By Lemma 3·6 and Lemma 6·33, we may lift a cyclic subgroup of order 24 from $N_{O^{5'}(\mathrm{Aut}_{\mathcal{F}}(P))}(\mathrm{Aut}_{E_1}(P))$ to $\mathrm{Aut}_{\mathcal{F}}(E_1)$. This subgroup acts faithfully on $Z(E_1)$ and so injects into $\mathrm{Aut}(E_1)/C_{\mathrm{Aut}(E_1)}(Z(E_1)) \cong \mathrm{GL}_2(5)$. Since $\mathrm{Aut}_{\mathcal{F}}(E_1)/C_{\mathrm{Aut}_{\mathcal{F}}(E_1)}(Z(E_1))$ is a $5'$-group containing a cyclic subgroup of order 24, we deduce that $\mathrm{Aut}_{\mathcal{F}}(E_1)/C_{\mathrm{Aut}_{\mathcal{F}}(E_1)}(Z(E_1))$ has order at most 48 and contains a cyclic subgroup of order 24 of index at most 2.

Write $N := N_{\mathrm{Aut}(E_1)}(\mathrm{Aut}_{\mathcal{F}}(E_1))C_{\mathrm{Aut}(E_1)}(Z(E_1))$ so that $N/C_{\mathrm{Aut}(E_1)}(Z(E_1))$ has order 48, and N contains $\mathrm{Aut}_{\mathcal{F}}(E_1)$. Since $|\mathrm{Aut}_{\mathcal{G}}(E_1)|_{5'} = |\mathrm{Aut}(E_1)|_{5'}$ (see [**44**, appendix A], we have that $|C_{\mathrm{Aut}(E_1)}(Z(E_1))|_{5'} = 6$. In particular, $C_{\mathrm{Aut}(E_1)}(Z(E_1))$ is solvable and we conclude that N is solvable.

Since $\mathrm{Aut}_{\mathcal{D}^*}(E_1)C_{\mathrm{Aut}(E_1)}(Z(E_1))/C_{\mathrm{Aut}(E_1)}(Z(E_1))$ has order 48 (and $\mathrm{GL}_2(5)$ has a unique conjugacy class of groups of order 48 with a cyclic subgroup of index 2), we deduce that $\mathrm{Aut}_{\mathcal{D}^*}(E_1)$ is $\mathrm{Aut}(E_1)$-conjugate to a subgroup of N. Hence, $\mathrm{Out}_{\mathcal{D}^*}(E_1)$ is $\mathrm{Out}(E_1)$-conjugate to a subgroup of $N/\mathrm{Inn}(E_1)$. But $|N/\mathrm{Inn}(E_1)|_{5'} = 2^5.3^2 = |\mathrm{Out}_{\mathcal{D}^*}(E_1)|$ and so $\mathrm{Out}_{\mathcal{D}^*}(E_1)$ is $\mathrm{Out}(E_1)$-conjugate to a Hall $5'$-subgroup of $N/\mathrm{Inn}(E_1)$. Since $\mathrm{Out}_{\mathcal{F}}(E_1)$ is a $5'$-group, $\mathrm{Out}_{\mathcal{F}}(E)$ lies in a Hall $5'$-subgroup of $N/\mathrm{Inn}(E_1)$ and we deduce that $\mathrm{Out}_{\mathcal{F}}(E_1)$ is $\mathrm{Out}(E_1)$-conjugate to a subgroup of $\mathrm{Out}_{\mathcal{D}^*}(E_1)$.

LEMMA 6·35. *There is a unique conjugacy class of cyclic subgroups of order 24 in $\mathrm{Out}_{\mathcal{D}^*}(E_1)$ whose Sylow 3-subgroups act non-trivially on $Z(E_1)$ and $\Phi(E_1)/Z(E_1)$. This class contains two subgroups.*

Proof. We note that the Sylow 3-subgroups of $C_{\mathrm{Out}_{\mathcal{D}^*}(E_1)}(Z(E_1))$ and of $C_{\mathrm{Out}_{\mathcal{D}^*}(E_1)}(\Phi(E_1)/Z(E_1))$ are normal in $\mathrm{Out}_{\mathcal{D}^*}(E_1)$. Indeed, these are the unique subgroups of order 3 which are normal in $\mathrm{Out}_{\mathcal{D}^*}(E_1)$. The rest of the calculation is performed computationally (see [**44**, appendix A]).

