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Product structure of graph classes with bounded treewidth

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Abstract

We show that many graphs with bounded treewidth can be described as subgraphs of the strong product of a graph with smaller treewidth and a bounded-size complete graph. To this end, define the *underlying treewidth* of a graph class \mathcal{G} to be the minimum non-negative integer c such that, for some function f , for every graph $G \in \mathcal{G}$ there is a graph H with $\text{tw}(H) \leq c$ such that G is isomorphic to a subgraph of $H \boxtimes K_{f(\text{tw}(G))}$. We introduce disjointed coverings of graphs and show they determine the underlying treewidth of any graph class. Using this result, we prove that the class of planar graphs has underlying treewidth 3; the class of $K_{s,t}$ -minor-free graphs has underlying treewidth s (for $t \geq \max\{s, 3\}$); and the class of K_t -minor-free graphs has underlying treewidth $t - 2$. In general, we prove that a monotone class has bounded underlying treewidth if and only if it excludes some fixed topological minor. We also study the underlying treewidth of graph classes defined by an excluded subgraph or excluded induced subgraph. We show that the class of graphs with no H subgraph has bounded underlying treewidth if and only if every component of H is a subdivided star, and that the class of graphs with no induced H subgraph has bounded underlying treewidth if and only if every component of H is a star.

Keywords: Minors; treewidth; product structure

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1. Introduction

Graph product structure theory describes complicated graphs as subgraphs of strong products¹ of simpler building blocks. The building blocks typically have bounded treewidth, which is the standard measure of how similar a graph is to a tree. Examples of graph classes that can be described

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¹The *strong product* of graphs A and B , denoted by $A \boxtimes B$, is the graph with vertex-set $V(A) \times V(B)$, where distinct vertices $(v, x), (w, y) \in V(A) \times V(B)$ are adjacent if $v = w$ and $xy \in E(B)$, or $x = y$ and $vw \in E(A)$, or $vw \in E(A)$ and $xy \in E(B)$.



this way include planar graphs [29, 73], graphs of bounded Euler genus [23, 29], graphs excluding a fixed minor [29], and various non-minor-closed classes [31, 41]. These results have been the key to solving several open problems regarding queue layouts [29], nonrepetitive colouring [28], p -centered colouring [25], adjacency labelling [27, 36], twin-width [3, 11], and comparable box dimension [33].

This paper shows that graph product structure theory can even be used to describe graphs of bounded treewidth in terms of simpler graphs. Here the building blocks are graphs of smaller treewidth and complete graphs of bounded size. For example, a classical theorem by the referee of [19] can be interpreted as saying that every graph G of treewidth k and maximum degree Δ is contained² in $T \boxtimes K_{O(k\Delta)}$ for some tree T .

This result motivates the following definition. The *underlying treewidth* of a graph class \mathcal{G} is the minimum $c \in \mathbb{N}_0$ such that, for some function f , for every graph $G \in \mathcal{G}$ there is a graph H with $\text{tw}(H) \leq c$ such that G is contained in $H \boxtimes K_{f(\text{tw}(G))}$. If there is no such c , then \mathcal{G} has *unbounded* underlying treewidth. We call f the *treewidth-binding function*. For example, the above-mentioned result in [19] says that any graph class with bounded degree has underlying treewidth at most 1 with treewidth-binding function $O(k)$.

This paper introduces disjointed coverings of graphs and shows that they are intimately related to underlying treewidth; see Section 3. Indeed, we show that disjointed coverings characterise the underlying treewidth of any graph class (Theorem 11). The remainder of the paper uses disjointed coverings to determine the underlying treewidth of several graph classes of interest, with a small treewidth-binding function as a secondary goal.

Minor-closed classes: We prove that every planar graph of treewidth k is contained in $H \boxtimes K_{O(k^2)}$ where H is a graph of treewidth 3. Moreover, this bound on the treewidth of H is best possible. Thus the class of planar graphs has underlying treewidth 3 (Theorem 21). We prove the following generalisations of this result: the class of graphs embeddable on any fixed surface has underlying treewidth 3 (Theorem 22); the class of K_t -minor-free graphs has underlying treewidth $t - 2$ (Theorem 18); and for $t \geq \max\{s, 3\}$ the class of $K_{s,t}$ -minor-free graphs has underlying treewidth s (Theorem 19). In all these results, the treewidth-binding function is $O(k^2)$ for fixed s and t .

Monotone classes: We characterise the monotone graph classes with bounded underlying treewidth. We show that a monotone graph class \mathcal{G} has bounded underlying treewidth if and only if \mathcal{G} excludes some fixed topological minor (Theorem 28). In particular, we show that for $t \geq 5$ the class of K_t -topological-minor-free graphs has underlying treewidth t (Theorem 25). The characterisation for monotone classes has immediate consequences. For example, it implies that the class of 1-planar graphs has unbounded underlying treewidth. On the other hand, for any $k \in \mathbb{N}$, we show that the class of outer k -planar graphs has underlying treewidth 2 (Theorem 46), which generalises the well-known fact that outerplanar graphs have treewidth 2.

We use our result for disjointed coverings to characterise the graphs H for which the class of H -free graphs has bounded underlying treewidth. In particular, the class of H -free graphs has bounded underlying treewidth if and only if every component of H is a subdivided star (Theorem 29). For specific graphs H , including paths and disjoint unions of paths, we precisely determine the underlying treewidth of the class of H -free graphs.

Hereditary classes: We characterise the graphs H for which the class of graphs with no induced subgraph isomorphic to H has bounded underlying treewidth. The answer is precisely when every component of H is a star, in which case the underlying treewidth is at most 2. Moreover, we characterise the graphs H for which the class of graphs with no induced subgraph isomorphic to H has underlying treewidth 0, 1 or 2 (Theorem 38).

Universal graphs: A graph U is *universal* for a graph class \mathcal{G} if $U \in \mathcal{G}$ and U contains every graph in \mathcal{G} . This definition is only interesting when considering infinite graphs. For each $k \in \mathbb{N}$

²A graph G is *contained* in a graph X if G is isomorphic to a subgraph of X .

there is a universal graph \mathcal{T}_k for the class of countable graphs of treewidth k . Huynh, Mohar, Šámal, Thomassen, and Wood [42] gave an explicit construction for \mathcal{T}_k , and showed how product structure theorems for finite graphs lead to universal graphs. Their results imply that for any hereditary class \mathcal{G} of countable graphs, if the class of finite graphs in \mathcal{G} has underlying treewidth c with treewidth-binding function f , then every graph in \mathcal{G} of treewidth at most k is contained in $\mathcal{T}_c \boxtimes K_{f(k)}$. This result is applicable to all the classes above. For example, every countable K_t -minor-free graph of treewidth k is contained in $\mathcal{T}_{t-2} \boxtimes K_{O(k^2)}$.

The definition of underlying treewidth suggests an underlying version of any graph parameter. An extended version of this paper [12] explores this idea, focusing on underlying chromatic number. It also includes details of some straightforward proofs omitted from this version.

The rest of this paper is organised as follows. Section 2 introduces some standard structural graph theory notions that will be useful. Section 3 presents disjointed coverings, our main technical tool that characterises underlying treewidth. Section 4 defines two graphs that provide lower bounds on the underlying treewidth of many graph classes. Sections 5–8 address the underlying treewidth of graph classes defined by excluded minors, topological minors, subgraphs, and induced subgraphs, respectively. Finally, Section 9 considers graphs defined by their drawings.

2. Preliminaries

2.1. Basic definitions

See [17] for graph-theoretic definitions not given here. We consider simple, finite, undirected graphs G with vertex-set $V(G)$ and edge-set $E(G)$. A *graph class* is a collection of graphs closed under isomorphism. A graph class is *hereditary* if it is closed under taking induced subgraphs. A graph class is *monotone* if it is closed under taking subgraphs. A graph H is a *minor* of a graph G if H is isomorphic to a graph obtained from a subgraph of G by contracting edges. A graph G is *H -minor-free* if H is not a minor of G . A graph class \mathcal{G} is *minor-closed* if every minor of each graph in \mathcal{G} is also in \mathcal{G} .

The class of planar graphs is minor-closed. More generally, the class of graphs embeddable on a given surface (that is, a closed compact 2-manifold) is minor-closed. The *Euler genus* of a surface with h handles and c cross-caps is $2h + c$. The *Euler genus* of a graph G is the minimum $g \in \mathbb{N}_0$ such that there is an embedding of G in a surface of Euler genus g ; see [56] for more about graph embeddings in surfaces. A graph is *linklessly embeddable* if it has an embedding in \mathbb{R}^3 with no two linked cycles; see [66] for a survey and precise definitions. The class of linklessly embeddable graphs is also minor-closed.

A graph \tilde{G} is a *subdivision* of a graph G if \tilde{G} can be obtained from G by replacing each edge vw by a path P_{vw} with endpoints v and w (internally disjoint from the rest of \tilde{G}). If each P_{vw} has t internal vertices, then \tilde{G} is the *t -subdivision* of G . If each P_{vw} has at most t internal vertices, then \tilde{G} is a ($\leq t$)-*subdivision* of G . A graph H is a *topological minor* of G if a subgraph of G is isomorphic to a subdivision of H . A graph G is *H -topological-minor-free* if H is not a topological minor of G .

A *clique* in a graph is a set of pairwise adjacent vertices. Let $\omega(G)$ be the size of the largest clique in a graph G . An *independent set* in a graph is a set of pairwise non-adjacent vertices. Let $\alpha(G)$ be the size of the largest independent set in a graph G . Let $\chi(G)$ be the chromatic number of G . Note that $|V(G)| \leq \chi(G)\alpha(G)$. A graph G is *d -degenerate* if every non-empty subgraph of G has a vertex of degree at most d . A greedy algorithm shows that $\chi(G) \leq d + 1$ for every d -degenerate graph G .

Let P_n be the n -vertex path. For a graph G and $\ell \in \mathbb{N}$, let ℓG be the union of ℓ vertex-disjoint copies of G . Let \widehat{G} be the graph obtained from G by adding one dominant vertex.

Let $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{N}_0 := \{0, 1, \dots\}$. All logarithms in this paper are binary.

