Moduli **1**, e5 (2024) 1–19
doi:10.1112/mod.2024.8

KSB stability is automatic in codimension ≥ 3

János Kollár and Sándor J. Kovács

Abstract

KSB stability holds at codimension 1 points trivially, and it is quite well understood at codimension 2 points because we have a complete classification of 2-dimensional slc singularities. We show that it is automatic in codimension 3.

Contents

1. Introduction

The right framework for a moduli theory of canonical models of varieties of general type was established in [KSB88], at least in characteristic 0 and over Noetherian bases, both of which we assume from now on. The resulting notion, now called *KSB stability*, works with finitely presented, flat morphisms $g: X \to B$ that satisfy three requirements.

- (Global condition) $\omega_{X/B}$ is relatively ample, and g is projective,
- (Fiberwise condition) the fibers X_b are semi-log-canonical, and
- (Local stability condition) $\omega_{X/B}^{[m]}$ is flat over B and commutes with base changes $B' \to B$ for every $m \in \mathbb{Z}$.

Received 11 July 2023, accepted 5 June 2024.
2020 Mathematics Subject Classification 14J10 (Primary), 14D20, 14J17 (Secondary)
Keywords: KSB stability, slc singularities, Du Bois singularities, liftable local cohomology.

If g satisfies the last two, then it is called *locally KSB stable*. See [Kol23] for a detailed discussion of the resulting moduli theory, especially [Kol23, Sec. 6.2].

Note that the local stability condition is automatic at codimension 1 point and is quite well understood at codimension 2 points because we have a complete classification of 2-dimensional slc singularities; see [KSB88] and [Kol23, Sec. 2.2]. Our aim is to show that local stability is automatic in codimension ≥ 3. The simplest version is the following.

THEOREM 1.1. *Let $g: X \to B$ be a flat morphism of finite type over a field of characteristic 0. Let $Z \subset X$ be a closed subset such that* $\operatorname{codim}(Z_b \subset X_b) \geq 3$ *for every $b \in B$, and set $U := X \setminus Z$.*

Assume that the fibers X_b are semi-log-canonical and that $g|_U : U \to B$ is locally KSB stable. Then $g: X \to B$ is locally KSB stable.

If the fibers X_b are CM, the claim follows from [Kol23, 10.73]. Being CM is a deformation invariant property for projective, locally stable families by [KK10]; see also [Kol23, 2.67]. In particular, the theorem was known to hold for varieties in those connected components of the KSB moduli space that contain a canonical model of a smooth variety.

If B is reduced, the theorem is proved in [Kol13a]; see also [Kol23, 5.6]. Thus it remains to deal with the case when $B = \operatorname{Spec} A$ for an Artinian ring A, which implies the theorem for any B.

For applications, and even for the proof of Theorem 1.1, we need a form that strengthens it in two significant ways. First, we deal with pairs $(X, \Delta = \sum a_i D_i)$, where $a_i \in \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, 1\right\}$ for every i; these are frequently called *standard* coefficients. Second, and this is more important, we assume g to be flat only in codimension ≤ 2.

THEOREM 1.2. *Let $g: X \to B$ be a morphism of finite type and of pure relative dimension over a field of characteristic 0, and let $\Delta = \sum a_i D_i$, where the D_i are relative Mumford $\mathbb{Z}$-divisors. Let $Z \subset X$ be a closed subset and set $U := X \setminus Z$. Assume that*

(1.2.1) $a_i \in \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, 1\right\}$ *for every i,*

(1.2.2) $\operatorname{codim}(Z_b \subset X_b) \geq 3$ *for every $b \in B$,*

(1.2.3) $g|_U : U \to B$ *is flat and the fibers $(U_b, \Delta|_{U_b})$ are semi-log-canonical,*

(1.2.4) $\omega_{U/B}^{[m]}(\sum_i \lfloor m a_i \rfloor D_i|_U)$ *is flat over B and commutes with base changes for every $m \in \mathbb{Z}$,*

(1.2.5) $\operatorname{depth}_Z X \geq 2$ *and*

(1.2.6) *the normalization $(\overline{X}_b, \overline{C}_b + \overline{\Delta}_b) \to X_b$ is log canonical for every $b \in B$, where $\overline{C}_b$ denotes the conductor of the normalization $\overline{X}_b \to X_b$; see* [Kol13b, 5.2].

Then

(1.2.7) $g: X \to B$ *is flat,*

(1.2.8) *the fibers (X_b, Δ_b) are semi-log-canonical and*

(1.2.9) $\omega_{X/B}^{[m]}(\sum_i \lfloor m a_i \rfloor D_i)$ *is flat over B and commutes with base changes for every $m \in \mathbb{Z}$.*

Remark 1.3.

(1.3.1) As in [Kol23, 4.68], D is a *relative Mumford divisor* if at every generic point of $X_b \cap D$, the fiber X_b is smooth and D is Cartier.

https://doi.org/10.1112/mod.2024.8 Published online by Cambridge University Press

(1.3.2) The notation $\operatorname{depth}_Z X$ stands for $\operatorname{depth}_Z \mathscr{O}_X := \inf\{\operatorname{depth}_z \mathscr{O}_X \mid z \in Z\}$. This terminology is used, for instance, in [EGA-IV/2, (5.10.1)] and [Kol23, 10.3].

(1.3.3) The condition (1.2.5) is easy to ensure by replacing $\mathscr{O}_X$ with the push-forward of $\mathscr{O}_U$ if necessary. If B is S_2 then (1.2.5) holds iff X is S_2.

(1.3.4) Assumption (1.2.6) is a weakening of the fiberwise condition; the two are equivalent iff X_b is S_2. In many applications, including the proof of Theorem 1.2, at the beginning we know only (1.2.6), but we eventually conclude that (X_b, Δ_b) is slc.

(1.3.5) The following may be a better way of formulating (1.2.6). Let $j : U \hookrightarrow X$ be the natural embedding, and set $\widetilde{X}_b := \operatorname{Spec}_{X_b} j_* \mathscr{O}_{U_b}$, which is the *demi-normalization* and also the *S_2-hull* of the fiber X_b; see [Kol13b, Sec. 5.1] and [Kol23, Sec. 9.1]. Then $\widetilde{X}_b \to X_b$ is a universal homeomorphism that is an isomorphism over U_b. Now (1.2.6) holds iff the induced pair $(\widetilde{X}_b, \widetilde{\Delta}_b)$ is slc.

(1.3.6) If $a_i \in \{\frac{2}{3}, \frac{3}{4}, \dots\}$, then (1.2.4) is the same as the main assumption of KSB stabilty, with standard coefficients as defined in [Kol23, 6.21.3].

If we allow $a_i = \frac{1}{2}$, then the above definition treats the pairs (X, D), $(X, \frac{1}{2}D + \frac{1}{2}D)$ and $(X, \frac{1}{2}(2D))$ as different objects. Note that $\omega_X(\sum \lfloor a_i \rfloor D_i)$ is $\omega_X(D)$ in the first case but is ω_X in the other two cases. Thus, replacing $1 \cdot D_i$ with $\frac{1}{2}D_i + \frac{1}{2}D_i$ ensures the extra condition on the $\{D_i : a_i = 1\}$ in [Kol23, 6.22.3].

This way of handling the coefficient $\frac{1}{2}$ case may not be natural from the point of view of moduli, but it seems necessary; see [Kol23, Secs. 8.1–2] for a discussion of the general notion of such *marked pairs*.

(1.3.7) The definition of KSB stabilty with standard coefficients also requires the D_i to be flat by [Kol23, 6.21.1]. We do not know whether this is automatic in codimensions ≥ 3; see Corollary 4.3 for a special case.

(1.3.8) We comment on other versions of stability in Section 5.

§1.4 *Sketch of an approach to* Theorem 1.2. Assume for simplicity that we are over $\mathbb{C}$, $B = \operatorname{Spec} A$ for an Artinian ring A and that the closed fiber X_k is projective. As in [KK10], the proof relies on the Du Bois property (see Remark 1.9) of slc varieties, which implies that the natural maps

$$H^i(X_k^{\mathrm{an}}, \mathbb{C}) \twoheadrightarrow H^i(X_k^{\mathrm{an}}, \mathscr{O}_{X_k^{\mathrm{an}}}) \qquad \text{are surjective.} \tag{1.4.1}$$

If g is also flat, these imply that the $R^i g_* \mathscr{O}_X$ are (locally) free by [DJ74]. Using this for various cyclic covers, [Kol23, 2.68] shows that $\omega_{X/B}$ is flat over B and commutes with base changes $B' \to B$.

An inspection of these proofs shows that, in order to get the flatness of $\omega_{X/B}$, we need (1.4.1) only for $i = n, n-1$, where $n := \dim X_k$. This is where the codimension 3 condition enters first. As we noted in (1.3.5), the demi-normalization $\widetilde{X}_k$ of X_k is slc, and $\widetilde{X}_k \to X_k$ is a universal homeomorphism that is an isomorphism over U_k. Thus

$$H^i(X_k^{\mathrm{an}}, \mathbb{C}) \simeq H^i(\widetilde{X}_k^{\mathrm{an}}, \mathbb{C}) \quad \text{for every } i, \text{ and}$$
$$H^i(X_k^{\mathrm{an}}, \mathscr{O}_{X_k^{\mathrm{an}}}) \simeq H^i(\widetilde{X}_k^{\mathrm{an}}, \mathscr{O}_{\widetilde{X}_k^{\mathrm{an}}}) \quad \text{for } i = n, n-1.$$

https://doi.org/10.1112/mod.2024.8 Published online by Cambridge University Press

It follows that (1.4.1) holds for $i=n,n-1$, although X_k is not (yet known to be) Du Bois; see also Theorem 4.2. One also sees that it is enough if g is flat at points of dimension $\geq n-2$. Therefore we get that $\omega_{X/B}$ is flat over B.