The next result is computed in MAGMA (see [**44**, appendix A]).

PROPOSITION 6·36. *Let $\mathcal{F}$ be a saturated fusion system supported on E_1. Then $\mathcal{E}(\mathcal{F}) \subseteq \{E_3^{\mathcal{D}^*}\}$.*

Again, we provide some explanation for this without formal proof. The MAGMA calculation performed, as documented in [**44**, appendix A], and the existence of $\mathcal{D}^*$ yields that $\mathcal{E}(\mathcal{F}) \subseteq \{E_3^{\mathrm{Aut}(E_1)}\} = \{E_3^{\mathrm{Aut}_{\mathcal{D}}(E_1)}\}$. By Lemma 6·34, and as we are only interested in classifying fusion systems up to isomorphism, we arrange that $\mathrm{Aut}_{\mathcal{F}}(E_1)$ is contained in $\mathrm{Aut}_{\mathcal{D}^*}(E_1)$.

Let $P_1, P_2 \in \mathcal{E}(\mathcal{F})$ so that P_1 and P_2 are $\mathcal{D}$-conjugate to E_3. Further, suppose that P_1 and P_2 are not $\mathcal{D}^*$-conjugate. Writing K_{P_i} for the lift to $\mathrm{Aut}_{\mathcal{F}}(E_1)$ of $N_{O^{5'}(\mathrm{Aut}_{\mathcal{F}}(P_i))}(\mathrm{Aut}_{E_1}(P_i))$, we see that

$$K_{P_i}C_{\mathrm{Aut}_{\mathcal{D}}(E_1)}(Z(E_1)) \le N_{\mathrm{Aut}_{\mathcal{D}}(E_1)}(\{P_i^{\mathcal{F}}\}) \le \mathrm{Aut}_{\mathcal{D}^*}(E_1)$$

for $i \in \{1, 2\}$. In particular, we see that $K_{P_1}C_{\mathrm{Aut}_{\mathcal{D}}(E_1)}(Z(E_1)) = K_{P_2}C_{\mathrm{Aut}_{\mathcal{D}}(E_1)}(Z(E_1))$. Let $\alpha \in \mathrm{Aut}_{\mathcal{D}}(E_1) \setminus \mathrm{Aut}_{\mathcal{D}^*}(E_1)$ such that $P_1\alpha = P_2$. Then $N_{\mathrm{Aut}_{\mathcal{D}}(E_1)}(\{P_1^{\mathcal{F}}\})\alpha = N_{\mathrm{Aut}_{\mathcal{D}}(E_1)}(\{P_2^{\mathcal{F}}\})$. Hence, either $N_{\mathrm{Aut}_{\mathcal{D}}(E_1)}(\{P_1^{\mathcal{F}}\}) = \mathrm{Aut}_{\mathcal{D}^*}(E_1)$ and α normalises $\mathrm{Aut}_{\mathcal{D}^*}(E_1)$ or $N_{\mathrm{Aut}_{\mathcal{D}}(E_1)}(\{P_1^{\mathcal{F}}\}) = K_{P_1}C_{\mathrm{Aut}_{\mathcal{D}}(E_1)}(Z(E_1))$ and α normalises $K_{P_1}C_{\mathrm{Aut}_{\mathcal{D}}(E_1)}(Z(E_1))$. Either way, we have that $\alpha \in \mathrm{Aut}_{\mathcal{D}^*}(E_1)$, a contradiction.

Hence, $\mathcal{E}(\mathcal{F}) \subseteq \{P^{\mathcal{D}^*}\}$ where P is some $\mathcal{D}$-conjugate of E_3. It remains to show that P and E_3 are $\mathcal{D}^*$-conjugate. Assume for a contradiction that this is not the case. We may lift a cyclic subgroup of order 24 from $N_{O^{5'}(\mathrm{Aut}_{\mathcal{F}}(P))}(\mathrm{Aut}_{E_1}(P))$ to $\mathrm{Aut}_{\mathcal{F}}(E_1)$, and denote it K_P.