2.2. Tree-decompositions

For a tree T , a *T -decomposition* of a graph G is a collection $\mathcal{W} = (W_x : x \in V(T))$ of subsets of $V(G)$ indexed by the nodes of T such that (i) for every edge $vw \in E(G)$, there exists a node $x \in V(T)$ with

$v, w \in W_x$; and (ii) for every vertex $v \in V(G)$, the set $\{x \in V(T) : v \in W_x\}$ induces a (connected) subtree of T . Each set W_x in \mathcal{W} is called a *bag*. The *width* of \mathcal{W} is $\max\{|W_x| : x \in V(T)\} - 1$. A *tree-decomposition* is a T -decomposition for any tree T . The *treewidth* $\text{tw}(G)$ of a graph G is the minimum width of a tree-decomposition of G . Treewidth is the standard measure of how similar a graph is to a tree. Indeed, a connected graph has treewidth 1 if and only if it is a tree. Treewidth is of fundamental importance in structural and algorithmic graph theory; see [6, 40, 62] for surveys.

We use the following well-known facts about treewidth. Every (topological) minor H of a graph G satisfies $\text{tw}(H) \leq \text{tw}(G)$. In every tree-decomposition of a graph G , each clique of G appears in some bag. Thus $\text{tw}(G) \geq \omega(G) - 1$ and $\text{tw}(K_n) = n - 1$. If $\{v_1, \dots, v_k\}$ is a clique in a graph G_1 and $\{w_1, \dots, w_k\}$ is a clique in a graph G_2 , and G is the graph obtained from the disjoint union of G_1 and G_2 by identifying v_i and w_i for each $i \in \{1, \dots, k\}$, then $\text{tw}(G) = \max\{\text{tw}(G_1), \text{tw}(G_2)\}$. For any graph G , we have $\text{tw}(\widehat{G}) = \text{tw}(G) + 1$ and $\text{tw}(\ell G) = \text{tw}(G)$ for any $\ell \in \mathbb{N}$, implying $\text{tw}(\widehat{\ell G}) = \text{tw}(G) + 1$. Finally, every graph G is $\text{tw}(G)$ -degenerate, implying $\chi(G) \leq \text{tw}(G) + 1$.

2.3. Partitions

To describe our main results in Section 1, it is convenient to use the language of graph products. However, to prove our results, it is convenient to work with the equivalent notion of graph partitions, which we now introduce.

For graphs G and H , an H -partition of G is a partition $(V_x : x \in V(H))$ of $V(G)$ indexed by the nodes of H , such that for every edge vw of G , if $v \in V_x$ and $w \in V_y$, then $x = y$ or $xy \in E(H)$. We say that H is the *quotient* of such a partition. The *width* of such an H -partition is $\max\{|V_x| : x \in V(H)\}$. For $c \in \mathbb{N}_0$, an H -partition where $\text{tw}(H) \leq c$ is called a c -tree-partition. The c -tree-partition-width of a graph G , denoted $\text{tpw}_c(G)$, is the minimum width of a c -tree-partition of G .

It follows from the definitions that a graph G has an H -partition of width at most ℓ if and only if G is contained in $H \boxtimes K_\ell$. Thus, $\text{tpw}_c(G)$ equals the minimum $\ell \in \mathbb{N}_0$ such that G is contained in $H \boxtimes K_\ell$ for some graph H with $\text{tw}(H) \leq c$. Hence, the underlying treewidth of a graph class \mathcal{G} equals the minimum $c \in \mathbb{N}_0$ such that, for some function f , every graph $G \in \mathcal{G}$ has c -tree-partition-width at most $f(\text{tw}(G))$. We henceforth use this as our working definition of underlying treewidth.

If a graph G has an H -partition for some graph H of treewidth c , then we may assume that H is edge-maximal of treewidth c . So H is a c -tree (which justifies the ‘ c -tree-partition’ terminology). Such graphs H are chordal. Chordal partitions are well studied with several applications [42, 63, 68, 74, 75]. For example, van den Heuvel and Wood [75] proved that every K_t -minor-free graph has a $(t - 2)$ -tree-partition in which each part induces a connected subgraph with maximum degree at most $t - 2$ (amongst other properties). Our results give chordal partitions with bounded-size parts (for graphs of bounded treewidth).

Before continuing, we review work on the $c = 1$ case. A *tree-partition* is a T -partition for some tree T . The *tree-partition-width*³ of G , denoted by $\text{tpw}(G)$, is the minimum width of a tree-partition of G . Thus $\text{tpw}(G) = \text{tpw}_1(G)$, which equals the minimum $\ell \in \mathbb{N}_0$ for which G is contained in $T \boxtimes K_\ell$ for some tree T . Tree-partitions were independently introduced by Seese [69] and Halin [39], and have since been widely investigated [7, 8, 19, 20, 24, 34, 76, 77]. Applications of tree-partitions include graph drawing [13, 16, 30, 32, 80], nonrepetitive graph colouring [2], clustered graph colouring [1], monadic second-order logic [51], network emulations [4, 5, 9, 37], size Ramsey number [26, 43], statistical learning theory [81], and the edge-Erdős-Pósa property [14, 38, 60]. Planar-partitions and other more general structures have also been studied [18, 21, 22, 63, 80].

Bounded tree-partition-width implies bounded treewidth, as noted by Seese [69]. This fact easily generalises for c -tree-partition-width; see [12] for a proof.

³Tree-partition-width has also been called *strong treewidth* [8, 69].

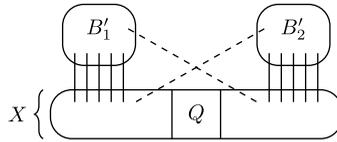


Figure 1. A disjointed partition with $c = 2$, where non-edges are dashed.

Observation 1. For every graph G and $c \in \mathbb{N}_0$, we have $\text{tw}(G) \leq (c + 1)\text{tpw}_c(G) - 1$.

Of course, $\text{tw}(T) = \text{tpw}(T) = 1$ for every tree T . But in general, $\text{tpw}(G)$ can be much larger than $\text{tw}(G)$. For example, fan graphs on n vertices have treewidth 2 and tree-partition-width $\Omega(\sqrt{n})$; see Lemma 12 below. On the other hand, the referee of [19] showed that if the maximum degree and treewidth are both bounded, then so is the tree-partition-width, which is one of the most useful results about tree-partitions.

Lemma 2 ([19]). For $k, \Delta \in \mathbb{N}$, every graph of treewidth less than k and maximum degree at most Δ has tree-partition-width at most $24k\Delta$.

This bound is best possible up to the multiplicative constant [77]. Note that bounded maximum degree is not necessary for bounded tree-partition-width (for example, stars). Ding and Oporowski [20] characterised graph classes with bounded tree-partition-width in terms of excluded topological minors. We give an alternative characterisation, which says that graph classes with bounded tree-partition-width are exactly those that have bounded treewidth and satisfy a further ‘disjointedness’ condition. Furthermore, this result naturally generalises for c -tree-partition-width and thus for underlying treewidth.

3. Disjointed coverings

This section introduces disjointed coverings and shows that they can be used to characterise bounded c -tree-partition-width and underlying treewidth (Theorem 11). On a high level, disjointed coverings are simply a weakening of c -tree-partitions. As such, they are often easier to construct than c -tree-partitions. This is important since disjointed coverings can in fact be used to construct c -tree-partitions (Lemma 8).

Here is the intuition behind disjointed coverings. An important property of any c -tree G is that for any set S of $c + 1$ vertices and any component X of $G - S$, there is a set Q of at most c vertices in X such that no component of $X - Q$ is adjacent to all of S . Given a c -tree-partition of a graph, an analogous property holds for the parts of the partition. Weakening this property slightly and allowing the parts of the partition to overlap leads to the following definition of disjointed coverings.

An ℓ -covering of a graph G is a set $\beta \subseteq 2^{V(G)}$ such that $|B| \leq \ell$ for every $B \in \beta$, and $\cup\{B : B \in \beta\} = V(G)$.⁴ If $B_1 \cap B_2 = \emptyset$ for all distinct $B_1, B_2 \in \beta$, then β is an ℓ -partition. As illustrated in Figure 1, an ℓ -covering β of a graph G is (c, d) -disjointed if for every c -tuple $(B_1, \dots, B_c) \in \beta^c$ and every component X of $G - (B_1 \cup \dots \cup B_c)$ there exists $Q \subseteq V(X)$ with $|Q| \leq d$ such that for each component Y of $X - Q$, for some $i \in \{1, \dots, c\}$ we have $V(Y) \cap N_G(B'_i) = \emptyset$, where $B'_i := B_i \setminus (B_1 \cup \dots \cup B_{i-1})$. Note that we can take $Q = \emptyset$ if some $B'_i = \emptyset$, since $N_G(\emptyset) = \emptyset$.

Let β be an ℓ -covering of a graph G . For $t \in \mathbb{N}$, let $\beta[t] := \{\cup B : B \subseteq \beta, |B| \leq t\}$. So $\beta[t]$ is a $t\ell$ -covering of G . For a function $f : \mathbb{N} \rightarrow \mathbb{R}^+$ we say that β is (c, f) -disjointed if $\beta[t]$ is $(c, f(t))$ -disjointed for every $t \in \mathbb{N}$.

⁴Our definition of ℓ -covering differs from the standard usage where it refers to a covering in which each element of the ground set is covered ℓ times.

While (c, d) -disjointed coverings are conceptually simpler than (c, f) -disjointed coverings, we show they are roughly equivalent (Theorem 4). Moreover, (c, f) -disjointed coverings are essential for the main proof (Lemma 8) and give better bounds on the c -tree-partition-width, leading to smaller treewidth-binding functions when determining the underlying treewidth of several graph classes of interest (for K_t -minor-free graphs for example).

Note that we often consider the singleton partition $\beta := \{\{v\} : v \in V(G)\}$ of a graph G , which is (c, f) -disjointed if and only if, for every $t \in \mathbb{N}$, every t -partition of G is $(c, f(t))$ -disjointed.

This section characterises c -tree-partition-width in terms of (c, d) -disjointed coverings (or partitions) and (c, f) -disjointed coverings (or partitions). The following observation deals with the $c = 0$ case.

Observation 3. *The following are equivalent for any graph G and $d \in \mathbb{N}$:*

- (a) G has a $(0, d)$ -disjointed covering;
- (b) every covering of G is $(0, d)$ -disjointed;
- (c) each component of G has at most d vertices;
- (d) G has 0-tree-partition-width at most d .

Observation 3 implies that a graph class \mathcal{G} has underlying treewidth 0 if and only if there is a function f such that every component of every graph $G \in \mathcal{G}$ has at most $f(\text{tw}(G))$ vertices.

We prove the following characterisation of bounded c -tree-partition-width (which is new even in the $c = 1$ case).