Interestingly, this approach does not seem to imply that X is flat over B, much less the full Theorem 1.2. A possible explanation is that ω_X is insensitive to codimension 2:

LEMMA 1.5. *Let $\pi: Y \to X$ be a quasi-finite morphism that is an isomorphism at points of codimension ≤ 1. Then $\pi_*\omega_Y \simeq \omega_X$.*

Proof. Let $\imath: U \hookrightarrow X$ be the largest open subset such that $\pi' := \pi|_{\pi^{-1}U}$ is an isomorphism between $\pi^{-1}U$ and U. Let $\jmath: \pi^{-1}U \hookrightarrow Y$ denote the embedding. By assumption $\operatorname{codim}(Y \setminus \pi^{-1}U, Y) \geq 2$ and $\operatorname{codim}(X \setminus U, X) \geq 2$. Therefore, because ω_X and ω_Y are S_2-sheaves (cf. [KM98, 5.69]), it follows that

$$\pi_*\omega_Y \simeq \pi_*\jmath_*\omega_{\pi^{-1}U} \simeq \imath_*\pi'_*\omega_{\pi^{-1}U} \simeq \imath_*\omega_U \simeq \omega_X.$$

□

In order to prove Theorem 1.2, we use the techniques of [KK20], and establish the following local, Du Bois version (see Remark 1.9).

THEOREM 1.6. *Let B be a local scheme over a field of characteristic* 0, *and let $f:(X,x) \to B$ be a local morphism that is essentially of finite type. Let X_k be the fiber of f over the closed point of B, let $Z \subseteq X_k$ be a closed subset of codimension ≥ 3, and set $\jmath: U_k := X_k \setminus Z \hookrightarrow X_k$. Assume that*

(1.6.1) *f is flat along U_k, and*

(1.6.2) $\operatorname{Spec} \jmath_*\mathscr{O}_{U_k}$ *is Du Bois.*

Then $\omega_{X/B}$ is flat over B and commutes with arbitrary base change.

Theorem 1.6 will be proved as a combination of Theorem 3.16 and Theorem 4.2.

As before, the method does not seem to imply that X is flat over $\operatorname{Spec} S$, not even if we assume that $\operatorname{depth}_Z X \geq 2$, as in (1.2.5). However, we do not have a counterexample.

Note that, without the Du Bois assumption, such examples are easy to get:

Example 1.7. Let $\{C_i : i \in I\}$ be a finite set of smooth, projective curves. Fix $d_i > 0$ such that $d_i \leq \deg \omega_{C_i}$ for some $i \in I$ and that $d_j > \deg \omega_{C_j}$ for some $j \in I$. Set $Y := \times_i C_i$ and consider a line bundle $L = \boxtimes_i L_i$ on Y, where $\deg L_i = d_i$.

The affine cone over Y with conormal bundle L (cf. [Kol13b, 3.8]) is

$$C_a(Y,L) := \operatorname{Spec}_k \oplus_{m\in\mathbb{Z}} H^0(Y, L^m).$$

By the $i=0$ case of [Kol13b, 3.13.2], its dualizing sheaf is the sheafification of the module

$$\oplus_{m\in\mathbb{Z}} H^0(Y, \omega_Y \otimes L^m).$$

The mth graded pieces are

$$\otimes_{i\in I} H^0(C_i, L_i^m) \quad \text{and} \quad \otimes_{i\in I} H^0(C_i, \omega_{C_i} \otimes L_i^m).$$

Note that if $d_i \leq \deg \omega_{C_i}$ then $h^0(C_i, L_i)$ depends on the choice of L_i, not only on $\deg L_i$.

By contrast, we claim that $h^0(Y, \omega_Y \otimes L^m)$ depends only on the degrees of the L_i and the m. Indeed, if $m \leq -1$ then $\omega_{C_j} \otimes L_j^m$ has negative degree, so $H^0(Y, \omega_Y \otimes L^m) = 0$. If $m = 0$ then

https://doi.org/10.1112/mod.2024.8 Published online by Cambridge University Press

there is no dependence on the L_i, and for $m \geq 1$

$$h^0(C_i, \omega_{C_i} \otimes L_i^m) = m \deg L_i + g(C_i) - 1.$$

Now set $B := \times_i \mathrm{Pic}^{d_i}(C_i)$, and note that $Y \times B \simeq \times_i \left(C_i \times \mathrm{Pic}^{d_i}(C_i)\right)$. Let P_i denote the universal degree d_i line bundle on $C_i \times \mathrm{Pic}^{d_i}(C_i)$, and let $P = \boxtimes P_i$ on $Y \times B$. Further, let $\pi : Y \times B \to B$ be the projection, and consider the universal cone

$$X_B := C_a(Y \times B, P) := \mathrm{Spec}_B \oplus_{m \geq 0} \pi_* P^m$$

over B. As we noted, the $h^0(Y, \omega_Y \otimes P_b^m)$ are independent of $b \in B$, so the dualizing sheaf of X_B is flat over B. However, $h^0(Y, P_b)$ does depend on $b \in B$; thus the structure sheaf is not flat over B. Note that $h^1(Y, P_b)$ also depends on $b \in B$, and when $h^1(Y, P_b) \neq 0$, then $C_a(Y, P_b)$, the normalization of the fiber of X_B over b, is not Du Bois by [GK14, 2.5].

We also prove that KSB stability is automatic in codimension 3 in a different manner, namely, that it is enough to check it on general hyperplane sections.

COROLLARY 1.8. *Let $g : X \to B$ be a quasi-projective morphism of pure relative dimension $n \geq 3$ over a field of characteristic 0, and $\Delta = \sum a_i D_i$, where the D_i are relative Mumford $\mathbb{Z}$-divisors. Assume that*

(1.8.1) $a_i \in \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, 1\right\}$ *for every i,*

(1.8.2) $\mathrm{depth}_x X \geq \min\{2, \mathrm{codim}(x, g^{-1}(g(x))\}$ *for every $x \in X$,*

(1.8.3) *the normalization $(\overline{X}_b, \overline{C}_b + \overline{\Delta}_b) \to X_b$ is log canonical for every $b \in B$ and*

(1.8.4) *general relative surface sections of $(X, \Delta) \to B$ are locally KSB stable.*

Then (1.2.7)–(1.2.9) hold.

Proof. By [Kol23, 9.17] we may assume that B is Artinian. Then the relative pluricanonical sheaves $\omega_{X/B}^{[m]}(\sum_i \lfloor m a_i \rfloor D_i)$ are S_2. This continues to hold after first tensoring with line bundles and then restricting to general surface sections $Y := H_1 \cap \cdots H_{n-2} \subset X$; for the latter, see [Kol23, 10.18]. Thus

$$\omega_{Y/B}^{[m]}(\textstyle\sum_i \lfloor m a_i \rfloor D_i|_Y) \simeq \omega_{X/B}^{[m]}(\sum_j H_j + \sum_i \lfloor m a_i \rfloor D_i)|_Y.$$

Now by [Mat89, p.177] or [Kol23, 10.56], the $\omega_{X/B}^{[m]}(\sum_i \lfloor m a_i \rfloor D_i)$ are flat over B outside a subset of codimesion ≥ 3. Thus they are flat everywhere by Theorem 1.2. Over Artin rings, flat modules are free [StacksProject, Tag 051G], so commuting with base change holds; see also [Kol23, 9.17]. □

Remark 1.9. The precise definition of Du Bois singularities, introduced by Steenbrink [Ste83], is quite involved. It starts with the construction of the Du Bois complex; see [DB81, GNPP88], which has a natural filtration and agrees with the usual de Rham complex if X is nonsingular. For our purposes the important part is the 0^{th} associated graded Du Bois complex of X, which is denoted by $\underline{\Omega}_X^0$. This comes with the natural morphism $\mathscr{O}_X \to \underline{\Omega}_X^0$, and a separated scheme of finite type over $\mathbb{C}$ is said to have Du Bois singularities if this natural morphism is a quasi-isomorphism. For more details on the definition of Du Bois singularities and their relevance to higher dimensional geometry, see [Kol13b, Chap.6].

As we already mentioned in (1.4.1), for a proper complex variety X with Du Bois singularities, the natural morphism

https://doi.org/10.1112/mod.2024.8 Published online by Cambridge University Press

$$H^i(X^{\mathrm{an}}, \mathbb{C}) \twoheadrightarrow H^i(X^{\mathrm{an}}, \mathscr{O}_{X^{\mathrm{an}}}) \tag{1.9.1}$$

is surjective. (At least heuristically, one may think of Du Bois singularities as the largest class for which this holds, cf. [Kov12].)

The surjectivity in (1.9.1) enables one to use topological arguments to control the sheaf cohomology groups $H^i(X, \mathscr{O}_X)$. It is a key element of Kodaira-type vanishing theorems [Kol87, Kol95, Sec. 12, Kov00, KSS10] and leads to various results on deformations of Du Bois schemes [DJ74, KK10, KS16b].

The obvious candidate for a local analog of (1.4.1) is the map on local cohomologies

$$H^i_x(X^{\mathrm{an}}, \mathbb{C}) \to H^i_x(X^{\mathrm{an}}, \mathscr{O}_{X^{\mathrm{an}}}). \tag{1.9.2}$$

However, this map is never surjective for $i = \dim X$. In fact, if X is smooth of dimension $n \geq 2$, then $H^n_x(X^{\mathrm{an}}, \mathbb{C})$ is trivial, but $H^n_x(X^{\mathrm{an}}, \mathscr{O}_{X^{\mathrm{an}}})$ is infinite dimensional.

To get the right notion, one should look at the natural morphisms

$$\mathbb{C}_{X^{\mathrm{an}}} \xrightarrow{\sigma} \mathscr{O}_{X^{\mathrm{an}}} \xrightarrow{\varrho} \underline{\Omega}^0_{X^{\mathrm{an}}} \tag{1.9.3}$$

The general theory implies that the composition $\varrho \circ \sigma$ induces surjectivity on (hyper)cohomology for any proper X. If X has Du Bois singularities, then ϱ is a quasi-isomorphism, and the surjectivity in (1.9.1) follows.

Note that ϱ may be represented by a map between coherent sheaves; thus it is possible to work with ϱ entirely algebraically. Eventually, this suggests that the correct local replacement of (1.4.1) is the (a priori stronger) quasi-isomorphism of ϱ; see also [Kov99, Lemma 2.2]. This turns out to be equivalent to the local Du Bois isomorphisms

$$H^i_x(X, \mathscr{O}_X) \simeq \mathbb{H}^i_x(X, \underline{\Omega}^0_X) \quad \text{for } i \in \mathbb{N} \text{ and } x \in X. \tag{1.9.4}$$

At the end this leads to the local cohomology lifting property, the key technical ingredient in [KK20]; see Definition 3.2.

Notation 1.10. $\mathbb{H}^i$ stands for $\mathscr{R}^i\Gamma$, the i^{th} derived functor of Γ, the functor of sections; and $\mathbb{H}^i_x$ stands for $\mathscr{R}^i\Gamma_x$, the i^{th} derived functor Γ_x, the functor of sections with support at x, i.e., the i^{th} local cohomology functor with support at x on the derived category of quasi-coherent sheaves on X.