Then, by Lemma 6·35, K_P is $\mathcal{D}^*$ conjugate the cyclic subgroup of order 24 which is induced by lifted morphisms from $N_{O^{5'}(\mathrm{Aut}_{\mathcal{F}}(E_3))}(\mathrm{Aut}_{E_1}(E_3))$. Since $\mathcal{E}(\mathcal{F}) \subseteq \{P^{\mathcal{D}^*}\}$, we may as well assume that these groups are equal. Hence, we may apply Proposition 3·9 to $\mathcal{F}$, with $V = E_3$ and $\Delta = O^{5'}(\mathrm{Aut}_{\mathcal{D}^*}(E_3))$. It easy to see that we verify the hypothesis there, and so we may construct a saturated fusion system on E_1 in which both E_3 and P are essential. But by the above, this is a contradiction and we see that $\mathcal{E}(\mathcal{F}) \subseteq \{E_3^{\mathcal{D}^*}\}$.

THEOREM E. *Suppose that $\mathcal{F}$ is saturated fusion system on E_1 such that $E_1 \ntrianglelefteq \mathcal{F}$. Then $\mathcal{F}$ is either isomorphic to a subsystem of $\mathcal{D}^*$ of $5'$-index, of which there are five, or isomorphic to a subsystem of $\mathcal{L}_{\mathcal{P}}$ of $5'$-index, of which there are four.*

Proof. Since we are only interested in determining $\mathcal{F}$ up to isomorphism, and as $E_1 \ntrianglelefteq \mathcal{F}$, applying Proposition 6·36, we have that $E_3 \in \mathcal{E}(\mathcal{F}) \subseteq \{E_3^{\mathcal{D}^*}\}$. Note that Lemma 6·23 holds upon replacing E_3 by any $\mathcal{D}^*$ conjugate of E_3 and so $\mathcal{E}(\mathcal{F})$ and $N_{\mathcal{F}}(E_1)$ determine $\mathcal{F}$ completely, by the Alperin–Goldschmidt theorem. By Lemma 6·34 we arrange that $\mathrm{Aut}_{\mathcal{F}}(E_1)$ is a subgroup of $\mathrm{Aut}_{\mathcal{D}^*}(E_1)$. Let $P \in \{E_3^{\mathcal{D}^*}\}$ with P not conjugate to E_3 by any element of E_1.

Let K be a Hall $5'$-subgroup of $N_{O^{5'}(\mathrm{Aut}_{\mathcal{F}}(E_3))}(\mathrm{Aut}_{E_1}(E_3))$ so that K is cyclic of order 24. We note that a Sylow 3-subgroup acts non-trivially on $Z(E_1)$ and $\mathrm{Aut}_{E_1}(E_3) \cong \Phi(E_1)/Z(E_1)$. By Lemma 3·6, we let $\widehat{K}$ be the lift of K to $\mathrm{Aut}_{\mathcal{F}}(E_1)$. Then, by Lemma 6·35, in $\mathrm{Out}_{\mathcal{D}^*}(E_1)$ there is a unique conjugacy class of cyclic subgroups of order 24 whose Sylow 3-subgroup is not contained in $C_{\mathrm{Out}_{\mathcal{D}^*(E_1)}}(Z(E_1))$ or $C_{\mathrm{Out}_{\mathcal{D}^*(E_1)}}(\Phi(E_1)/Z(E_1))$. Indeed, $\widehat{K}\mathrm{Inn}(E_1)/\mathrm{Inn}(E_1)$ belongs in this class and again by Lemma 6·35, we have two candidates for $\widehat{K}\mathrm{Inn}(E_1)$ in $\mathrm{Aut}_{\mathcal{D}^*}(E_1)$ (one coming from the lift of automorphisms of E_3 and one coming from the lift of automorphisms of P).