Theorem 4. *For fixed $c \in \mathbb{N}_0$, the following are equivalent for a graph class \mathcal{G} with bounded treewidth:*

- (a) \mathcal{G} has bounded c -tree-partition-width;
- (b) for some $d, \ell \in \mathbb{N}$, every graph in \mathcal{G} has a (c, d) -disjointed ℓ -partition;
- (c) for some $d, \ell \in \mathbb{N}$, every graph in \mathcal{G} has a (c, d) -disjointed ℓ -covering;
- (d) for some $\ell \in \mathbb{N}$ and function f , every graph in \mathcal{G} has a (c, f) -disjointed ℓ -partition;
- (e) for some $\ell \in \mathbb{N}$ and function f , every graph in \mathcal{G} has a (c, f) -disjointed ℓ -covering.

Proof. Observation 3 handles the $c = 0$ case. Now assume that $c \geq 1$. Lemma 6 below says that (a) implies (b). Since every ℓ -partition is an ℓ -covering, (b) implies (c), and (d) implies (e). Lemma 5 below says that (c) implies (d). Finally, Lemma 8 below says that (e) implies (a). \square

By definition, every (c, f) -disjointed ℓ -covering is $(c, f(1))$ -disjointed. The next lemma gives a qualitative converse to this.

Lemma 5. *Let $\ell, c, d \in \mathbb{N}$, and let β be a (c, d) -disjointed ℓ -covering of a graph G . Then β is (c, f) -disjointed, where $f(t) := dt^c$ for each $t \in \mathbb{N}$.*

Proof. Fix $t \in \mathbb{N}$. Let $B_1, \dots, B_c \in \beta[t]$. Let X be a component of $G - (B_1 \cup \dots \cup B_c)$. For each $i \in \{1, \dots, c\}$, let \mathcal{B}_i be a set of at most t elements of β whose union is B_i . Let $\mathcal{F} := \mathcal{B}_1 \times \dots \times \mathcal{B}_c$, and for each $y = (A_1, \dots, A_c) \in \mathcal{F}$, define Q_y as follows. Let X_y the component of $G - (A_1 \cup \dots \cup A_c)$ containing X . Since β is (c, d) -disjointed, there exists $Q_y \subseteq V(X_y)$ of size at most d such that for every component Y of $X_y - Q_y$ there is some $i \in \{1, \dots, c\}$ such that $V(Y) \cap N_G(A_i \setminus (A_1 \cup \dots \cup A_{i-1})) = \emptyset$. Now let $Q := \bigcup_{y \in \mathcal{F}} Q_y$, and note that $|Q| \leq d|\mathcal{F}| \leq dt^c$.

Suppose for contradiction that for some component Y of $X - Q$ and each $i \in \{1, \dots, c\}$, there is a vertex $b_i \in N_G(Y) \cap B'_i$, where $B'_i := B_i \setminus (B_1 \cup \dots \cup B_{i-1})$. Let $y = (A_1, \dots, A_c) \in \mathcal{F}$ be such that $(b_1, \dots, b_c) \in A_1 \times \dots \times A_c$, and consider that component Y' of $X_y - Q_y$ containing Y . By the definition of Q_y , there is some $i \in \{1, \dots, c\}$ such that Y' contains no neighbour of a vertex in $A_i \setminus (A_1 \cup \dots \cup A_{i-1})$. In particular, all neighbours of vertices of Y are either vertices of Y' or neighbours of vertices of Y' , so b_i is not a neighbour of any vertex of Y , a contradiction. \square

Now we prove that having a (c, d) -disjointed partition is necessary for bounded c -tree-partition-width.

Lemma 6. *For all $c, \ell \in \mathbb{N}_0$, every graph G with c -tree-partition-width ℓ has a $(c, c\ell)$ -disjointed ℓ -partition.*

Proof. By assumption, G has an H -partition $\beta = (V_h : h \in V(H))$ where H is a graph of treewidth at most c and $|V_h| \leq \ell$ for all h . We first show that the singleton partition of H is (c, c) -disjointed. Let $v_1, \dots, v_c \in V(H)$ and let X be a component of $H - \{v_1, \dots, v_c\}$. Let $(W_x : x \in V(T))$ be a tree-decomposition of H where $|W_x| \leq c + 1$ for all $x \in V(T)$. We may assume that $W_x \neq W_y$ whenever $x \neq y$. For each $i \in \{1, \dots, c\}$, let T_i be the subtree of T induced by $\{x \in V(T) : v_i \in W_x\}$.

First suppose that $V(T_i) \cap V(T_j) = \emptyset$ for some $i, j \in \{1, \dots, c\}$. Let $z \in V(T_i)$ be the closest node (in T) to T_j . Let $Q := W_z \cap X$. Note that $Q \subseteq W_z \setminus \{v_i\}$ so $|Q| \leq c$. Any path from v_i to v_j in H passes through W_z , so each component of $X - Q$ is disjoint from $N_H(v_i)$ or $N_H(v_j)$.

Now assume that $V(T_i) \cap V(T_j) \neq \emptyset$ for all $i, j \in \{1, \dots, c\}$. Let T_X be the subgraph of T induced by $\{x \in V(T) : V(X) \cap W_x \neq \emptyset\}$. Since X is connected, T_X is a subtree of T . Suppose that $V(T_i) \cap V(T_X) = \emptyset$ for some i . Since $N_H(v_i) \subseteq \bigcup \{W_x : x \in V(T_i)\}$, it follows that $N_H(v_i) \cap V(X) = \emptyset$ and so we may take $Q := \emptyset$ in this case. Now assume that $V(T_i) \cap V(T_X) \neq \emptyset$ for all $i \in \{1, \dots, c\}$. By the Helly property, $\tilde{T} := T_1 \cap \dots \cap T_c \cap T_X$ is a non-empty subtree of T . For $x \in V(\tilde{T})$, we have $|W_x| \leq c + 1$ and so $W_x = \{v_1, \dots, v_c, u\}$ for some $u \in V(X)$. First suppose that $|V(\tilde{T})| \geq 2$. Then there are adjacent $x, y \in V(\tilde{T})$ with $W_x = \{v_1, \dots, v_c, u\}$ and $W_y = \{v_1, \dots, v_c, v\}$ for $u, v \in V(X)$. Since $W_x \neq W_y$, we have $u \neq v$ and thus there is no (u, v) -path in $H - \{v_1, \dots, v_c\}$, contradicting the connectedness of X . Hence \tilde{T} consists of a single vertex z ; thus $W_z = \{v_1, \dots, v_c, u\}$ for some $u \in V(X)$. Let $Q := \{u\}$ and consider a component Y of $X - Q$. Let T_Y be the subtree of T induced by $\{y \in V(T) : V(Y) \cap W_y \neq \emptyset\}$. Since T_Y is connected and does not contain z , it is disjoint from some T_i . As above, $N_H(v_i) \cap V(Y) = \emptyset$, as required.

We have shown that the singleton partition of H is (c, c) -disjointed. Now focus on G . By assumption, β is an ℓ -partition of G . Let V_{v_1}, \dots, V_{v_c} be parts in β , and let X be a component of $G - (V_{v_1} \cup \dots \cup V_{v_c})$. Then $X \subseteq \bigcup \{V_h : h \in X'\}$ where X' is a component of $H - \{v_1, \dots, v_c\}$. Since H is (c, c) -disjointed, there exists $Q' \subseteq V(X')$ of size at most c such that each component $X' - Q'$ is disjoint from some $N_H(v_i)$. Let $Q := \bigcup \{V_h : h \in Q'\}$, which has size at most $c\ell$. Each component of $X - Q$ is disjoint from some $N_G(V_{v_i})$. \square

Note that (c, f) -disjointedness is preserved when restricting to a subgraph.

Lemma 7. *If β is a (c, f) -disjointed ℓ -covering of a graph G , then for every subgraph \tilde{G} of G , the restriction $\tilde{\beta} := \{B \cap V(\tilde{G}) : B \in \beta\}$ is a (c, f) -disjointed ℓ -covering of \tilde{G} .*

Proof. Fix $t \in \mathbb{N}$. Let $\tilde{B}_1, \dots, \tilde{B}_c \in \tilde{\beta}[t]$ and let \tilde{X} be a component of $\tilde{G} - (\tilde{B}_1 \cup \dots \cup \tilde{B}_c)$. For each $i \in \{1, \dots, c\}$, there is a subset $S_i \subseteq \beta$ of size at most t such that $\tilde{B}_i = \bigcup_{B \in S_i} (B \cap V(\tilde{G}))$. Let $(B_1, \dots, B_c) := (\bigcup S_1, \dots, \bigcup S_c)$, and let β'' be the $t\ell$ -covering of G given by $\beta \cup \{B_1, \dots, B_c\}$. Let X be the component of $G - (B_1 \cup \dots \cup B_c)$ which contains \tilde{X} , and for each $i \in \{1, \dots, c\}$ let $B'_i := B_i \setminus (B_1 \cup \dots \cup B_{i-1})$. Since β is (c, f) -disjointed, there is a subset Q of $V(X)$ of size at most $f(t)$ such that each component of $X - Q$ disjoint from $N_G(B'_i)$ for some $i \in \{1, \dots, c\}$. Let $\tilde{Q} := Q \cap V(\tilde{X})$, and note that $|\tilde{Q}| \leq |Q| \leq f(t)$. Each component of $\tilde{X} - \tilde{Q}$ is contained in a component of $X - Q$, and hence is disjoint from $N_{\tilde{G}}(\tilde{B}_i \setminus (\tilde{B}_1 \cup \dots \cup \tilde{B}_{i-1})) \subseteq N_G(B'_i)$ for some $i \in \{1, \dots, c\}$. Hence $\tilde{\beta}$ is (c, f) -disjointed. \square

The next lemma lies at the heart of the paper.

Lemma 8. *Let $k, c, \ell \in \mathbb{N}$ and $f : \mathbb{N} \rightarrow \mathbb{R}^+$. For any graph G , if $\text{tw}(G) < k$ and G has a (c, f) -disjointed ℓ -covering, then G has c -tree-partition-width $\text{tpw}_c(G) \leq \max\{12\ell k, 2c\ell f(12k)\}$.*

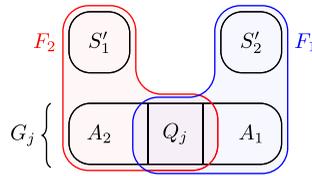


Figure 2. The graphs F_1 and F_2 in the case $c = 2$.

We prove Lemma 8 via the following induction hypothesis.