2. Filtrations on modules over Artinian local rings

We recall the following notation from [KK20].

2.1 Maximal filtrations

Let $(S, \mathfrak{m}, k)$ be an Artinian local ring and let N be a finite S-module with a filtration $N = N_0 \supsetneq N_1 \supsetneq \cdots \supsetneq N_q \supsetneq N_{q+1} = 0$ such that $N_j \big/ N_{j+1} \simeq k$ as S-modules for each $j = 0, \ldots, q$. Further, let $f : (X, x) \to (\operatorname{Spec} S, \mathfrak{m})$ be a local morphism, and denote the fiber of f over $\mathfrak{m}$ by X_k. It then follows that for each $j = 0, \ldots, q$,

$$f^*\left(N_j \big/ N_{j+1}\right) \simeq \mathscr{O}_{X_k}. \tag{2.1.1}$$

https://doi.org/10.1112/mod.2024.8 Published online by Cambridge University Press

2.2 Filtering S

In particular, considering S as a module over itself, we choose a filtration of S by ideals $S = I_0 \supsetneq I_1 \supsetneq \cdots \supsetneq I_q \supsetneq I_{q+1} = 0$ such that $I_j \big/ I_{j+1} \simeq k$ as S-modules for all $0 \le j \le q$. Observe that in this case, $I_1 = \mathfrak{m}$ and for every j there exists a $t_j \in I_j$ such that the composition $S \xrightarrow{t_j \cdot} I_j \longrightarrow I_j \big/ I_{j+1}$ induces an isomorphism $S \big/ \mathfrak{m} \simeq I_j \big/ I_{j+1}$. In particular, $\mathit{ann}\left(I_j \big/ I_{j+1}\right) = \mathfrak{m}$. Finally, let $S_j := S \big/ I_j$. Note that $S_1 = S \big/ \mathfrak{m}$ and $S_{q+1} = S$.

2.3 Filtering ω_S

Applying Grothendieck duality to the closed embedding given by the surjection $S \twoheadrightarrow S_j$ implies that $\omega_{S_j} \simeq \operatorname{Hom}_S(S_j, \omega_S)$, and we obtain injective S-module homomorphisms $\varsigma_j : \omega_{S_j} \hookrightarrow \omega_{S_{j+1}}$ induced by the natural surjection $S_{j+1} \twoheadrightarrow S_j$. Using the fact that the canonical module of an Arinian local ring, in particular ω_S, is an injective module and applying the functor $\operatorname{Hom}_S(\_\_, \omega_S)$ to the short exact sequence of S-modules

$$0 \longrightarrow I_j \big/ I_{j+1} \longrightarrow S_{j+1} \longrightarrow S_j \longrightarrow 0,$$

we obtain another short exact sequence of S-modules:

$$0 \longrightarrow \omega_{S_j} \xrightarrow{\varsigma_j} \omega_{S_{j+1}} \longrightarrow \operatorname{Hom}_S(k, \omega_S) \simeq k \longrightarrow 0. \tag{2.3.1}$$

Therefore, we obtain a filtration of $N = \omega_S$ by the submodules $N_j := \omega_{S_{q+1-j}}$ as in (2.1), where $q+1 = \operatorname{length}_S(S) = \operatorname{length}_S(\omega_S)$. The composition of the embeddings in (2.3.1) will be denoted by $\varsigma := \varsigma_q \circ \cdots \circ \varsigma_1 : \omega_{S_1} \hookrightarrow \omega_{S_{q+1}} = \omega_S$.

Recall that the *socle* of a module M over a local ring $(S, \mathfrak{m}, k)$ is

$$\operatorname{Soc} M := (0 : \mathfrak{m})_M = \{x \in M \mid \mathfrak{m} \cdot x = 0\} \simeq \operatorname{Hom}_S(k, M). \tag{2.3.2}$$

Soc M is naturally a k-vector space and $\dim_k \operatorname{Soc} \omega_S = 1$ by the definition of the canonical module. In particular, $\operatorname{Soc} \omega_S \simeq k$, which is the only S-submodule of ω_S isomorphic to k.

Let us recall [KK20, Lemma 3.4], which will be important later:

LEMMA 2.4. *Using the notation from (2.2) and (2.3), we have that*

$$\operatorname{im} \varsigma = \operatorname{Soc} \omega_S = I_q \omega_S. \tag{2.4.1}$$

Remark 2.4.2. Note that this is not simply stating that the modules in (2.4.1) are isomorphic but that they are equal as submodules of ω_S.

3. Families over Artinian local rings

We will frequently use the following notation.

Notation 3.1. Let A be a noetherian ring, $(R, \mathfrak{m})$ a noetherian local A-algebra, $I \subset R$ a nilpotent ideal and $(T, \mathfrak{n}) := (R/I, \mathfrak{m}/I)$, with natural morphism $\alpha : R \twoheadrightarrow T$.

DEFINITION 3.2. *Let A be a noetherian ring, and let $(T, \mathfrak{n})$ be a noetherian local A-algebra, with $i \in \mathbb{N}$ fixed. We say that T has liftable i^{th} local cohomology over A if for any noetherian local A-algebra $(R, \mathfrak{m})$ and nilpotent ideal $I \subset R$ such that $R/I \simeq T$, the natural morphism on local cohomology*

https://doi.org/10.1112/mod.2024.8 Published online by Cambridge University Press

$$H^i_{\mathfrak{m}}(R) \twoheadrightarrow H^i_{\mathfrak{n}}(T)$$

is surjective. Finally, if T has liftable i^{th} local cohomology over A for every $i \in \mathbb{N}$, then we say that T has liftable local cohomology over A [KK20].

We say that T has liftable i^{th} local cohomology, resp. liftable local cohomology, if it has the relevant property over $\mathbb{Z}$.

Remark 3.3. Notice that using the above notation, if $\phi: A' \to A$ is a ring homomorphism from another noetherian ring A', then if T has liftable i^{th} local cohomology over A', then it also has liftable i^{th} local cohomology over A. In particular, if T has liftable i^{th} local cohomology over $\mathbb{Z}$, then it has liftable i^{th} local cohomology over any noetherian ring A justifying the above terminology.

Furthermore, if $A = k$ is a field of characteristic 0, then the notions of having liftable i^{th} local cohomology over k and over $\mathbb{Z}$ are equivalent. This follows in one direction by the above and in the other direction by the Cohen structure theorem [StacksProject, Tag 032A].

DEFINITION 3.4. *We extend this definition to schemes: Let (X,x) be a local scheme over a noetherian ring A. Then we say that (X,x) has liftable i^{th} local cohomology over A if $\mathscr{O}_{X,x}$ has liftable i^{th} local cohomology over A. If $f: X \to Z$ is a morphism of schemes, then we say that X has liftable i^{th} local cohomology over Z if (X,x) has liftable i^{th} local cohomology over A for each $x \in X$ and for each $\operatorname{Spec} A \subseteq Z$ open affine neighbourhood of $f(x) \in Z$. This also extends the notion of liftable local cohomology in the obvious way.*

LEMMA 3.5. *Let $X \to Y \to Z$ be morphisms schemes. If X has liftable i^{th} local cohomology over Z, then X has liftable i^{th} local cohomology over Y as well.*

In particular, if X has liftable i^{th} local cohomology over a field k, then it has liftable i^{th} local cohomology over any other k-scheme to which it admits a morphism. In addition if $\operatorname{char} k = 0$, then X has liftable i^{th} local cohomology.

Proof. This follows from the definitions and Remark 3.3. □

Let us recall the following simple lemma from [KK20, Lemma 4.4]:

LEMMA 3.6. *Using Notation 3.1, let M be an R-module such that there exists a surjective R-module homomorphism $\phi: M \twoheadrightarrow T$. Assume that the induced natural homomorphism $H^i_{\mathfrak{m}}(R) \twoheadrightarrow H^i_{\mathfrak{n}}(T)$ is surjective for some $i \in \mathbb{N}$. Then the induced homomorphism on local cohomology*

$$H^i_{\mathfrak{m}}(M) \twoheadrightarrow H^i_{\mathfrak{m}}(T) \simeq H^i_{\mathfrak{n}}(T) \tag{3.6.1}$$

is surjective for the same i. In particular, if $(T,\mathfrak{n})$ has liftable local cohomology over A, then the homomorphism in (3.1) is surjective for every $i \in \mathbb{N}$.

We will also need the following.

LEMMA 3.7. *Let $\mathcal{D}_i$ be the derived category of an abelian category $\mathcal{A}_i$ for $i = 1,2$ and $\Phi: \mathcal{D}_1 \to \mathcal{D}_2$ a triangulated functor, and define $\Phi^i := h^i \circ \Phi: \mathcal{D}_1 \to \mathcal{A}_2$. Let $\mathsf{A} \in \operatorname{Ob} \mathcal{D}_1$ such that $h^j(\mathsf{A}) = 0$ for $j > d$ for some $d \in \mathbb{Z}$, and assume that there exists an $m \in \mathbb{N}$ such that $\Phi^i(h^j(\mathsf{A})) = 0$ for $i > m$ and for each $j \in \mathbb{Z}$. Then $\Phi^i(\mathsf{A}) = 0$ for $i > m + d$.*

Proof. Consider the conjugate spectral sequence associated to A and Φ:

$$E_2^{p,q} = \Phi^p(h^q(\mathsf{A})) \Rightarrow \Phi^{p+q}(\mathsf{A}).$$

https://doi.org/10.1112/mod.2024.8 Published online by Cambridge University Press

By the assumptions, $E_2^{p,q}=0$ if either $p>m$ or $q>d$, which implies that $E_2^{p,q}=0$ for $p+q>m+d$. This implies the desired statement. ☐

DEFINITION 3.8. *Let $f:X\to Y$ be a morphism. Then f is said to be flat in codimension t if there exists a closed subset $Z\subseteq X$ such that* $\operatorname{codim}(Z\cap X_y, X_y)\geq t+1$ *for every $y\in Y$ and $f|_{X\setminus Z}$ is flat.*

In the proof of the next statement, we will use the *canonical truncation* of cochain complexes of objects of an abelian category, which has the property that its cohomology objects are the same as the original complex up to or above the given index. We follow the notation and terminology of [StacksProject, Tag 0118]. In particular, for any complex $C^\bullet$ and any $r\in\mathbb{Z}$, we have the following distinguished triangle of complexes,

$$\tau_{\leq r}(C^\bullet)\longrightarrow C^\bullet\longrightarrow \tau_{\geq r+1}(C^\bullet)\xrightarrow{+1} \tag{3.8.1}$$

COROLLARY 3.9. *Let $(S,\mathfrak{m},k)$ be an Artinian local ring, N a finite S-module, (X,x) a local scheme of dimension n and $f:(X,x)\to(\operatorname{Spec} S,\mathfrak{m})$ a local morphism. Assume that f is flat in codimension $t-1$. Then the natural morphism*

$$\mathcal{R}^i\Gamma_x(\mathcal{L}f^*N)\xrightarrow{\simeq} H_x^i(f^*N)$$

is an isomorphism for $i\geq n-t$.