We enumerate the possible overgroups of $\widehat{K}\mathrm{Inn}(E_1)/\mathrm{Inn}(E_1) \cong \mathrm{C}_{24}$ in $\mathrm{Out}_{\mathcal{D}^*}(E_1) \cong \mathrm{Sym}(3) \times \mathrm{C}_{24} : 2$. These are the groups of shape

$$\mathrm{C}_{24},\ \mathrm{C}_{24}:2,\ 3 \times \mathrm{C}_{24},\ (3 \times \mathrm{C}_{24}):2,\ \mathrm{Sym}(3) \times \mathrm{C}_{24},\ 3 \times (\mathrm{C}_{24}:2) \text{ and } \mathrm{Sym}(3) \times (\mathrm{C}_{24}:2).$$

Note that there are three subgroups of shape $\mathrm{C}_{24}:2$, all conjugate, and every other group is unique. Finally, we note that $\mathrm{Out}_{O^{5'}(\mathcal{L}_{\mathcal{P}})}(E_1) \cong \mathrm{C}_{24}$, $\mathrm{Out}_{O^{5'}(\mathcal{D}^*)}(E_1) \cong 3 \times \mathrm{C}_{24}$ and $\mathrm{Out}_{\mathcal{L}}(E_1) \cong (3 \times \mathrm{C}_{24}):2$.

Suppose first that $P \notin \mathcal{E}(\mathcal{F})$. Hence, $\mathcal{E}(\mathcal{F}) = \{E_3^{E_1}\}$ and so $\mathrm{Aut}_{\mathcal{F}}(E_1) \le N_{\mathrm{Aut}_{\mathcal{F}}(E_1)}(E_3)\mathrm{Inn}(E_1)$. In particular, $\mathrm{Out}_{\mathcal{F}}(E_1) \le \mathrm{Out}_{\mathcal{L}}(E_1) \cong (3 \times \mathrm{C}_{24}):2$. There are four choices for $\mathrm{Out}_{\mathcal{F}}(E_1)$ up to conjugacy, and so there are four choices for $\mathrm{Aut}_{\mathcal{F}}(E_1)$ and these choices correspond exactly with $\mathrm{Aut}_{\mathcal{Y}}(E_1)$ where $\mathcal{Y}$ is a subsystem of $5'$-index in $\mathcal{L}_{\mathcal{P}}$ described in Proposition 6·31 and Proposition 6·32. By the Alperin–Goldschmidt theorem, there is $\alpha \in \mathrm{Aut}(E_1)$ such that $N_{\mathcal{F}^\alpha}(E_1) = N_{\mathcal{F}}(E_1)^\alpha = N_{\mathcal{Y}}(E_1)$. If $\mathcal{E}(\mathcal{F}^\alpha) = \{E_3^{E_1}\}$ then we have that $\mathcal{F}^\alpha = \mathcal{Y}$ by an earlier observation so that $\mathcal{F} \cong \mathcal{Y}$. Hence, we have that $P \in \mathcal{E}(\mathcal{F}^\alpha)$. Then there is $\beta \in \mathrm{Aut}_{\mathcal{D}^*}(E_1)$ such that $P\beta = E_3$ and $N_{\mathcal{Y}}(E_1)^\beta = N_{\mathcal{Y}}(E_1)$. Hence, by an earlier observation using the Alperin–Goldschmidt theorem, we have that $\mathcal{F}^{\alpha\beta} = \mathcal{Y}$ and so $\mathcal{F} \cong \mathcal{Y}$.

Therefore, we may assume that $\mathcal{E}(\mathcal{F}) = \{E_3^{\mathcal{D}^*}\}$. Let $\beta \in \mathrm{Aut}_{\mathcal{D}^*}(E_1)$ with $E_3\beta = P$. Then $\beta \notin \mathrm{Aut}_{\mathcal{L}}(E_1)$, $\widehat{K}\mathrm{Inn}(E_1) \neq \widehat{K}\beta\mathrm{Inn}(E_1)$ and $\widehat{K}\beta|_P \le O^{5'}(\mathrm{Aut}_{\mathcal{F}}(P))$. In particular, by Lemma 6·33, we have that $\langle \widehat{K}, \widehat{K}\beta \rangle \le \mathrm{Aut}_{\mathcal{F}}(E_1)$ and we infer that $\mathrm{Out}_{\mathcal{F}}(E_1)$ is an overgroup of $\mathrm{Out}_{O^{5'}(\mathcal{D}^*)}(E_1) \cong 3 \times \mathrm{C}_{24}$ Thus, there are five choices for $\mathrm{Out}_{\mathcal{F}}(E_1)$ up to conjugacy, and so there are five choices for $\mathrm{Aut}_{\mathcal{F}}(E_1)$ and these choices correspond exactly with $\mathrm{Aut}_{\mathcal{Y}}(E_1)$