Lemma 9. *Let $k, c, \ell \in \mathbb{N}$ and let $f : \mathbb{N} \rightarrow \mathbb{R}^+$. Let G be a graph of treewidth less than k and let $\beta \subseteq 2^{V(G)}$ be a (c, f) -disjointed ℓ -covering of G . Let $S_1, \dots, S_{c-1}, R \subseteq V(G)$, where $S_i \in \beta[12k]$ for each $i \in \{1, \dots, c-1\}$ and $4k \leq |R| \leq f(12k)$. Then there exists a c -tree-partition $(V_x : x \in V(H))$ of G of width at most $W := \max\{12\ell k, 2c\ell f(12k)\}$, and there exists a c -clique $\{x_1, \dots, x_{c-1}, y\}$ of H such that $V_{x_i} = S_i \setminus (S_1 \cup \dots \cup S_{i-1})$ for each $i \in \{1, \dots, c-1\}$, and $R \setminus (S_1 \cup \dots \cup S_{c-1}) \subseteq V_y$ with $|V_y| \leq 2\ell(|R| - 2k)$.*

Proof. We proceed by induction on $|V(G)|$. Let $S := S_1 \cup \dots \cup S_{c-1}$.

Case 0. $V(G) = R \cup S$: Let H be the complete graph on vertices x_1, \dots, x_{c-1}, y . Let $V_{x_i} := S_i \setminus (S_1 \cup \dots \cup S_{i-1})$ for each i and let $V_y := R$. Then $(V_x : x \in V(H))$ is a c -tree-partition of G with width at most W and $|V_y| = |R| \leq 2(|R| - 2k) \leq 2\ell(|R| - 2k)$. From now on assume that $G - (R \cup S)$ is non-empty.

Case 1. $4k \leq |R| \leq 12k$: Since β is an ℓ -covering, and $|R| \leq 12k$, we can pick $S_c \in \beta[12k]$ such that $R \subseteq S_c$ and $|S_c| \leq \ell|R| \leq 2\ell(|R| - 2k)$.

Let G_1, \dots, G_a be the connected components of $G - (S \cup S_c)$. For each $i \in \{1, \dots, c\}$, let $S'_i := S_i \setminus (S_1 \cup \dots \cup S_{i-1})$. To complete this case, we first prove the following.

Claim 1. *For each $j \in \{1, \dots, a\}$, the subgraph $G[V(G_j) \cup S \cup S_c]$ has a c -tree-partition $(V_h^j : h \in V(H_j))$ of width at most W such that there is a c -clique $K = \{x_1, \dots, x_c\}$ in H_j , where $S'_i = V_{x_i}^j$ for each $i \in \{1, \dots, c\}$.*

Proof. If $|V(G_j)| < 4k$, then take H_j to be the complete graph on vertices x_1, \dots, x_c, z with the partition $V_{x_i}^j := S'_i$ for $i \in \{1, \dots, c\}$ and $V_z^j := V(G_j)$. Then this gives us the desired c -tree-partition of $G[V(G_j) \cup S \cup S_c]$. So assume $|V(G_j)| \geq 4k$. Note that $\beta[12k]$ is a $(c, f(12k))$ -disjointed $12k\ell$ -covering containing S_1, \dots, S_c , so there is a subset $Q'_j \subseteq V(G_j)$ of size at most $f(12k)$ and there is a partition $\{A_1, \dots, A_c\}$ of $V(G_j - Q'_j)$ such that each A_i is a union of vertex sets of components of $G_j - Q'_j$ that do not intersect $N_G(S'_i)$. Let Q_j be a set such that $Q'_j \subseteq Q_j \subseteq V(G_j)$ and $4k \leq |Q_j| \leq f(12k)$. As illustrated in Figure 2, consider the subgraph

$$\begin{aligned} F_i &:= G[A_i \cup Q_j \cup S_1 \cup \dots \cup S_{i-1} \cup (S_{i+1} \setminus S'_i) \cup \dots \cup (S_c \setminus S'_i)] \\ &= G[A_i \cup Q_j \cup S'_1 \cup \dots \cup S'_{i-1} \cup S'_{i+1} \cup \dots \cup S'_c]. \end{aligned}$$

By Lemma 7, the restriction of β to $V(F_i)$ is a (c, f) -disjointed ℓ -covering of F_i . Apply induction to F_i with the sets $S_1, \dots, S_{i-1}, S_{i+1} \setminus S'_i, \dots, S_c \setminus S'_i$ in place of the sets S_1, \dots, S_{c-1} and the set Q_j in the place of R . For each $i \in \{1, \dots, c\}$, this gives a graph L_i of treewidth at most c containing a c -clique $\{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_c, y\}$ such that F_i has an L_i -partition $(V_x^{j,i} : x \in V(L_i))$ with $V_{x_m}^{j,i} = S'_m$ for all $m \in \{1, \dots, i-1, i+1, \dots, c\}$ and $Q_j \setminus ((S \cup S_c) \setminus S'_i) \subseteq V_y^{j,i}$ where

$$|V_y^{j,i}| \leq 2\ell(|Q_j| - 2k) \leq 2\ell f(12k).$$

Let L_i^+ be L_i together with a vertex x_i adjacent to the clique $\{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_c, y\}$. So $\text{tw}(L_i^+) \leq c$. Set $V_{x_i}^{j,i} := S'_i$. Then L_i^+ contains the $(c + 1)$ -clique $K^+ := \{x_1, \dots, x_c, y\}$ and $(V_h^{j,i} : h \in V(L_i^+))$ is an L_i^+ -partition of $G[A_i \cup Q_j \cup S \cup S_c]$. Now we may assume that $V(L_1^+), \dots, V(L_c^+)$ pairwise intersect in exactly the clique K^+ . Let $H_j := L_1^+ \cup \dots \cup L_c^+$. Since each L_i^+ has treewidth at most c , so does H_j . For $x \notin K^+$, set $V_x^j := V_x^{j,i}$ for the unique i for which $x \in V(L_i^+)$, for $i \in \{1, \dots, c\}$ set $V_{x_i}^j := S'_i$, and set $V_y^j := \bigcup_{i \in \{1, \dots, c\}} V_y^{j,i}$. Since $|V_y^j| \leq 2c\ell f(12k)$, the partition $(V_x^j : x \in V(H_j))$ has width at most W . Setting $K := \{x_1, \dots, x_c\}$, the claim follows. \square

We may assume that $V(H_1), \dots, V(H_a)$ pairwise intersect in exactly the clique K . Let $H := H_1 \cup \dots \cup H_a$. For $x \in V(H)$, setting $V_x := V_x^j$ if $x \in V(H_j)$ is well defined, and yields an H -partition $(V_h : h \in V(H))$ of G . Since each H_i has treewidth at most c , so does H . Let $y := x_c$. Then $R \setminus S \subseteq S'_c = V_y$, and, as noted at the start of this case, $|V_y| = |S'_c| \leq |S_c| \leq 2\ell(|R| - 2k)$. Hence, the width of this partition is at most W , as required.

Case 2. $12k < |R| \leq f(12k)$: Since $\text{tw}(G) < k$, by the separator lemma of Robertson and Seymour [65, (2.6)], there is a partition (A, B, C) of $V(G)$ with no edges between A and B , where $|C| \leq k$ and $|A \cap R|, |B \cap R| \leq \frac{2}{3}|R \setminus C|$. Let $G_1 := G[A \cup C]$ and $G_2 := G[B \cup C]$. Let $R_1 := (R \cap A) \cup C$ and $R_2 := (R \cap B) \cup C$. Since $|R| \geq 12k$,

$$|R_1| = |A \cap R| + |C| \leq \frac{2}{3}|R| + k < |R| \quad \text{and}$$

$$|R_1| \geq |R| - |B \cap R| \geq |R| - \frac{2}{3}|R \setminus C| \geq \frac{1}{3}|R| \geq 4k.$$

Hence, $4k \leq |R_1| \leq f(12k)$ and similarly $4k \leq |R_2| \leq f(12k)$. Also $|V(G) - V(G_1)| = |B| \geq |R_2| - |C| \geq 4k - k > 0$, so $|V(G_1)| < |V(G)|$ and likewise for G_2 . Fix $j \in \{1, 2\}$. Let β_j be the restriction of β to $V(G_j)$. By Lemma 7, β_j is a (c, f) -disjointed ℓ -covering of G_j . Let $S'_i := S_i \cap V(G_j)$ for each $i \in \{1, \dots, c - 1\}$; note that each set S'_i is a union of at most $12k$ elements of β_j .

Apply induction to G_j with S'_i in place of S_i and R_j in place of R . Thus, there is a graph H_j of treewidth at most c , a c -clique $\{x_1, \dots, x_{c-1}, y\}$ of H_j , and an H_j -partition $(V_x^j : x \in V(H_j))$ of G_j of width at most W such that $V_{x_i}^j = S'_i \setminus (S_1^j \cup \dots \cup S_{i-1}^j)$ for each i and $R_j \setminus (S_1^j \cup \dots \cup S_{c-1}^j) \subseteq V_y^j$ with $|V_y^j| \leq 2\ell(|R_j| - 2k)$. We may assume that the intersection of $V(H_1)$ and $V(H_2)$ is equal to the clique $K := \{x_1, \dots, x_{c-1}, y\}$. Let H be the union of H_1 and H_2 and consider the H -partition $(V_x : x \in V(H))$ of G given by $V_x := V_x^j$ for $x \in V(H_j) \setminus V(H_{3-j})$ and $V_x := V_x^1 \cup V_x^2$ for $x \in K$. As H_1 and H_2 both have treewidth at most c , so does H .

Now $\{x_1, \dots, x_{c-1}, y\}$ is a c -clique, $V_{x_i} = S_i \setminus (S_1 \cup \dots \cup S_{i-1})$ and $R \setminus S \subseteq V_y$ with

$$|V_y| \leq |V_y^1| + |V_y^2| \leq 2\ell(|R_1| + |R_2| - 4k) \leq 2\ell(|R| + 2|C| - 4k) \leq 2\ell(|R| - 2k).$$

The other parts do not change and so we have the desired H -partition of G . \square

We are now ready to prove Lemma 8.

Proof of Lemma 8. Let β be a (c, f) -disjointed ℓ -covering of G . If $|V(G)| < 4k$, then the trivial partition $\{V(G)\}$ satisfies the claim. Otherwise $|V(G)| \geq 4k$. Let $R \subseteq V(G)$ with $|R| = 4k$. Let S_1, \dots, S_{c-1} be arbitrary elements of β . Since β is an ℓ -covering, $|S_i| \leq \ell \leq 12\ell k$ for each i . Now Lemma 8 follows from Lemma 9. \square

Lemmas 5 and 8 imply the following result:

Corollary 10. *Let $k, c, d, \ell \in \mathbb{N}$. For any graph G , if $\text{tw}(G) < k$ and G has a (c, d) -disjointed ℓ -covering, then G has c -tree-partition-width $\text{tpw}_c(G) \leq 2cd\ell(12k)^c$.*

Note that the singleton partition of any graph with maximum degree Δ is $(1, \Delta)$ -disjointed. So Corollary 10 with $c = \ell = 1$ and $d = \Delta$ implies Lemma 2 (even with the same constant 24).