Proof. As f is flat in codimension $t-1$, it follows that $\dim\operatorname{supp}\mathcal{L}^jf^*N\leq n-t$ for each $j<0$. This implies that $H_x^i(\mathcal{L}^jf^*N)=0$ for $i>n-t$ and $j<0$. Let $\mathsf{A}:=\tau_{\leq -1}(\mathcal{L}f^*N)$ and $\mathsf{B}:=\tau_{\geq 0}(\mathcal{L}f^*N)$. Then (3.8.1) gives a distinguished triangle of complexes of $\mathscr{O}_X$-modules,

$$\mathsf{A}\longrightarrow \mathcal{L}f^*N\longrightarrow \mathsf{B}\xrightarrow{+1}.$$

Furthermore, $h^j(\mathsf{A})=\mathcal{L}^jf^*N$ for $j<0$ and $h^j(\mathsf{A})=0$ for $j\geq 0$; hence Lemma 3.7 (for A, $\Phi=\mathcal{R}\Gamma_x$, $m=n-t$ and $d=-1$) implies that $\mathcal{R}^i\Gamma_x(\mathsf{A})=0$ for $i>n-t-1$. Finally, $\mathsf{B}\simeq_{\text{qis}} f^*N$, so the desired statement follows. ☐

PROPOSITION 3.10. *Let $(S,\mathfrak{m},k)$ be an Artinian local ring, (X,x) a local scheme of dimension n and $f:(X,x)\to(\operatorname{Spec} S,\mathfrak{m})$ a local morphism. Assume that f is flat in codimension $t-1$. Let N be a finite S-module with a filtration as in (2.1), and assume that (X_k,x), where X_k is the fiber of f over the closed point of $\operatorname{Spec} S$ and has liftable i^{th} local cohomology for $i\geq n-t$ over S. Then for each $i>n-t$ and for each j, the natural sequence of morphisms induced by the embeddings $N_{j+1}\hookrightarrow N_j$ forms a short exact sequence,*

$$0\longrightarrow H_x^i(f^*N_{j+1})\longrightarrow H_x^i(f^*N_j)\longrightarrow H_x^i\left(f^*\left(N_j\big/N_{j+1}\right)\right)\simeq H_x^i\left(\mathscr{O}_{X_k}\right)\longrightarrow 0.$$

Proof. Because $\mathcal{ann}\left(N_j\big/N_{j+1}\right)=\mathfrak{m}$, there is a natural surjective morphism

$$f^*N_j\otimes\mathscr{O}_{X_k}\twoheadrightarrow f^*\left(N_j\big/N_{j+1}\right).$$

By Lemma 3.6 and (2.1.1), the natural homomorphism

$$H_x^i(f^*N_j)\twoheadrightarrow H_x^i\left(f^*\left(N_j\big/N_{j+1}\right)\right)\simeq H_x^i\left(\mathscr{O}_{X_k}\right) \tag{3.10.1}$$

is surjective for all $i\geq n-t$. Next, consider the distinguished triangle

$$\mathcal{L}f^*N_{j+1}\longrightarrow \mathcal{L}f^*N_j\longrightarrow \mathcal{L}f^*\left(N_j\big/N_{j+1}\right)\xrightarrow{+1},$$

https://doi.org/10.1112/mod.2024.8 Published online by Cambridge University Press

and the induced long exact cohomology sequence for the functor $\mathcal{R}\Gamma_x$. By Corollary 3.9 the terms of that long exact sequence may be replaced by terms in the form of $H^i_x(f^*(\quad))$ for $i \geq n-t$, and hence the statement follows from (3.10.1). $\square$

3.11. The exceptional inverse image of the structure sheaves. Let $(S, \mathfrak{m}, k)$ be an Artinian local ring with a filtration by ideals as in (2.2). Further, let $f: X \to \operatorname{Spec} S$ be a morphism that is essentially of finite type and $f_j = f|_{X_j}: X_j := X \times_{\operatorname{Spec} S} \operatorname{Spec} S_j \to \operatorname{Spec} S_j$, where $S_j = S/I_j$ as defined in (2.2), e.g., $X_{q+1} = X$ and $X_1 = X_k$, the fiber of f over the closed point of S. By a slight abuse of notation, we will denote $\omega_{\operatorname{Spec} S}$ with ω_S as well, but it will be clear from the context which one is meant at any given time.

Using the description of the exceptional inverse image functor via the residual/dualizing complexes [Con00, (3.3.6)] (cf. [R&D66, 3.4(a)], [StacksProject, Tag 0E9L]):

$$f^! = \mathcal{R}\mathcal{H}om_X(\mathcal{L}f^* \mathcal{R}\mathcal{H}om_S(\_, \omega^\bullet_S), \omega^\bullet_X) \tag{3.11.1}$$

and because S is Artinian, $\omega^\bullet_{S_j} \simeq \omega_{S_j}$ for each j, and we have that

$$\omega^\bullet_{X_j/S_j} \simeq f_j^! \mathcal{O}_{\operatorname{Spec} S_j} \simeq \mathcal{R}\mathcal{H}om_{X_j}(\mathcal{L}f_j^* \omega_{S_j}, \omega^\bullet_{X_j}). \tag{3.11.2}$$

In the rest of this section, we will use the following notation and assumptions.

Assumption 3.12. *Let $(S, \mathfrak{m}, k)$ be an Artinian local ring, (X, x) a local scheme of dimension n and $f: (X, x) \to (\operatorname{Spec} S, \mathfrak{m})$ a local morphism. Assume that f is flat in codimension $t-1$ and that (X_k, x), where X_k is the fiber of f over the closed point of* $\operatorname{Spec} S$ *and has liftable i^{th} local cohomology for $i \geq n-t$ over S.*

Theorem 3.13. *For each $i > n-t$ and each $j \in \mathbb{N}$,*

(i) *there exists a natural surjective morphism $\varrho_{i,j}: \mathsf{h}^{-i}(\omega^\bullet_{X_{j+1}/S_{j+1}}) \twoheadrightarrow \mathsf{h}^{-i}(\omega^\bullet_{X_j/S_j})$,*

(ii) *there exists a natural surjective morphism $\varrho^i = \varrho_{i,1} \circ \cdots \circ \varrho_{i,q}: \mathsf{h}^{-i}(\omega^\bullet_{X/S}) \twoheadrightarrow \mathsf{h}^{-i}(\omega^\bullet_{X_k})$*

(iii) *the natural morphisms $\varrho_{i,j}$ fit into a short exact sequence,*

$$0 \longrightarrow \mathsf{h}^{-i}(\omega^\bullet_{X_k}) \longrightarrow \mathsf{h}^{-i}(\omega^\bullet_{X_{j+1}/S_{j+1}}) \xrightarrow{\varrho_{i,j}} \mathsf{h}^{-i}(\omega^\bullet_{X_j/S_j}) \longrightarrow 0,$$

(iv) $\ker \varrho_{i,j} = I_j \mathsf{h}^{-i}(\omega^\bullet_{X_{j+1}/S_{j+1}}) \simeq I_j \mathsf{h}^{-i}(\omega^\bullet_{X/S}) \big/ I_{j+1} \mathsf{h}^{-i}(\omega^\bullet_{X/S})$,

(v) $\mathsf{h}^{-i}(\omega^\bullet_{X_j/S_j}) \simeq \mathsf{h}^{-i}(\omega^\bullet_{X/S}) \big/ I_j \mathsf{h}^{-i}(\omega^\bullet_{X/S}) \simeq \mathsf{h}^{-i}(\omega^\bullet_{X/S}) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_j}$ *and*

(vi) $\ker \varrho^i = \mathfrak{m} \mathsf{h}^{-i}(\omega^\bullet_{X/S})$.

Proof. Let $N = \omega_S$ and consider the filtration on N given by $N_j = \omega_{S_{q+1-j}}$, cf. (2.3), (2.3.1). Further, let $(\ )^\wedge$ denote the completion at x (the closed point of X). Then by Proposition 3.10, for each $i > n-t$ and each j, there exists a short exact sequence

$$0 \longrightarrow H^i_x(f^* \omega_{S_j}) \longrightarrow H^i_x(f^* \omega_{S_{j+1}}) \longrightarrow H^i_x\left(f^*\left(\omega_{S_{j+1}} \big/ \omega_{S_j}\right)\right) \longrightarrow 0. \tag{3.13.1}$$

Notice that $f^* \omega_{S_j} \simeq f_j^* \omega_{S_j}$. Combining this observation for both j and $j+1$ with Corollary 3.9 yields that this short exact sequence may also be written as

$$0 \longrightarrow \mathcal{R}\Gamma^i_x(\mathcal{L}f_j^* \omega_{S_j}) \longrightarrow \mathcal{R}\Gamma^i_x(\mathcal{L}f_{j+1}^* \omega_{S_{j+1}}) \longrightarrow \mathcal{R}\Gamma^i_x\left(f^*\left(\omega_{S_{j+1}} \big/ \omega_{S_j}\right)\right) \longrightarrow 0. \tag{3.13.2}$$

https://doi.org/10.1112/mod.2024.8 Published online by Cambridge University Press

Applying local duality [StacksProject, Tag 0AAK] to (3.13.2) gives the short exact sequence

$$0 \longrightarrow \mathcal{E}xt_X^{-i}\left(f^*\left(\omega_{S_{j+1}}\big/\omega_{S_j}\right), \omega_X^\bullet\right)^\frown \longrightarrow \mathcal{E}xt_X^{-i}(\mathcal{L}f_{j+1}^*\omega_{S_{j+1}}, \omega_X^\bullet)^\frown \longrightarrow \mathcal{E}xt_X^{-i}(\mathcal{L}f_j^*\omega_{S_j}, \omega_X^\bullet)^\frown \longrightarrow 0.$$