where $\mathcal{Y}$ is a subsystem of $5'$-index in $\mathcal{D}^*$ described in and Proposition 6·29. By the Alperin–Goldschmidt theorem, there is $\alpha \in \mathrm{Aut}(S)$ such that $N_{\mathcal{F}^\alpha}(E_1) = N_{\mathcal{F}}(E_1)^\alpha = N_{\mathcal{Y}}(E_1)$. Since $\mathcal{E}(\mathcal{F}^\alpha) = \mathcal{E}(\mathcal{Y}) = \{E_3^{\mathcal{D}^*}\}$, by an earlier observation we have that $\mathcal{F}^\alpha = \mathcal{Y}$ so that $\mathcal{F} \cong \mathcal{Y}$.

We provide the following tables summarising the actions induced by the fusion systems described in Theorem E on their centric-radical subgroups. Table 6 and Table 7 treat those subsystems of $\mathcal{D}$ which are not "pruned", while Table 8 and Table 9 deals with the remainder. The entry "-" indicates that the subgroup is no longer centric-radical in the subsystem, and an entry decorated with "†" specifies that there are two conjugacy classes of E_3 in this subsystem which are fused upon enlarging to $\mathcal{D}$.

Table 6. *$\mathcal{D}$-conjugacy classes of radical-centric subgroups of E_1*

P	$\lvert P\rvert$	$\mathrm{Out}_{\mathcal{D}^*}(P)$	$\mathrm{Out}_{O^{5'}(\mathcal{D}^*).2_1}(P)$	$\mathrm{Out}_{O^{5'}(\mathcal{D}^*).2_2}(P)$
E_1	5^8	$\mathrm{Sym}(3) \times (\mathrm{C}_{24} : 2)$	$\mathrm{Sym}(3) \times \mathrm{C}_{24}$	$3 \times (\mathrm{C}_{24} : 2)$
E_3	5^4	$(3 \times \mathrm{SL}_2(25)).2$	$3 \times \mathrm{SL}_2(25)$	$3 \times \mathrm{SL}_2(25)$

Table 7. *$\mathcal{D}$-conjugacy classes of radical-centric subgroups of E_1*

P	$\lvert P\rvert$	$\mathrm{Out}_{\mathcal{L}}(P)$	$\mathrm{Out}_{O^{5'}(\mathcal{D}^*)}(P)$
E_1	5^8	$(3 \times \mathrm{C}_{24}) : 2$	$3 \times \mathrm{C}_{24}$
E_3	5^4	$(3 \times \mathrm{SL}_2(25)).2^\dagger$	$3 \times \mathrm{SL}_2(25)^\dagger$

Table 8. *$\mathcal{L}$-conjugacy classes of radical-centric subgroups of E_1*

P	$\lvert P\rvert$	$\mathrm{Out}_{\mathcal{L}_{\mathcal{P}}}(P)$	$\mathrm{Out}_{O^{5'}(\mathcal{D}^*)_{\mathcal{P}}}(P)$
E_1	5^8	$(3 \times \mathrm{C}_{24}) : 2$	$3 \times \mathrm{C}_{24}$
E_3	5^4	$(3 \times \mathrm{SL}_2(25)).2$	$3 \times \mathrm{SL}_2(25)$

Table 9. *$\mathcal{L}$-conjugacy classes of radical-centric subgroups of E_1*

P	$\lvert P\rvert$	$\mathrm{Out}_{O^{5'}(\mathcal{L}_{\mathcal{P}}).2}(P)$	$\mathrm{Out}_{O^{5'}(\mathcal{L}_{\mathcal{P}})}(P)$
E_1	5^8	$\mathrm{C}_{24} : 2$	C_{24}
E_3	5^4	$\mathrm{SL}_2(25).2$	$\mathrm{SL}_2(25)$

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