Indeed, the proof of Lemma 8 in the case of graphs with bounded degree is equivalent to the proof of Lemma 2.

To conclude this section, Lemmas 6 and 8 imply the following characterisation of underlying treewidth.

Theorem 11. *The underlying treewidth of a graph class \mathcal{G} equals the minimum $c \in \mathbb{N}_0$ such that, for some function $g : \mathbb{N} \rightarrow \mathbb{N}$, every graph $G \in \mathcal{G}$ has a $(c, g(\text{tw}(G)))$ -disjointed $g(\text{tw}(G))$ -partition.*

4. Lower bounds

We now define two graphs that provide lower bounds on the underlying treewidth of various graph classes. Recall that $\widehat{\ell G}$ is the graph obtained from ℓ disjoint copies of a graph G by adding one dominant vertex. For $c, \ell \in \mathbb{N}$, define graphs $G_{c,\ell}$ and $C_{c,\ell}$ recursively as follows. First, $G_{1,\ell} := P_{\ell+1}$ is the path on $\ell + 1$ vertices, and $C_{1,\ell} := K_{1,\ell}$ is the star with ℓ leaves. Further, for $c \geq 2$, let $G_{c,\ell} := \widehat{\ell G_{c-1,\ell}}$ and $C_{c,\ell} := \widehat{\ell C_{c-1,\ell}}$. Note that $C_{c,\ell}$ is the closure of the rooted complete ℓ -ary tree of height c . Here, the closure of rooted tree T is the graph G with $V(G) = V(T)$ and $vw \in E(G)$ whenever v is an ancestor of w or w is an ancestor of v .

The next lemma collects together some useful and well-known properties of $G_{c,\ell}$ and $C_{c,\ell}$.

Lemma 12. *For all $c, \ell \in \mathbb{N}$,*

- (i) $\text{tw}(G_{c,\ell}) = \text{tw}(C_{c,\ell}) = c$;
- (ii) *for any ℓ -partition of $G \in \{G_{c,\ell}, C_{c,\ell}\}$, there is a $(c + 1)$ -clique in G whose vertices are in distinct parts;*
- (iii) $G_{c,\ell}$ and $C_{c,\ell}$ both have $(c - 1)$ -tree-partition-width greater than ℓ ;
- (iv) $G_{2,\ell}$ is outerplanar, $G_{3,\ell}$ is planar, $G_{4,\ell}$ is linklessly embeddable;
- (v) $G_{c,\ell}$ is $K_{c, \max\{c,3\}}$ -minor-free;
- (vi) $C_{c,\ell}$ does not contain P_4 as an induced subgraph;
- (vii) $C_{c,\ell}$ does not contain P_n as a subgraph for $n \geq 2^{c+1}$.

Proof. Since $\text{tw}(\widehat{\ell G}) = \text{tw}(G) + 1$ for any graph G and $\ell \in \mathbb{N}$, part (i) follows by induction.

We establish (ii) by induction on c . In the case $c = 1$, every ℓ -partition of $P_{\ell+1}$ or $K_{1,\ell}$ contains an edge whose endpoints are in different parts, and we are done. Now assume the claim for $c - 1$ ($c \geq 2$) and let $G \in \{G_{c-1,\ell}, C_{c-1,\ell}\}$. Consider an ℓ -partition of $\widehat{\ell G}$. At most $\ell - 1$ copies of G contain a vertex in the same part as the dominant vertex v . Thus, some copy G_0 of G contains no vertices in the same part as v . By induction, G_0 contains a c -clique K whose vertices are in distinct parts. Since v is dominant, $K \cup \{v\}$ satisfies the induction hypothesis.

Let $G \in \{G_{c,\ell}, C_{c,\ell}\}$. Consider an H -partition of G of width at most ℓ . By (ii), G contains a $(c + 1)$ -clique whose vertices are in distinct parts. So $\omega(H) \geq c + 1$, implying $\text{tw}(H) \geq c$. This establishes (iii).

Observe that $G_{2,\ell}$ is outerplanar (called a *fan* graph). The disjoint union of outerplanar graphs is outerplanar and the graph obtained from any outerplanar graph by adding a dominant vertex is planar; thus $G_{3,\ell}$ is planar. The disjoint union of planar graphs is planar, and the graph obtained from any planar graph by adding a dominant vertex is linklessly embeddable; thus $G_{4,\ell}$ is linklessly embeddable. This proves (iv).

We next show that $G_{c,\ell}$ is $K_{c, \max\{c,3\}}$ -minor-free. $G_{1,\ell}$ is a path and so has no $K_{1,3}$ -minor. $G_{2,\ell}$ is outerplanar and so has no $K_{2,3}$ -minor. Let $c \geq 3$ and assume the result holds for smaller c . Suppose that $G_{c,\ell}$ contains a $K_{c,c}$ -minor. Since $K_{c,c}$ is 2-connected, some copy of $G_{c-1,\ell}$ in $G_{c,\ell}$ contains a $K_{c-1,c}$ -minor. This contradiction establishes (v).

We show that $C_{c,\ell}$ contains no induced P_4 by induction on c . First, $C_{1,\ell} = K_{1,\ell}$ does not contain P_4 . Next, suppose that $C_{c-1,\ell}$ does not contain an induced P_4 . P_4 does not have a dominant vertex

and so any induced P_4 in $C_{c,\ell}$ must lie entirely within one copy of $C_{c-1,\ell}$. In particular, $C_{c,\ell}$ does not contain an induced P_4 . This proves (vii).

Finally, Nešetřil and Ossona de Mendez [57] proved (vii). □

The underlying treewidth of the class of graphs of treewidth at most k is obviously at most k . Lemma 12 (i) and (iii) imply the following.

Corollary 13. *The underlying treewidth of the class of graphs of treewidth at most k equals k .*

Corollary 14. *The classes $\{G_{c,\ell} : c, \ell \in \mathbb{N}\}$ and $\{C_{c,\ell} : c, \ell \in \mathbb{N}\}$ both have unbounded underlying treewidth.*

Proof. Suppose that $\{G_{c,\ell} : c, \ell \in \mathbb{N}\}$ has underlying treewidth b . Thus, for some function f , for all $c, \ell \in \mathbb{N}$, we have $\text{tpw}_b(G_{c,\ell}) \leq f(\text{tw}(G_{c,\ell})) = f(c)$. In particular, with $c := b + 1$ and $\ell := f(c)$, we have $\text{tpw}_{c-1}(G_{c,\ell}) \leq \ell$, which contradicts Lemma 12 (iii). The proof for $\{C_{c,\ell} : c, \ell \in \mathbb{N}\}$ is analogous. □

The graphs $G_{c,\ell}$ and $C_{c,\ell}$ are common in the graph-theory literature, and are particularly important for clustered colouring [35, 44, 58, 75], as we now explain. In an (improperly) vertex-coloured graph, a *monochromatic component* is a connected component of the subgraph induced by all the vertices of one colour. A graph G is *c-colourable with clustering ℓ* if each vertex can be assigned one of c colours such that each monochromatic component has at most ℓ vertices. The *clustered chromatic number* of a graph class \mathcal{G} is the minimum $c \in \mathbb{N}$ such that, for some $\ell \in \mathbb{N}$, every graph in \mathcal{G} has a c -colouring with clustering ℓ . See [79] for a survey on clustered graph colouring. Note that a graph G is c -colourable with clustering ℓ if and only if G is contained in $H \boxtimes K_\ell$ for some graph H with $\chi(H) \leq c$.

Consider a graph class \mathcal{G} with underlying treewidth c , where every graph in \mathcal{G} has treewidth at most k . Thus every graph $G \in \mathcal{G}$ is contained in $H \boxtimes K_\ell$ for some graph H with $\text{tw}(H) \leq c$, where $\ell := \max\{f(0), \dots, f(k)\}$ and f is from the definition of underlying treewidth. Since $\chi(H) \leq \text{tw}(H) + 1$ for every graph H , it follows that \mathcal{G} has clustered chromatic number at most $c + 1$. This means that lower bounds on the clustered chromatic number of a graph class with bounded treewidth provide lower bounds on the underlying treewidth of the class. In particular, it is known that $G_{c,\ell}$ and $C_{c,\ell}$ are not c -colourable with clustering ℓ (see [79]), which implies Lemma 12 (iii) by the above connection. See [58] for more examples of graphs G that are not c -colourable with clustering ℓ , implying $\text{tpw}_{c-1}(G) > \ell$.

5. Excluding a minor

This section uses disjointed partitions to determine the underlying treewidth of several minor-closed classes of interest.

The next definition enables K_t -minor-free graphs and $K_{s,t}$ -minor-free graphs to be handled simultaneously. For $s, t \in \mathbb{N}$, let $\mathcal{K}_{s,t}$ be the class of graphs G for which there is a partition $\{A, B\}$ of $V(G)$ such that $|A| = s$ and $|B| = t$; $vw \in E(G)$ for all $v \in A$ and $w \in B$; and $G[B]$ is connected. Obviously, every graph in $\mathcal{K}_{s,t}$ contains $K_{s,t}$. Similarly, we obtain K_t as a minor of any $G \in \mathcal{K}_{t-2,t}$ by contracting a matching between A and B of size $t - 2$ whose end-vertices are distinct from the end-vertices of some edge of $G[B]$.