Since completion is faithfully flat [StacksProject, Tag 00MC], this implies that there are short exact sequences

$$\begin{aligned} 0 \longrightarrow \mathcal{E}xt_X^{-i}\left(f^*\left(\omega_{S_{j+1}}\big/\omega_{S_j}\right), \omega_X^\bullet\right) \longrightarrow \\ \longrightarrow \mathcal{E}xt_X^{-i}\left(\mathcal{L}f_{j+1}^*\omega_{S_{j+1}}, \omega_X^\bullet\right) \longrightarrow \mathcal{E}xt_X^{-i}\left(\mathcal{L}f_j^*\omega_{S_j}, \omega_X^\bullet\right) \longrightarrow 0. \end{aligned} \tag{3.13.3}$$

By Grothendieck duality

$$\mathcal{R}\mathcal{H}om_X(\mathcal{L}f_j^*\omega_{S_j}, \omega_X^\bullet) \simeq \mathcal{R}\mathcal{H}om_{X_j}(\mathcal{L}f_j^*\omega_{S_j}, \omega_{X_j}^\bullet),$$

and hence $\mathcal{E}xt_X^{-i}\left(\mathcal{L}f_j^*\omega_{S_j}, \omega_X^\bullet\right) \simeq \mathsf{h}^{-i}(\omega_{X_j/S_j}^\bullet)$ for each i, j, by (3.11.2). Therefore, defining $\varrho_{i,j}$ as the surjective morphism in (3.13.3) implies (i). Composing the surjective morphisms in (3.13.1) for all j implies that the natural morphism

$$\mathsf{h}^{-i}(\omega_{X/S}^\bullet) \simeq \mathcal{E}xt_X^{-i}\left(f^*\omega_S, \omega_X^\bullet\right) \xrightarrow{\varrho^i} \mathcal{E}xt_X^{-i}\left(f^*\omega_{S_q}, \omega_X^\bullet\right) \simeq \mathsf{h}^{-i}(\omega_{X_k}^\bullet)$$

is surjective, and hence (ii) follows as well.

By (2.3.1) $f^*\left(\omega_{S_{j+1}}\big/\omega_{S_j}\right) \simeq \mathscr{O}_{X_k}$, and hence $\mathcal{E}xt_X^{-i}\left(f^*\left(\omega_{S_{j+1}}\big/\omega_{S_j}\right), \omega_X^\bullet\right) \simeq \mathsf{h}^{-i}(\omega_{X_k}^\bullet)$, (3.13.3) also implies (iii).

Composing the injective maps in (3.13.1) for all j shows that the embedding $\varsigma : \omega_{S_1} \hookrightarrow \omega_S$ induces an embedding on local cohomology:

$$H_x^i(f^*\omega_{S_1}) \subseteq H_x^i(f^*\omega_S). \tag{3.13.4}$$

Next we prove (iv) for $j = q$ first. Because $\mathsf{h}^{-i}(\omega_{X_q/S_q}^\bullet)$ is supported on X_q, it follows that

$$I_q\mathsf{h}^{-i}(\omega_{X/S}^\bullet) \subseteq K := \ker \mathsf{h}^{-i}(\varrho_q)$$

Recall from (2.2) that there exists a $t_q \in I_q$ such that $I_q = St_q \simeq S\big/\mathfrak{m}$ and from Lemma 2.4 that $I_q\omega_S = \operatorname{Soc}\omega_S$. It follows that for $j = q$, the short exact sequence of (2.3.1) takes the form

$$0 \longrightarrow \omega_{S_q} \longrightarrow \omega_S \xrightarrow{\tau} \operatorname{Soc}\omega_S \longrightarrow 0, \tag{3.13.5}$$

where $\tau : \omega_S \twoheadrightarrow \operatorname{Soc}\omega_S \subset \omega_S$ may be identified with multiplication by t_q on ω_S. Applying f^* and taking local cohomology, we obtain the sequence

$$0 \longrightarrow H_x^i(f^*\omega_{S_q}) \longrightarrow H_x^i(f^*\omega_S) \xrightarrow{H_x^i(\tau)} H_x^i\left(f^*\operatorname{Soc}\omega_S\right) \longrightarrow 0, \tag{3.13.6}$$

which coincides with (3.13.1) for $j = q$, and hence it is exact. Further note that the morphism $H_x^i(\tau)$ may also be identified with multiplication by t_q on $H_x^i(f^*\omega_S)$. By Lemma 2.4 and (3.13.4), the natural morphism $H_x^i(\varsigma) : H_x^i\left(f^*\operatorname{Soc}\omega_S\right) = H_x^i(I_qf^*\omega_S) = H_x^i(f^*\omega_{S_1}) \to H_x^i(f^*\omega_S)$ is injective. Because $H_x^i(\tau)$, i.e., multiplication by t_q on $H_x^i(f^*\omega_S)$, is surjective onto $H_x^i\left(f^*\operatorname{Soc}\omega_S\right)$, it follows that

$$H_x^i\left(f^*\operatorname{Soc}\omega_S\right) \xrightarrow[H_x^i(\varsigma)]{\simeq} \operatorname{im} H_x^i(\varsigma) = I_qH_x^i(f^*\omega_S) \hookrightarrow H_x^i(f^*\omega_S), \tag{3.13.7}$$

https://doi.org/10.1112/mod.2024.8 Published online by Cambridge University Press

i.e., $H^i_x(f^*\mathrm{Soc}\,\omega_S)$ coincides with $I_qH^i_x(f^*\omega_S)$ as submodules of $H^i_x(f^*\omega_S)$. Next let E be an injective hull of $\kappa(x)=\mathscr{O}_{X,x}\big/\mathfrak{m}_{X,x}$, and consider a morphism $\phi: H^i_x(f^*\mathrm{Soc}\,\omega_S)\to E$. As E is injective, ϕ extends to a morphism $\widetilde{\phi}: H^i_x(f^*\omega_S)\to E$. If $a\in H^i_x(f^*\omega_S)$, then $t_qa\in I_qH^i_x(f^*\omega_S)=H^i_x(f^*\mathrm{Soc}\,\omega_S)$, so

$$t_q\widetilde{\phi}(a)=\widetilde{\phi}(t_qa)=\phi(t_qa)=\left(\phi\circ H^i_x(\tau)\right)(a)$$

Therefore, $\phi\circ H^i_x(\tau)=t_q\widetilde{\phi}$. Similarly, if $\psi: H^i_x(f^*\omega_S)\to E$ is an arbitrary morphism, then setting $\phi=\psi|_{H^i_x(f^*\mathrm{Soc}\,\omega_S)}: H^i_x(f^*\mathrm{Soc}\,\omega_S)\to E$ and applying the same computation as above, with $\widetilde{\phi}$ replaced by ψ, shows that $\phi\circ H^i_x(\tau)=t_q\psi$. It follows that the embedding induced by $H^i_x(\tau)$,

$$\alpha:\mathrm{Hom}_{\mathscr{O}_{X,x}}(H^i_x(f^*\mathrm{Soc}\,\omega_S),E)\hookrightarrow \mathrm{Hom}_{\mathscr{O}_{X,x}}(H^i_x(f^*\omega_S),E) \tag{3.13.8}$$

identifies $\mathrm{Hom}_X(H^i_x(f^*\mathrm{Soc}\,\omega_S),E)$, with $I_q\mathrm{Hom}_X(H^i_x(f^*\omega_S),E)$. By local duality it follows that

$$\left(\ker\left[\varrho_{i,q}:\mathsf{h}^{-i}(\omega^\bullet_{X/S})\twoheadrightarrow\mathsf{h}^{-i}(\omega^\bullet_{X_q/S_q})\right]\Big/ I_q\mathsf{h}^{-i}(\omega^\bullet_{X/S})\right)\otimes\widehat{\mathscr{O}}_{X,x}=0$$

and hence, because completion is faithfully flat, this implies (iv) in the case $j=q$. Running through the same argument with S replaced by S_{j+1} gives the equality in (iv) for all j. In addition, (iv) for $j=q$ implies (v) for $j\geq q$. Assuming that (v) holds for $j=r+1$ implies the isomorphism in (vi) for $j=r$. In turn, the entire (iv) for $j=r$, combined with (v) for $j=r+1$, implies (v) for $j=r$. Therefore, (iv) and (v) follow by descending induction on j, and then (vi) follows from (iv) and the definition of ϱ^i. □

We will also need the following simple lemma from [KK20, 4.11].

LEMMA 3.14. *Let R be a ring, M an R-module, $t\in R$ and $J=(t)\subseteq R$. Assume that $(0:J)_M=(0:J)_R\cdot M$. Then the natural morphism $J\otimes_R M\xrightarrow{\simeq} JM$ is an isomorphism.*

The the following proposition and its proof are essentially the same as that of [KK20, Prop. 4.12]. We include it here because the original situation here is slightly different than [KK20], although the difference in the original situation does not influence anything in this particular proof.

PROPOSITION 3.15. *Using the same notation as above,*

(i) $I_j\otimes\mathsf{h}^{-i}(\omega^\bullet_{X/S})\simeq I_j\mathsf{h}^{-i}(\omega^\bullet_{X/S})$,

(ii) *for any* $l\in\mathbb{N}$, $I_j\big/I_{j+l}\otimes\mathsf{h}^{-i}(\omega^\bullet_{X/S})\simeq I_j\mathsf{h}^{-i}(\omega^\bullet_{X/S})\big/I_{j+l}\mathsf{h}^{-i}(\omega^\bullet_{X/S})$, *and*

(iii) *for any* $l\in\mathbb{N}$, $\mathfrak{m}^l\big/\mathfrak{m}^{l+1}\otimes\mathsf{h}^{-i}(\omega^\bullet_{X/S})\simeq\mathfrak{m}^l\mathsf{h}^{-i}(\omega^\bullet_{X/S})\big/\mathfrak{m}^{l+1}\mathsf{h}^{-i}(\omega^\bullet_{X/S})$.

Proof. Notice that since $H^i_x(f^*\mathrm{Soc}\,\omega_S)$ is both a quotient and a submodule of $H^i_x(f^*\omega_S)$, there are two natural maps between $\mathrm{Hom}_{\mathscr{O}_{X,x}}(H^i_x(f^*\mathrm{Soc}\,\omega_S),E)$ and $\mathrm{Hom}_{\mathscr{O}_{X,x}}(H^i_x(f^*\omega_S),E)$. Regarding $H^i_x(f^*\mathrm{Soc}\,\omega_S)$ as a quotient module via $H^i_x(\tau)$, we get the embedding $\alpha=(\_\_)\circ H^i_x(\tau)$ in (3.13.8) and consider it a submodule on the restriction map

$$\begin{aligned}\beta:\mathrm{Hom}_{\mathscr{O}_{X,x}}(H^i_x(f^*\omega_S),E)&\longrightarrow\mathrm{Hom}_{\mathscr{O}_{X,x}}(H^i_x(f^*\,\mathrm{Soc}\,\omega_S),E).\\ \phi&\longmapsto\phi|_{H^i_x(f^*\,\mathrm{Soc}\,\omega_S)}\end{aligned}$$

https://doi.org/10.1112/mod.2024.8 Published online by Cambridge University Press

These maps are of course not inverses to each other. In fact, we have already established (cf. (3.13.8)) that $\phi|_{H^i_x(f^*\operatorname{Soc}\omega_S)} \circ H^i_x(\tau) = t_q\phi$, and hence the composition $\alpha\circ\beta$ is simply multiplication by t_q:

$$\begin{array}{ccc} \phi \in \operatorname{Hom}_{\mathscr{O}_{X,x}}(H^i_x(f^*\omega_S), E) & \xrightarrow{\beta} & \operatorname{Hom}_{\mathscr{O}_{X,x}}(H^i_x(f^*\operatorname{Soc}\omega_S), E) \\ & \searrow^{\alpha\circ\beta} & \simeq \downarrow \alpha \\ & & t_q\phi \in I_q \operatorname{Hom}_{\mathscr{O}_{X,x}}(H^i_x(f^*\omega_S), E). \end{array} \tag{3.15.1}$$

This implies, (cf. (3.13.4) and (3.13.7)), that ϱ^i may be identified with multiplication by t_q on $\mathsf{h}^{-i}(\omega^\bullet_{X/S})$. Together with Theorem 3.12(vi) this implies that

$$(0:I_q)_{\mathsf{h}^{-i}(\omega^\bullet_{X/S})} = \ker\varrho^i = \mathfrak{m}\mathsf{h}^{-i}(\omega^\bullet_{X/S}) = (0:I_q)_S\cdot \mathsf{h}^{-i}(\omega^\bullet_{X/S}),$$

and hence the natural morphism. The exceptional inverse image of the structure sheaves

$$I_q\otimes_S \mathsf{h}^{-i}(\omega^\bullet_{X/S}) \xrightarrow{\simeq} I_q\mathsf{h}^{-i}(\omega^\bullet_{X/S}) \tag{3.15.2}$$

is an isomorphism by Lemma 3.14. Now assume, by induction, that (i) holds for S_q in place of S. In particular, keeping in mind that $S_q = S\big/I_q$, the natural map

$$I_j\big/I_q\otimes_{S_q}\mathsf{h}^{-i}(\omega^\bullet_{X_q/S_q}) \xrightarrow{\simeq} \left(I_j\big/I_q\right)\mathsf{h}^{-i}(\omega^\bullet_{X_q/S_q}) \tag{3.15.3}$$

is an isomorphism for all j. Consider the short exact sequence (cf. Theorem 3.13(v)),

$$0\longrightarrow I_q\mathsf{h}^{-i}(\omega^\bullet_{X/S})\longrightarrow \mathsf{h}^{-i}(\omega^\bullet_{X/S})\longrightarrow \mathsf{h}^{-i}(\omega^\bullet_{X_q/S_q})\longrightarrow 0$$

and apply $I_j\big/I_q\otimes_S(\_\_)$. The image of $I_j\big/I_q\otimes_S I_q\mathsf{h}^{-i}(\omega^\bullet_{X/S})$ in $I_j\big/I_q\otimes_S\mathsf{h}^{-i}(\omega^\bullet_{X/S})$ is 0 and hence by (3.15.3),the natural map

$$\boxed{I_j\big/I_q\otimes_S\mathsf{h}^{-i}(\omega^\bullet_{X/S})} \simeq I_j\big/I_q\otimes_{S_q}\mathsf{h}^{-i}(\omega^\bullet_{X_q/S_q}) \xrightarrow{\simeq} \left(I_j\big/I_q\right)\mathsf{h}^{-i}(\omega^\bullet_{X_q/S_q}) \simeq$$
$$\simeq \left(I_j\big/I_q\right)\mathsf{h}^{-i}(\omega^\bullet_{X/S})\big/I_q\mathsf{h}^{-i}(\omega^\bullet_{X/S}) \simeq \boxed{I_j\mathsf{h}^{-i}(\omega^\bullet_{X/S})\big/I_q\mathsf{h}^{-i}(\omega^\bullet_{X/S})}.$$

is an isomorphism. This, combined with (3.15.2) and the 5-lemma, implies (i). Then (ii) is a direct consequence of (i) and the fact that tensor product is right exact.

Finally, recall, that the choice of filtration in (2.2) was fairly unrestricted. In particular, we may assume that the filtration $I.$ of S is chosen so that for all $l\in\mathbb{N}$, there exists a $j(l)$ such that $I_{j(l)} = \mathfrak{m}^l$. Applying (ii) for this filtration implies (iii). □

The following theorem is an easy combination of the results of this section.

THEOREM 3.16. *Let $(S,\mathfrak{m},k)$ be an Artinian local ring, (X,x) a local scheme of dimension n, and $f:(X,x)\to(\operatorname{Spec}S,\mathfrak{m})$ a local morphism. Assume that f is flat in codimension $t-1$ and that (X_k,x), where X_k is the fiber of f over the closed point of* Spec S, *has liftable i^{th} local cohomology for $i\geq n-t$ over S. Then for each $i>n-t$, $\mathsf{h}^{-i}(\omega^\bullet_{X/S})$ is flat over* Spec S. *In particular, if $t>0$, then $\omega_{X/S}$ is flat over* Spec S *and commutes with arbitrary base change.*

https://doi.org/10.1112/mod.2024.8 Published online by Cambridge University Press

Proof. Flatness follows from Proposition 3.15(iii) and [StacksProject, Tag 0AS8]. If $t > 0$, then this implies that $\omega_{X/S}$ is flat over Spec S. Furthermore, it commutes with arbitrary base change by Theorem 3.13(ii) and [Kol23, 9.17]. □

4. Du Bois singularities and liftable local cohomology

In this section we prove a criterion for a local scheme to have liftable i^{th} local cohomology for $i \geq n-t$. As before, $\mathbb{H}^i_x$ denotes $\mathcal{R}^i\Gamma_x$, the i^{th} derived functor of Γ_x, the functor of sections with support at x, i.e., the i^{th} local cohomology functor with support at x on the derived category of quasi-coherent sheaves on X.

LEMMA 4.1. *Let (X,x) be a local scheme of dimension n which is essentially of finite type over a field of characteristic 0. Then $H^i_x(\mathscr{O}_X) \to \mathbb{H}^i_x(\underline{\Omega}^0_X)$ is surjective for each $i \in \mathbb{Z}$.*

Proof. This follows by applying Matlis duality to the map in [MSS17, Lemma 3.2] (cf. [Kov99, Lemma 2.2], [KS16a, Theorem 3.3], [KS16b, Theorem 3.2], [MSS17, Lemma 3.3]). □

THEOREM 4.2. *Let (X,x) be a local scheme of dimension n, which is essentially of finite type over a field of characteristic 0. Fix $t \in \mathbb{N}, t > 0$, and let $Z \subseteq X$ be a closed subset of codimension $t+2$. Further let $\sigma : Y \to X$ be an affine morphism which is an isomorphism over $U := X \setminus Z$. Assume that Y is Du Bois. Then*

(4.2.1) *$H^i_x(\mathscr{O}_X) \to \mathbb{H}^i_x(\underline{\Omega}^0_X)$ is an isomorphism for $i \geq n-t$, and*

(4.2.2) *X has liftable i^{th} local cohomology for $i \geq n-t$.*

Proof. Let $W = \sigma^{-1}(x) \subseteq Y$, and observe that there is an equality of functors:

$$\Gamma_x \circ \sigma_* = \Gamma_W.$$

Because σ is an affine morphism, with σ_* exact, we obtain an equality of derived functors:

$$\mathcal{R}\Gamma_x \circ \sigma_* = \mathcal{R}\Gamma_W. \tag{4.2.3}$$

Consider the short exact sequence

$$0 \longrightarrow \mathscr{O}_X \longrightarrow \sigma_*\mathscr{O}_Y \longrightarrow \mathscr{Q} \longrightarrow 0,$$

where $\mathscr{Q}$ is defined as the cokernel of the first non-zero morphism in this short exact sequence. Applying the functor $\mathcal{R}\Gamma_x$ and taking into account (4.2.1), we obtain the following distinguished triangle:

$$\mathcal{R}\Gamma_x\mathscr{O}_X \longrightarrow \mathcal{R}\Gamma_W\mathscr{O}_Y \longrightarrow \mathcal{R}\Gamma_x\mathscr{Q} \xrightarrow{+1}$$

The assumption implies that $\mathscr{Q}$ is supported on Z, so $H^i_x(\mathscr{Q}) = 0$ for $i > n-t-2$ and hence

$$H^i_x(\mathscr{O}_X) \simeq H^i_W(\mathscr{O}_Y) \quad \text{for } i \geq n-t. \tag{4.2.4}$$

Next, consider the following diagram:

$$\begin{array}{ccc} \mathscr{O}_X & \longrightarrow & \mathcal{R}\sigma_*\mathscr{O}_Y \\ \downarrow & & \downarrow \\ \underline{\Omega}^0_X & \longrightarrow & \mathcal{R}\sigma_*\underline{\Omega}^0_Y. \end{array}$$

https://doi.org/10.1112/mod.2024.8 Published online by Cambridge University Press

Applying $\mathcal{R}\Gamma_x$ to each element and using (4.2.1) and (4.2.4) leads to the following:

$$\begin{array}{ccc} H^i_x(\mathscr{O}_X) & \xrightarrow[\text{(for } i\geq n-t)]{\simeq} & H^i_W(\mathscr{O}_Y) \\ \downarrow\!\!\!\downarrow & & \downarrow \simeq \\ \mathbb{H}^i_x(\underline{\Omega}^0_X) & \longrightarrow & \mathbb{H}^i_W(\underline{\Omega}^0_Y) \end{array} \tag{4.2.5}$$

The top horizontal arrow is an isomorphism for $i \geq n-t$, and the right vertical arrow is an isomorphism for all i because Y is Du Bois. It follows that the diagonal map is also an isomorphism and, in particular, injective for $i \geq n-t$. In particular the left vertical arrow is also injective for $i \geq n-t$. It is surjective for each i by Lemma 4.1 and hence an isomorphism for $i \geq n-t$. This proves (4.2.1).