Lemma 15. *Let G be a graph with no minor in $\mathcal{K}_{s,t}$. Assume $\{A, B\}$ is a partition of $V(G)$ such that $G[B]$ is connected and every vertex in B has at least s neighbours in A . Then $|B| \leq \delta|A|$ for some $\delta = \delta(s, t)$.*

Proof. The following proof technique is well-known [53, 59]. Assign vertices in B to pairs of vertices in A as follows. Initially, no vertices in B are assigned. If there is an unassigned vertex $v \in B$ adjacent to distinct vertices $x, y \in A$ and no vertex in B is already assigned to $\{x, y\}$, then assign

v to $\{x, y\}$. Repeat this operation until no more vertices in B can be assigned. Let B_1 and B_2 be the sets of assigned and unassigned vertices in B , respectively. Let G' be the graph obtained from $G[A \cup B_1]$ by contracting vx into x , for each vertex $v \in B_1$ assigned to $\{x, y\}$. Thus G' is a minor of G with $V(G') = A$ and $|E(G')| \geq |B_1|$. For each vertex $v \in B_2$, the set $N_G(v) \cap A$ is a clique in G' (otherwise v would have been assigned to a pair of non-adjacent neighbours) of size at least s . For each s -clique C in G' , there are at most $t - 1$ vertices $v \in B_2$ with $C \subseteq N_G(v)$, otherwise G has a minor in $\mathcal{K}_{s,t}$ (since $G[B]$ is connected). Since G' has no minor in $\mathcal{K}_{s,t}$, we have $|E(G')| \leq \delta|A|$ for some $\delta = \delta(s, t)$; see [45–50, 61, 71, 72] for explicit bounds on δ . Thus $|B_1| \leq \delta|A|$. Moreover, G' is 2δ -degenerate. Every d -degenerate graph on n vertices has at most $\binom{d}{s-1}n$ cliques of order s [78, Lemma 18]. Thus $|B_2| \leq (t - 1)\binom{2\delta}{s-1}|A|$. Hence $|B| = |B_1| + |B_2| \leq (\delta + (t - 1)\binom{2\delta}{s-1})|A|$. \square

The following Erdős–Pósa type result is useful for showing disjointedness. It follows from well-known results in the literature; see [12] for details.

Lemma 16. *Let \mathcal{H} be a set of connected subgraphs of a graph G . Then, for every non-negative integer ℓ , either there are $\ell + 1$ vertex-disjoint graphs in \mathcal{H} or there is a set $Q \subseteq V(G)$ of size at most $\ell(\text{tw}(G) + 1)$ such that $G - Q$ contains no graph of \mathcal{H} .*

Lemma 17. *For fixed $s, t \in \mathbb{N}$, every graph G with no minor in $\mathcal{K}_{s,t}$ and of treewidth k has s -tree-partition-width $O(k^2)$.*

Proof. By Lemma 8 it suffices to show that the singleton partition of G is (s, f) -disjointed, where $f(n) := \delta sn(k + 1)$ and $\delta := \delta(s, t)$ from Lemma 15. Let S_1, \dots, S_s be subsets of $V(G)$ of size at most n , let $S := S_1 \cup \dots \cup S_s$, and for each $i \in \{1, \dots, s\}$ let $S'_i := S_i \setminus (S_1 \cup \dots \cup S_{i-1})$. Let X be a connected component of $G - S$. Let \mathcal{H} be the set of connected subgraphs H of X such that $H \cap N(S'_i) \neq \emptyset$ for each $i \in \{1, \dots, s\}$. Say \mathcal{R} is a maximum-sized set of pairwise disjoint subgraphs in \mathcal{H} . We may assume that $\bigcup\{V(R) : R \in \mathcal{R}\} = V(X)$. Let X' be the graph obtained from $G[S \cup V(X)]$ by contracting each subgraph $R \in \mathcal{R}$ into a vertex v_R . So $V(X') = S \cup \{v_R : R \in \mathcal{R}\}$. Since X is connected, $\{v_R : R \in \mathcal{R}\}$ induces a connected subgraph of X' . By construction, in X' , each vertex v_R has at least s neighbours in S . By Lemma 15, $|\mathcal{R}| \leq \delta|S|$. By Lemma 16, there is a set $Q \subseteq V(X)$ of size at most $\delta|S|(k + 1) \leq f(n)$ such that $X - Q$ contains no graph in \mathcal{H} . Thus each component Y of $X - Q$ satisfies $V(Y) \cap N_G(S'_i) = \emptyset$ for some $i \in \{1, \dots, s\}$. Hence, the singleton partition of G is (s, f) -disjointed. \square

We now determine the underlying treewidth of K_t - and $\mathcal{K}_{s,t}$ -minor-free graphs.

Theorem 18. *For fixed $t \in \mathbb{N}$ with $t \geq 2$, the underlying treewidth of the class of K_t -minor-free graphs equals $t - 2$. In particular, every K_t -minor-free graph of treewidth k has $(t - 2)$ -tree-partition-width $O(k^2)$.*

Proof. Since K_t is a minor of every graph in $\mathcal{K}_{t-2,t}$, Lemma 17 implies that every K_t -minor-free graph of treewidth k has $(t - 2)$ -tree-partition-width $O(k^2)$. Thus the underlying treewidth of the class of K_t -minor-free graphs is at most $t - 2$. Suppose for contradiction that equality does not hold. That is, for some function f , every K_t -minor-free graph G has $(t - 3)$ -tree-partition-width at most $f(\text{tw}(G))$. Let $\ell := f(t - 2)$. The graph $G_{t-2,\ell}$ in Lemma 12 has treewidth $t - 2$ and is thus K_t -minor-free. However, by Lemma 12, $\text{tpw}_{t-3}(G_{t-2,\ell}) > \ell = f(t - 2) = f(\text{tw}(G_{t-2,\ell}))$, which is the required contradiction. \square

Theorem 19. *For fixed $s, t \in \mathbb{N}$ with $t \geq \max\{s, 3\}$, the underlying treewidth of the class of $\mathcal{K}_{s,t}$ -minor-free graphs equals s . In particular, every $\mathcal{K}_{s,t}$ -minor-free graph G of treewidth k has s -tree-partition-width $O(k^2)$.*

Proof. Since $\mathcal{K}_{s,t}$ is a subgraph of every graph in $\mathcal{K}_{s,t}$, Lemma 17 implies that every $\mathcal{K}_{s,t}$ -minor-free graph G of treewidth k has s -tree-partition-width $O(k^2)$. Thus the underlying treewidth of the class of $\mathcal{K}_{s,t}$ -minor-free graphs is at most s . Suppose that it is at most $s - 1$. Thus for some

function f , every $K_{s,t}$ -minor-free graph G satisfies $\text{tpw}_{s-1}(G) \leq f(\text{tw}(G))$. Let $\ell := f(s)$. The graph $G_{s,\ell}$ given by Lemma 12 has treewidth s and is $K_{s,\max\{s,3\}}$ -minor-free, implying it is $K_{s,t}$ -minor-free (since $t \geq \max\{s, 3\}$). However, by Lemma 12, $\text{tpw}_{s-1}(G_{s,\ell}) > \ell = f(s) = f(\text{tw}(G_{s,\ell}))$, which is the required contradiction. \square

For $t \leq 2$, Theorem 19 is improved as follows:

Proposition 20. *For $s, t \in \mathbb{N}$ with $s \leq t \leq 2$, the underlying treewidth of the class of $K_{s,t}$ -minor-free graphs equals $s - 1$.*

Proof. Every $K_{1,1}$ -minor-free graph G has no edges, and so $\text{tpw}_0(G) \leq 1$. Every $K_{1,2}$ -minor-free graph G has at most one edge in each component, and so $\text{tpw}_0(G) \leq 2$. Each block of any $K_{2,2}$ -minor-free graph G is a triangle, an edge or an isolated vertex; so $\text{tpw}_1(G) \leq 2$. \square

Since planar graphs are K_5 - and $K_{3,3}$ -minor-free, Theorem 18 or Theorem 19 imply the next result (where the lower bound holds since the graph $G_{3,\ell}$ in Lemma 12 is planar).

Theorem 21. *The underlying treewidth of the class of planar graphs equals 3. In particular, every planar graph of treewidth k has 3-tree-partition-width $O(k^2)$.*

It follows from Euler’s formula that every graph with Euler genus at most g is $K_{3,2g+3}$ -minor-free. Thus Lemma 12 and Theorem 19 imply the following.

Theorem 22. *The underlying treewidth of the class of graphs embeddable on any fixed surface Σ equals 3. In particular, every graph embeddable in Σ and of treewidth k has 3-tree-partition-width $O(k^2)$.*

Since every linklessly embeddable graph is K_6 -minor-free and $K_{4,4}$ -minor-free [67], Theorem 18 or Theorem 19 imply the next result, where the lower bound follows from Lemma 12.

Theorem 23. *The underlying treewidth of the class of linklessly embeddable graphs equals 4. In particular, every linklessly embeddable graph of treewidth k has 4-tree-partition-width $O(k^2)$.*

It is an open problem to determine the underlying treewidth of a given minor-closed class \mathcal{G} . It is possible that the clustered chromatic number of \mathcal{G} equals the underlying treewidth of \mathcal{G} plus 1. This is true for any minor-closed class with underlying treewidth at most 1 by results of Norin, Scott, Seymour, and Wood [58] and Ding and Oporowski [20]; see [12]. See [58] for a conjectured value of the clustered chromatic number of \mathcal{G} . It follows from a result of DeVos, Ding, Oporowski, Sanders, Reed, Seymour, and Vertigan [15] that every minor-closed graph class \mathcal{G} with underlying treewidth c has clustered chromatic number at most $2(c + 1)$; see [12].

6. Excluding a topological minor

This section studies the underlying treewidth of graphs classes defined by an excluded topological minor. We conclude that a monotone graph class has bounded underlying treewidth if and only if it excludes some fixed topological minor.

Theorem 24. *For every fixed multigraph X with p vertices, every X -topological minor-free graph G of treewidth k has p -tree-partition-width $O(k^2)$.*

Proof. By Lemma 8 it suffices to show that the singleton partition of G is (p, f) -disjointed, where $f(n) \in O(kn)$. To this end, let S_1, \dots, S_p be subsets of $V(G)$ of size at most n , let $S := S_1 \cup \dots \cup S_p$, and for each $i \in \{1, \dots, p\}$ let $S'_i := S_i \setminus (S_1 \cup \dots \cup S_{i-1})$. We may assume that $V(X) = \{1, \dots, p\}$. Let \mathcal{H} be the set of connected subgraphs H of $G - S$ such that $V(H) \cap N_G(S'_i) \neq \emptyset$ for each $i \in \{1, \dots, p\}$.

Consider any set \mathcal{J} of pairwise vertex-disjoint graphs in \mathcal{H} of maximum size. Assign subgraphs in \mathcal{J} to pairs of vertices in S as follows. Initially, no subgraphs in \mathcal{J} are assigned. If there is an

unassigned subgraph $H \in \mathcal{J}$ adjacent to vertices $x \in S'_i$ and $y \in S'_j$, for some distinct $i, j \in \{1, \dots, p\}$, and no subgraph in \mathcal{J} is already assigned to $\{x, y\}$, then assign H to $\{x, y\}$. Repeat this operation until no more subgraphs in \mathcal{J} can be assigned.