Let $(R, \mathfrak{m})$ be a noetherian local ring and $I \subset R$ a nilpotent ideal such that $R/I \simeq \mathscr{O}_{X,x}$. In order to prove (4.2.2) we need that the induced natural morphism on local cohomology

$$H^i_{\mathfrak{m}}(R) \twoheadrightarrow H^i_x(\mathscr{O}_X) \tag{4.2.6}$$

is surjective for $i \geq n-t$. Let $X' := \operatorname{Spec} R$ and consider the following diagram:

$$\begin{array}{ccc} H^i_{\mathfrak{m}}(R) & \longrightarrow & H^i_x(\mathscr{O}_X) \\ \downarrow\!\!\!\downarrow & & \simeq \downarrow \text{ (for } i\geq n-t \text{ by (4.2.1))} \\ \mathbb{H}^i_{\mathfrak{m}}(\underline{\Omega}^0_{X'}) & \xrightarrow[\simeq]{} & \mathbb{H}^i(\underline{\Omega}^0_X) \end{array}$$

As above, the left vertical arrow is a surjection by Lemma 4.1. The bottom horizontal arrow is an isomorphism because $X'_{\text{red}} \simeq X_{\text{red}}$ and $\underline{\Omega}^0$ depends only on the reduced structure by definition, cf. [MSS17, p.2150]. Finally, the right vertical arrow is an isomorphism for $i \geq n-t$ by (4.2.1), and the combination of these implies (4.2.6) and hence (4.2.2). □

Proof of Theorem 1.6 It follows from Theorem 4.2 that the assumptions of Theorem 1.6 imply those of Theorem 3.16, which in turn implies the desired statement of Theorem 1.6 if S is Artinian.

If $\omega_{X/B}$ is known to commute with base changes, then one can check flatness over Artin subschemes of B by the local criterion of flatness.

The general case follows from [Kol23, 9.17], which is a variant of the local criterion of flatness, combined with obstruction theory. □

Proof of Theorem 1.2. We may assume that B is a local scheme with closed point $b \in B$. We will consider three, increasingly more general, cases.

Case I: $\Delta = 0$ *and* $\omega_{\widetilde{X}_b}$ *is locally free,* where $\pi : \widetilde{X}_b \to X_b$ is the demi-normalization as in (1.3.5).

Note that $\omega_{X/B}$ is flat and commutes with arbitrary base change by Theorem 1.6. By further localization we may assume that $\omega_{\widetilde{X}_b}$ is free. Because $\omega_{X_b} \simeq \pi_* \omega_{\widetilde{X}_b}$ by Lemma 1.5, we see that $\omega_{X/B}$ has a section σ such that σ_b does not vanish on U_b; hence $\sigma : \mathscr{O}_X \to \omega_{X/B}$ is an isomorphism away from a closed subset W for which $W_b \subset Z_b$. In particular, $\operatorname{depth}_{W_b} \mathscr{O}_X \geq 2$ by (1.2.5). Now we use the easy [Kol23, Lem.10.6] to conclude that $\mathscr{O}_X \simeq \omega_{X/B}$. Thus g is flat, $\omega_{X/B}$ is locally free and so are all of its powers.

https://doi.org/10.1112/mod.2024.8 Published online by Cambridge University Press

Case II: *$\Delta = D$ is a $\mathbb{Z}$-divisor and $\omega_{\widetilde{X}_b}(\widetilde{D}_b)$ is locally free.* Note that $\mathscr{O}_U(-D) \simeq \omega_{U/B}$ is flat over B and commutes with base changes by assumption. Thus Proposition 5.1 applies, so $\omega_{X/B}(D)$ is flat over B and commutes with base changes.

We may assume that $\omega_{\widetilde{X}_b}(\widetilde{D}_b)$ is free with generating section $\widetilde{\sigma}_b$. By Lemma 1.5 we can identify $\widetilde{\sigma}_b$ with a section σ_b of $\omega_{X_b}(D_b)$. By flatness it lifts to $\sigma : \mathscr{O}_X \to \omega_{X/B}(D)$, which is an isomorphism over U. By (1.2.5) (and the easy [Kol23, 10.6]), σ is an isomorphism. Thus $\omega_{X/B}(D)$ is locally free and so are its powers.

Case III: *The general case.* We may assume that X is local, and by [Kol23, 9.17] it is sufficient to prove the case when B is Artinian.

Write $\Delta = \sum_{i\in I} a_i D_i$, where $a_i = 1 - \frac{1}{i}$, $I \subset \{2, 3, 4, \dots, \infty\}$ is a finite subset and the D_i are reduced divisors.

Choose $m > 0$ such that $\omega^{[m]}_{U_b}(m\Delta_b) \sim \mathscr{O}_{U_b}$. The kernel of $\operatorname{Pic}(U) \to \operatorname{Pic}(U_b)$ is a k-vectorspace and is hence divisible and torsion free. Thus there is a unique line bundle L_U on U such that $L_{U_b} \sim \mathscr{O}_{U_b}$ and $\omega^{[m]}_{U/B}(m\Delta)[\otimes]L^m_U \sim \mathscr{O}_U$. Let L be the push-forward of L_U to X. Take the corresponding cyclic cover

$$\pi : Y := \operatorname{Spec}_X \textstyle\sum_{j=0}^{m-1} \omega^{[j]}_{X/B}(\sum_i \lfloor ja_i \rfloor D_i)[\otimes]L^{[j]} \to X.$$

Note that π ramifies along the D_i as follows. If $i \geq 3$, then π has ramification index i along D_i, and π is unramified along D_∞. The $i = 2$ case is somewehat special. Then π_b has ramification index 2 along an irreducible divisor $F_b \subset X_b$ if it has multiplicity 1 in $D_2|_b$, and Y_b is nodal along $\pi_b^{-1}(F_b)$ if F_b has multiplicity 2 in $D_2|_b$. Thus

$$K_{Y_b} + \pi_b^* D_\infty \sim_{\mathbb{Q}} \pi_b^*(K_{X_b} + \Delta_b).$$

In particular, $(Y, \pi^* D_\infty) \to B$ satisfies the assumptions (1.2.1)–(1.2.6). (Note that $Y \to B$ is known to be flat only over U, so requiring flatness only in codimension ≤ 2 is essential here.)

By duality, we get that

$$\begin{aligned} \pi_* \omega_{Y/B}(\pi^* D_\infty) &\simeq \textstyle\sum_{j=0}^{m-1} \omega^{[1-j]}_{X/B}(D_\infty - \sum_i \lfloor ja_i \rfloor D_i)[\otimes]L^{[-j]}, \quad \text{and} \\ (\pi_b)_* \omega_{\widetilde{Y}_b}(\pi_b^* D_\infty) &\simeq \textstyle\sum_{j=0}^{m-1} \omega^{[1-j]}_{X_b}(D_\infty|_b - \sum_i \lfloor ja_i \rfloor D_i|_b)[\otimes]L_b^{[-j]}. \end{aligned}$$

The $j = 1$ summand of $(\pi_b)_* \omega_{\widetilde{Y}_b}(\pi_b^* D_\infty)$ is trivial. Thus $\omega_{\widetilde{Y}_b}(\pi_b^* D_\infty)$ has a section that is nowhere zero on U_b, so $\omega_{\widetilde{Y}_b}(\pi_b^* D_\infty)$ is trivial. The previous case applies, and we conclude that all the

$$\omega^{[1-j]}_{X/B}(D_\infty - \textstyle\sum_i \lfloor ja_i \rfloor D_i)[\otimes]L^{[-j]}$$

are flat over B and commute with base changes.

The $j = 1$ summand is $L^{[-1]}$, whose restriction to X_b is trivial. By flatness, the constant 1 section of $L^{[-1]}|_{X_b}$ lifts to a section of $L^{[-1]}$; hence L is trivial.

Now fix $0 \leq r < m$ and set $1 - j = r - m$. Then we get that

$$\omega^{[r]}_{X/B}(D_\infty + \textstyle\sum_i (ma_i D_i - \lfloor (m - r + 1)a_i \rfloor) D_i) \simeq \omega^{[1-j]}_{X/B}(D_\infty - \sum_i \lfloor ja_i \rfloor D_i)[\otimes]L^{[-j]}$$

is flat over B and commutes with base changes. Now, observe that

$$\lfloor ra \rfloor + \lfloor (m - r + 1)a \rfloor = \begin{cases} m + 1 & \text{if} \quad a = 1, \quad \text{and} \\ m & \text{if} \quad a = \frac{c-1}{c} \quad \text{for some} \quad 1 < c \mid m. \end{cases}$$

This gives that

$$\omega^{[r]}_{X/B}(D_\infty + \textstyle\sum_i (ma_i - \lfloor (m - r + 1)a_i \rfloor) D_i) \simeq \omega^{[r]}_{X/B}(\sum_i \lfloor ra_i \rfloor D_i).$$

https://doi.org/10.1112/mod.2024.8 Published online by Cambridge University Press

Thus the $\omega^{[r]}_{X/B}(\sum_i \lfloor ra_i\rfloor D_i)$ are flat over B and commute with base changes. □

COROLLARY 4.3. *Using the notation and assumptions of Theorem 1.2, set $D_\infty := \sum_{i:a_i=1} D_i$. Then $\mathscr{O}_X(-D_\infty)$ and $\mathscr{O}_{D_\infty}$ are flat over B and commute with base changes.*

Proof. Arguing as in **Case III** above, we get that

$$\pi_*\omega_{Y/B} \simeq \sum_{j=0}^{m-1} \omega^{[1-j]}_{X/B}(-\sum_i \lfloor ja_i\rfloor D_i)[\otimes]L^{[-j]}.$$

We proved that L is trivial, so the $j=1$ summand is $\mathscr{O}_X(-D_\infty)$. It is thus flat over B with S_2 fibers. Therefore, the induced maps $\mathscr{O}_X(-D_\infty)|_{X_b} \to \mathscr{O}_{X_b}$ are injections; hence $\mathscr{O}_{D_\infty}$ is also flat over B and commutes with base changes. □

5. KSBA stability

It is possible that the analog of Theorem 1.2 holds for arbitrary KSBA stable pairs as in [Kol23, Sec. 8.2]. Note that by [Kol23, 7.5], K-flatness of divisors is automatic in codimension ≥ 3. This would mean that the whole theory of KSBA stability is determined in codimension 2.

The next result is a very small step in this direction. It shows that the reduced part of the boundary divisor behaves well in codimension ≥ 3.

PROPOSITION 5.1. *Let $g : X \to B$ be a morphism of finite type and of pure relative dimension over a field of characteristic 0, Δ be a relative Mumford $\mathbb{R}$-divisor and be $0 \leq D \leq \Delta$ a relative Mumford $\mathbb{Z}$-divisor. Let $Z \subset X$ be a closed subset, and set $U := X \setminus Z$. Assume that*

(5.1.1) $\operatorname{codim}(Z_b \subset X_b) \geq 3$ *for every* $b \in B$,

(5.1.2) $g|_U : U \to B$ *is flat with demi-normal fibers,*

(5.1.3) $\mathscr{O}_U(-D|_U)$ *is flat over B and commutes with base changes, and*

(5.1.4) *the demi-normalization $(\widetilde{X}_b, \widetilde{\Delta}_b)$ of (X_b, Δ_b) is semi-log-canonical for $b \in B$.*

Then $\omega_{X/B}(D)$ is flat over B and commutes with base changes.

Proof. Take two copies $(X_i, \Delta_i) \simeq (X, \Delta)$, and glue them together along $D_1 \simeq D_2$ to get

$$g_Y := (g_1 \amalg g_2) : Y := X_1 \amalg_{D_1 \simeq D_2} X_2 \to B.$$

Let $\pi : Y \to X$ be the projection. Set $\Delta_Y := \pi^*(\Delta - D)$, and consider the short exact sequence

$$0 \longrightarrow \mathscr{O}_{X_1}(-D_1) \longrightarrow \mathscr{O}_Y \longrightarrow \mathscr{O}_{X_2} \longrightarrow 0.$$

Because π is finite, the push-forward of this remains exact, and using the fact that $\pi|_{X_i}$ is an isomorphism, the natural morphism $\mathscr{O}_X \to \pi_*\mathscr{O}_Y$ provides a splitting of the push-forward of the above exact sequence. Therefore, $\pi_*\mathscr{O}_Y \simeq \mathscr{O}_X \oplus \mathscr{O}_X(-D)$, so $(Y, \Delta_Y) \to B$ is flat over $\pi^{-1}(U)$ with semi-log-canonical fibers. The demi-normalization of $(Y_b, \Delta_Y|_b)$ is the amalgamation of two copies of $(\widetilde{X}_b, \widetilde{\Delta}_b)$ along $\widetilde{D}_b$, hence semi-log-canonical. Thus $\omega_{Y/B}$ is flat over B and commutes with base changes by Theorem 1.6. Finally note that $\pi_*\omega_{Y/B} \simeq \omega_{X/B} \oplus \omega_{X/B}(D)$; thus $\omega_{X/B}(D)$ is flat over B and commutes with base changes. □

Remark 5.2. We claim that AFI stability, where we float all coefficients as in [Kol23, Sec. 8.3], is determined in codimension 2.

To see this, note that the boundary divisor Δ is necessarily $\mathbb{R}$-Cartier. Thus, for every point $x \in Z_b$ as in Theorem 1.2, either $x \notin \operatorname{supp}\Delta_b$, and then local stability holds by Theorem 1.2,

https://doi.org/10.1112/mod.2024.8 Published online by Cambridge University Press

or $x \in \operatorname{supp}\Delta_b$, and then x is not an lc center of X_b. Then $\operatorname{depth}_x \mathscr{O}_{\widetilde{X}_b} \geq 3$ by [Kol13b, 7.20] (cf. [Kov11] and [AH12]); hence local stability holds by [Kol23, 10.73].

CONFLICTS OF INTEREST

None.

FUNDING STATEMENT

János Kollár was supported in part by NSF Grant DMS-1901855. Sándor Kovács was supported in part by NSF Grants DMS-1951376,-2100389 and a Simons Fellowship.

JOURNAL INFORMATION

Moduli is published as a joint venture of the Foundation Compositio Mathematica and the London Mathematical Society. As not-for-profit organisations, the Foundation and Society reinvest 100% of any surplus generated from their publications back into mathematics through their charitable activities.

REFERENCES

[AH12] V. Alexeev and C. D. Hacon, *Non-rational centers of log canonical singularities*, J. Algebra **369** (2012), 1–15. MR 2959783.

[Con00] B. Conrad, *Grothendieck duality and base change*, Lecture Notes in Mathematics, vol. **1750**, Springer-Verlag, Berlin, 2000. MR 1804902.

[DB81] Ph. Du Bois, *Complexe de de Rham filtré d'une variété singulière*, Bull. Soc. Math. France **109** (1981), no. 1, 41–81. MR 613848.

[DJ74] Ph. Du Bois and P. Jarraud, *Une propriété de commutation au changement de base des images directes supérieures du faisceau structural*, C. R. Acad. Sci. Paris Sér. A **279** (1974), 745–747. MR 0376678.

[GK14] P. Graf and S. J. Kovács. *Potentially Du Bois spaces*, J. Singul. **8** (2014), 117–134.

[EGA-IV/2] Grothendieck, *Éléments de géométrie algébrique, IV. Étude locale des schémas et des morphismes de schémas II*, Inst. Hautes Études Sci. Publ. Math. (1965), no. **24**, 231. MR 0199181.

[GNPP88] F. Guillén, V. Navarro Aznar, P. Pascual Gainza, and F. Puerta, Hyperrésolutions cubiques et descente cohomologique, Lecture Notes in Mathematics, vol. **1335**, Springer-Verlag, Berlin, 1988, Papers from the Seminar on Hodge-Deligne Theory held in Barcelona, 1982. MR 972983.

[KK10] J. Kollár and S. J. Kovács, Log canonical singularities are Du Bois, J. Amer. Math. Soc. **23** (2010), no. 3, 791–813. MR 2629988.

[KK20] J. Kollár and S. J. Kovács, *Deformation of log canonical and F-pure singularities*, Algebr. Geom. **7** (2020), no. 6, 758–780.

[KM98] J. Kollár and S. Mori, Birational geometry of algebraic varieties, Cambridge Tracts in Mathematics, vol. **134**, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. MR 1658959.

[Kol87] J. Kollár, Vanishing theorems for cohomology groups, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., vol. 46, Amer. Math. Soc., Providence, RI, 1987, pp. 233–243. MR 927959.

[Kol95] J. Kollár, Shafarevich maps and automorphic forms, M. B. Porter Lectures, Princeton University Press, Princeton, NJ, 1995. MR 1341589.

[Kol13a] J. Kollár, *Grothendieck-Lefschetz type theorems for the local Picard group*, J. Ramanujan Math. Soc. **28A** (2013), 267–285. MR 3115196.

[Kol13b] J. Kollár, Singularities of the minimal model program, Cambridge Tracts in Mathematics, vol. **200**, Cambridge University Press, Cambridge, 2013, with the collaboration of Sándor J Kovács.

[Kol23] J. Kollár, Families of varieties of general type, Cambridge Tracts in Mathematics, vol. **231**, Cambridge University Press, Cambridge, 2023, With the collaboration of Klaus Altmann and Sándor Kovács. MR 4566297.

[Kov99] S. J. Kovács, *Rational, log canonical, Du Bois singularities: on the conjectures of Kollár and Steenbrink*, Compositio Math. **118** (1999), no. 2, 123–133. MR 1713307.

[Kov00] S. J. Kovács, *Rational, log canonical, Du Bois singularities. II. Kodaira vanishing and small deformations*, Compositio Math. **121** (2000), no. 3, 297–304. MR 1761628.

[Kov11] S. J. Kovács, Irrational centers, Pure Appl. Math. Q. **7** (2011), no. 4, Special Issue: In memory of Eckart Viehweg, 1495–1515. MR 2918171.

[Kov12] S. J. Kovács, The intuitive definition of Du Bois singularities, Geometry and arithmetic, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2012, pp. 257–266. MR 2987664.

[KS16a] S. J. Kovács and K. Schwede, Du Bois singularities deform, Minimal models and extremal rays (Kyoto, 2011), Adv. Stud. Pure Math., vol. **70**, Math. Soc. Japan, [Tokyo], 2016, pp. 49–65. MR 3617778.

[KS16b] S. J. Kovács and K. Schwede, *Inversion of adjunction for rational and Du Bois pairs*, Algebra Number Theory **10** (2016), no. 5, 969–1000.

[KSB88] J. Kollár and N. I. Shepherd-Barron, Threefolds and deformations of surface singularities, Invent. Math. **91** (1988), no. **2**, 299–338. MR 922803.

[KSS10] S. J. Kovács, K. E. Schwede, and K. E. Smith, The canonical sheaf of Du Bois singularities, Adv. Math. **224** (2010), no. 4, 1618–1640. MR 2646306.

[MSS17] L. Ma, K. Schwede, and K. Shimomoto, *Local cohomology of Du Bois singularities and applications to families*, Compos. Math. **153** (2017), no. 10, 2147–2170. MR 3705286.

[Mat89] H. Matsumura, Commutative ring theory, second ed., Cambridge Studies in Advanced Mathematics, vol. **8**, Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid. MR 1011461.

[R&D66] R. Hartshorne. Residues and duality, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20, Springer-Verlag, Berlin, 1966. MR 0222093.

[StacksProject] Stacks Project Authors. *Stacks project*, http://stacks.math.columbia.edu.

[Ste83] J. H. M. Steenbrink. *Mixed Hodge structures associated with isolated singularities, Singularities, Part 2 (Arcata, Calif., 1981)*, Proc. Sympos. Pure Math., vol. **40**, Amer. Math. Soc., Providence, RI, 1983, pp. 513–536. MR 713277.

János Kollár kollar@math.princeton.edu
Department of Mathematics, Princeton University, Princeton, NJ, USA

Sándor J. Kovács skovacs@uw.edu
Department of Mathematics, University of Washington, Seattle, WA, USA

https://doi.org/10.1112/mod.2024.8 Published online by Cambridge University Press