Let \mathcal{J}_1 and \mathcal{J}_2 be the sets of assigned and unassigned subgraphs in \mathcal{J} respectively. Let G' be the graph obtained from G as follows: for each $H \in \mathcal{J}_1$ assigned to $\{x, y\}$, contract an (x, y) -path through H down to a single edge xy . Delete any remaining vertices not in S . Thus G' is a topological minor of G with $V(G') = S$ and $|E(G')| \geq |\mathcal{J}_1|$. Consider a subgraph $H \in \mathcal{J}_2$. Since $H \in \mathcal{H}$ there are neighbours v_1, \dots, v_p of H with $v_i \in S'_i$ for each $i \in \{1, \dots, p\}$. Since $H \in \mathcal{J}_2$, the set $\{v_1, \dots, v_p\}$ is a clique in G' (otherwise H could have been assigned to some non-adjacent v_i and v_j). Charge H to (v_1, \dots, v_p) .

Suppose that there is a set \mathcal{J}' of at least $|E(X)|$ subgraphs in \mathcal{J}_2 charged to some clique (v_1, \dots, v_p) in G' with $v_i \in S'_i$. Label $\mathcal{J}' = \{H_e : e \in E(X)\}$. For $e = ij \in E(X)$, let P_e be a (v_i, v_j) -path through H_e . Thus $\{v_i P_e v_j : e = ij \in E(X)\}$ defines an X -topological minor in G with branch vertices v_1, \dots, v_p . This contradiction shows that there are fewer than $|E(X)|$ subgraphs in \mathcal{J}_2 charged to each clique (v_1, \dots, v_p) in G' with $v_i \in S'_i$.

Since G' is X -topological-minor-free, $|E(G')| \leq \gamma|S|$ for some $\gamma = \gamma(X)$; see [10, 70] for explicit bounds on γ . Thus $|\mathcal{J}_1| \leq \gamma|S|$. Moreover, G' is 2γ -degenerate. Every d -degenerate graph on n vertices has at most $\binom{d}{p-1}n$ cliques of order p [78, Lemma 18]. Thus $|\mathcal{J}_2| < |E(X)|\binom{2\gamma}{p-1}|S|$. Hence $|\mathcal{J}| = |\mathcal{J}_1| + |\mathcal{J}_2| < (\gamma + |E(X)|\binom{2\gamma}{p-1})|S|$. Define

$$f(n) := (k + 1) \left(\gamma + |E(X)|\binom{2\gamma}{p-1} \right) np.$$

Since $|S| \leq pn$, by Lemma 16, there is a set $Q \subseteq V(G - S)$ of size at most $f(n)$ such that no subgraph of $G - (S \cup Q)$ is in \mathcal{H} . So every component Y of $G - (S \cup Q)$ satisfies $Y \cap N_G(S'_i) = \emptyset$ for some $i \in \{1, \dots, p\}$. Hence the singleton partition of G is (p, f) -disjointed. \square

Theorem 25. *The underlying treewidth of the class of K_t -topological-minor-free graphs equals $t - 2$ if $t \in \{2, 3, 4\}$ and equals t if $t \geq 5$.*

Proof. For $t \in \{2, 3, 4\}$, a graph is K_t -topological-minor-free if and only if it has treewidth at most $t - 2$. So the result follows from Corollary 13. Now assume that $t \geq 5$.

Theorem 24 implies that the underlying treewidth of the class of K_t -topological-minor-free graphs is at most t . Suppose for the sake of contradiction that it is at most $t - 1$. That is, there is a function f such that every K_t -topological-minor-free graph G has $(t - 1)$ -tree-partition-width at most $f(\text{tw}(G))$. Let $\ell := f(t)$. Let G_1 be the graph $G_{t-2,\ell}$ from Lemma 12. So $\text{tw}(G_1) = t - 2$, implying G_1 is K_t -topological-minor-free, and for every H -partition of G_1 with width ℓ , there is a $(t - 1)$ -clique in G_1 whose vertices are in distinct parts.

Let $n \gg t, \ell$ be a sufficiently large integer as detailed below. Let G_2 be the graph with vertex-set $\{v_1, \dots, v_{t-1}\} \cup \{x_1, \dots, x_n\}$ where $\{v_1, \dots, v_{t-1}\}$ is a clique, x_1, \dots, x_n is a path, and $v_i x_j$ is an edge whenever $j = a(t - 1) + i$ for some $a \in \mathbb{N}_0$. Each subpath $x_{a(t-1)+1}, \dots, x_{a(t-1)+(t-1)}$ is called a *clump*. By construction, each vertex v_i has one neighbour in every clump. Since each vertex x_j has degree at most 3, the graph G_2 has exactly $t - 1$ vertices of degree at least 4. Thus G_2 contains no K_t -topological minor (since $t \geq 5$). Let $X_j := \{v_1, \dots, v_{t-1}, x_j, x_{j+1}\}$. Then $(X_1, X_2, \dots, X_{n-1})$ is a path-decomposition of G_2 with width t ; thus $\text{tw}(G_2) \leq t$.

Let G be obtained by pasting a copy of G_2 onto each $(t - 1)$ -clique C of G_1 , where the vertices v_1, \dots, v_{t-1} in G_2 are identified with C . Since each of G_1 and G_2 have treewidth at most t , so too does G . Since each of G_1 and G_2 contain no K_t -topological-minor, G contains no K_t -topological minor. Thus G has a $(t - 1)$ -tree-partition $\mathcal{P} = (V_h : h \in V(H))$ of width ℓ .

Since G_1 is a subgraph of G , there is a $(t - 1)$ -clique C in G_1 whose vertices are in distinct parts of \mathcal{P} . Let C' be the set of parts in \mathcal{P} intersecting C . Thus $|C'| = |C| = t - 1$. Let P_0 be the path in the copy of G_2 pasted onto C in G . At most $(t - 1)(\ell - 1)$ of the vertices in P_0 are in the parts of \mathcal{P}

in C' . Thus, for $n \gg t, \ell$, there is a subpath P_1 of P_0 that contains at least $(t - 1)\ell + 2$ clumps, and no vertex of P_1 is in the parts of \mathcal{P} in C' . Let A be the first clump in P_1 . Let A' be the parts of \mathcal{P} that intersect A . Since $G[A]$ is connected, $H[A']$ is connected. Since $|A'| \leq |A| = t - 1$, there are at most $(t - 1)\ell$ vertices in the parts in A' . Thus at most $(t - 1)\ell$ clumps in P_1 intersect parts in A' . Hence P_1 contains a clump B that intersects no part in A' . Let B' be the parts in \mathcal{P} that intersect B . Thus $A' \cap B' = \emptyset$. Since $G[B]$ is connected, $H[B']$ is connected. Since P_1 is connected and no vertex of P_1 is in a part in C' , there is a path Q in $H - C'$ from A' to B' , with no internal vertex of Q in $A' \cup B'$. Let H' be obtained from H by contracting $H[A']$ to a vertex a , contracting $H[B']$ to a vertex b , and contracting Q down to an edge ab . Each vertex in C' has a neighbour in A and in B . Thus $C' \cup \{a, b\}$ is a $(t + 1)$ -clique in H' . Thus H contains K_{t+1} as a minor, contradicting that $\text{tw}(H) \leq t - 1$. Therefore the underlying treewidth of the class of K_t -topological-minor-free graphs equals t . \square

The proof of Theorem 19 is easily adapted to show that for $s \leq 3$ and fixed t , every $K_{s,t}$ -topological-minor-free graph G of treewidth k has s -tree-partition-width $O(k^2)$; see [12]. So the underlying treewidth of the class of $K_{s,t}$ -topological minor-free graphs equals s if $s \leq 3$ and $t \geq \max\{s, 3\}$. But the proof does not generalise for $s \geq 4$. Determining the underlying treewidth of the class of $K_{s,t}$ -topological-minor-free graphs is an interesting open problem. Also, in the $s = 2$ case we can use Menger's Theorem instead of Lemma 16 to show that every $K_{2,t}$ -topological-minor-free graph of treewidth k has 2-tree-partition-width $O(t^2k)$; see [12]. A result of Leaf and Seymour [52, (4.4)] implies that every $K_{2,t}$ -minor-free graph has treewidth at most $3t/2$. Thus every $K_{2,t}$ -minor-free graph has 2-tree-partition-width $O(t^3)$. It is open whether every $K_{2,t}$ -minor-free graph has 2-tree-partition-width $O(t)$, which would be best possible by Observation 1. It is easily seen that every $K_{1,t}$ -minor-free graph has tree-partition-width $O(t)$; see [12].

We now show that c -tree-partition-width is well-behaved under subdivisions (for $c \geq 1$).

Lemma 26. *For $c, t \in \mathbb{N}$, if \tilde{G} is a subdivision of a graph G with $\text{tpw}_c(G) \leq t$, then $\text{tpw}_c(\tilde{G}) \leq t^2 + t$ if $c = 1$ and $\text{tpw}_c(\tilde{G}) \leq t$ if $c \geq 2$.*

Proof. We first prove the $c = 1$ case. If $t = 1$, then G is a forest. So \tilde{G} is a forest and the claim follows. Now assume $t \geq 2$. Let $(V_x : x \in V(T))$ be a tree-partition of G with width at most t . We construct a tree T' from T and a T' -partition of \tilde{G} iteratively as follows. Orient the edges of T so that each vertex has in-degree at most 1. Say an edge vw of G is replaced by the (v, w) -path $v_0, v_1, \dots, v_s, v_{s+1}$ in \tilde{G} . If v and w are in the same part V_x (where $x \in V(H)$), then add a path of bags $V_x, \{v_1, v_s\}, \{v_2, v_{s-1}\}, \dots$ to T' . If $v \in V_x$ and $w \in V_y$ then $xy \in E(T)$. Assume xy is directed from x to y . Add v_1 to V_y and add a path of bags $V_y, \{v_2, v_s\}, \{v_3, v_{s-2}\}, \dots$ to T' . Since $|\{uw \in E(G) : u \in V_x, w \in V_y\}| \leq t^2$, we have $|V_x| \leq t^2 + t$ for all $x \in V(T')$. This defines a tree-partition of \tilde{G} with width at most $t^2 + t$.

Now consider the $c \geq 2$ case. If $t = 1$, then $\text{tw}(G) \leq c$. Since subdividing edges does not increase treewidth, $\text{tw}(\tilde{G}) \leq c$, and the claim follows. Now assume $t \geq 2$. Let $(V_x : x \in V(H))$ be an H -partition of G with width at most t where $\text{tw}(H) \leq c$. We construct a graph H' of treewidth at most c from H and a H' -partition of \tilde{G} iteratively as follows. Say an edge vw of G is replaced by the (v, w) -path $v_0, v_1, \dots, v_s, v_{s+1}$ in \tilde{G} . If v and w are in the same part V_x (where $x \in V(H)$), then add a path of bags $V_x, \{v_1, v_s\}, \{v_2, v_{s-1}\}, \dots$ to H' . If $v \in V_x$ and $w \in V_y$ then $xy \in E(H)$; add a path of bags $\{v_1, v_s\}, \{v_2, v_{s-1}\}, \dots$ to H' , where $\{v_1, v_s\}$ is adjacent to x and y in H' . This defines a c -tree-partition of \tilde{G} with width at most t , as desired. \square

Lemma 27. *For all $c \in \mathbb{N}$, for every graph G and every subdivision G' of G ,*

$$\text{tpw}_c(G) \leq 4c^2 12^c (c \text{tpw}_c(G') + 1) \text{tpw}_c(G')^2 (\text{tw}(G') + 1)^c.$$

Proof. Let $\ell := \text{tpw}_c(G')$. By Lemma 6, G' has a $(c, c\ell)$ -disjointed ℓ -partition β' . Let $\beta := \{B' \cap V(G) : B' \in \beta'\}$ be the corresponding ℓ -partition of G . We show that β is $(c, 2c\ell(c\ell + 1))$ -disjointed. The result then follows from Corollary 10 as $\text{tw}(G') = \text{tw}(G)$.

Let $B_1, \dots, B_c \in \beta$ be distinct non-empty parts and let B'_1, \dots, B'_c be the corresponding parts in β' . Let $B := \bigcup\{B_i : i \in \{1, \dots, c\}\}$ and $B' := \bigcup\{B'_i : i \in \{1, \dots, c\}\}$. Observe that every vertex in $B' - B$ is an internal vertex of some subdivided edge in G . Let $\tilde{Q} \subseteq V(G)$ be the set of end-vertices of these subdivided edges, so $|\tilde{Q}| \leq 2|B' - B| < 2c\ell$.

Let X be a component of $G - B$ and let Y'_1, \dots, Y'_m be the components of $G' - B'$ that intersect X . Since β' is $(c, c\ell)$ -disjointed, for each j there is some $Q'_j \subseteq V(Y'_j)$ of size at most $c\ell$ such that every component of $Y'_j - Q'_j$ is disjoint from some $N_{G'}(B'_i)$. Define Q_j as follows. For each $w \in Q'_j$ either $w \in V(G)$ or w is an internal vertex of some subdivided edge uv in G . In the first case add w to Q_j , and in the second case add u and v to Q_j . Thus $|Q_j| \leq 2c\ell$.

Let $Q := (\bigcup\{Q_j : j \in \{1, \dots, m\}\} \cup \tilde{Q}) \cap V(X)$. Then $|Q| \leq 2m\ell + 2c\ell = 2c\ell(m + 1)$. Let Z be a component of $X - Q$. Consider any path P within Z : in G' this is subdivided to give a path P' . As P avoids \tilde{Q} , the path P' avoids B' . Hence $V(Z)$ is contained in some component of $G' - B'$ and so within some Y'_j . Further, P avoids Q_j , so P' avoids Q'_j . Hence $V(Z)$ is contained in some component Z' of $Y'_j - Q'_j$. By the definition of Q'_j , there is some i with $N_{G'}(B'_i) \cap V(Z') = \emptyset$.

We claim that $N_G(B_i) \cap V(Z) = \emptyset$. Indeed suppose $u \in B_i$ and $v \in V(Z)$ are adjacent and consider the corresponding (u, v) -path P' in G' . Since $v \notin \tilde{Q}$, the path $P' - u$ avoids B' and so is within Y'_j (as $v \in V(Z) \subseteq V(Y'_j)$). As $v \notin Q_j$, the path $P' - u$ avoids Q'_j . Hence $P' - u$ is a connected subgraph of $Y'_j - Q'_j$ so is within Z' (as $v \in V(Z) \subseteq V(Z')$). But then $N_{G'}(u) \cap V(Z') \neq \emptyset$, a contradiction. Hence β is $(c, 2c\ell(m + 1))$ -disjointed. We are left to show that $m \leq c\ell$.

For each $i \in \{1, \dots, m\}$, let $v_i \in V(Y'_i) \cap V(X)$. For distinct $i, j \in \{1, \dots, m\}$, let $P_{i,j}$ be a (v_i, v_j) -path in X . Since Y'_i and Y'_j are distinct components in $G' - B'$, there exist a vertex $w \in B' - B$ that is an internal vertex of a subdivided edge in $P_{i,j}$. Therefore, $m - 1 \leq |B' - B| < c\ell$. □

Theorem 28. *A monotone graph class \mathcal{G} has bounded underlying treewidth if and only if \mathcal{G} excludes some fixed topological minor.*

Proof. (\Leftarrow) Say \mathcal{G} excludes some fixed topological minor X on p vertices. By Theorem 24, every graph $G \in \mathcal{G}$ has p -tree-partition-width at most $O(\text{tw}(G)^2)$. Thus \mathcal{G} has underlying treewidth at most p .

(\Rightarrow) Say \mathcal{G} has underlying treewidth c . That is, there is a function f such that $\text{tpw}_c(G) \leq f(\text{tw}(G))$ for every graph $G \in \mathcal{G}$. For any $n \in \mathbb{N}$, by Lemma 12, there is a graph H of treewidth $c + 1$ and $\text{tpw}_c(H) > n$. Suppose for the sake of contradiction that some subdivision H' of H is in \mathcal{G} . Thus $\text{tpw}_c(H') \leq f(\text{tw}(H')) = f(\text{tw}(H)) = f(c + 1)$. By Lemma 27,

$$\begin{aligned} n < \text{tpw}_c(H) &\leq 4c^2 12^c (c \text{tpw}_c(H') + 1) \text{tpw}_c(H')^2 (\text{tw}(H') + 1)^c \\ &\leq 4c^2 12^c (cf(c + 1) + 1) (f(c + 1))^2 (c + 2)^c. \end{aligned}$$

We obtain a contradiction taking $n \gg f(c + 1)$. Thus no subdivision of H is in \mathcal{G} . Since \mathcal{G} is monotone, \mathcal{G} excludes H as a topological minor, as desired. □

7. Excluding a subgraph

For a graph H , a graph G is H -free if G contains no subgraph isomorphic to H . For a finite set of graphs \mathcal{H} , we say that G is \mathcal{H} -free if G is H -free for all $H \in \mathcal{H}$. Let \mathcal{G}_H be the class of H -free graphs and let $\mathcal{G}_{\mathcal{H}}$ be the class of \mathcal{H} -free graphs. This section characterises when $\mathcal{G}_{\mathcal{H}}$ has bounded underlying treewidth, and determines the exact underlying treewidth for several natural classes. A *spider* is a subdivision of a star and a *spider-forest* is a subdivision of a star-forest. For $s, t \in \mathbb{N}$ with

$s \geq 2$, the (s, t) -spider, denoted $S_{s,t}$, is the $(t - 1)$ -subdivision of $K_{1,s}$. If v is the centre of $S_{s,t}$, then each component of $S_{s,t} - v$ is called a leg.

Theorem 29. For all $\ell, n, s, t \in \mathbb{N}$ where $n, s \geq 3$ and $\ell \geq 2$, and for every finite set \mathcal{H} of graphs,

- (a) $\mathcal{G}_{\mathcal{H}}$ has bounded underlying treewidth if and only if \mathcal{H} contains a spider-forest;
- (b) the underlying treewidth of \mathcal{G}_{P_n} equals $\lfloor \log n \rfloor - 1$;
- (c) the underlying treewidth of $\mathcal{G}_{\ell P_n}$ equals $\lfloor \log n \rfloor$;
- (d) the underlying treewidth of $\mathcal{G}_{S_{s,t}}$ equals $\lfloor \log t \rfloor + 1$;
- (e) the underlying treewidth of $\mathcal{G}_{\ell S_{s,t}}$ equals $\lfloor \log t \rfloor + 2$.

We prove Theorem 29 through a sequence of lemmas.

Lemma 30. For every finite set \mathcal{H} of graphs, if $\mathcal{G}_{\mathcal{H}}$ has bounded underlying treewidth, then \mathcal{H} contains a spider-forest.

Proof. For the sake of contradiction, suppose that \mathcal{H} does not contain a spider-forest. Then each graph $H \in \mathcal{H}$ contains a cycle or two vertices with degree at least 3 in the same component of H . Let s_H be the minimum of the girth of H and the minimum distance between distinct vertices with degree at least 3 in the same component of H . Let $s := \max\{s_H : H \in \mathcal{H}\}$ and let \mathcal{G} be any graph class with unbounded underlying treewidth (such as that in Corollary 14). Let \mathcal{G}' be the class of $(s + 1)$ -subdivisions of graphs in \mathcal{G} . Then \mathcal{G}' is H -free for all $H \in \mathcal{H}$ and has unbounded underlying treewidth by Lemma 27, a contradiction. \square

Lemma 30 proves the necessity of (i) in Theorem 29.

We now work towards showing \mathcal{G}_H has bounded underlying treewidth when H is a spider. For a graph G and $\lambda \in \mathbb{N}$, let $G^{(\lambda)}$ be the graph with $V(G^{(\lambda)}) = V(G)$ and $uv \in E(G^{(\lambda)})$ whenever there are λ internally disjoint (u, v) -paths in G .

Lemma 31. For all $k, \lambda \in \mathbb{N}$ where $\lambda \geq k + 1$, if a graph G has treewidth at most k , then $\text{tw}(G^{(\lambda)}) \leq k$.

Proof. Let $\mathcal{W} := (W_x : x \in V(T))$ be a tree-decomposition of G with width k . We may assume that $W_x \neq W_y$ for each $xy \in E(T)$. We claim that \mathcal{W} is also a tree-decomposition of $G^{(\lambda)}$, for which it suffices to show that for every edge vw of $G^{(\lambda)}$, v and w are in a common bag of \mathcal{W} . To this end, suppose that $vw \in E(G^{(\lambda)})$ but there is no bag of \mathcal{W} containing both v and w . Then there is an edge $xy \in E(T)$ separating $\{z \in V(T) : v \in W_z\}$ and $\{z \in V(T) : w \in W_z\}$, which implies that $W_x \cap W_y$ is a set of at most k vertices separating v and w in G . On the other hand, since $vw \in E(G^{(\lambda)})$ and $\lambda \geq k + 1$, any vertex-set separating v and w must have at least $k + 1$ vertices, a contradiction. \square

Lemma 32. For all $s, t \in \mathbb{N}$ and $\lambda \geq 1 + s + st(2t + 1)$, if a graph G contains no $S_{s,2t+1}$ subgraph, then $G^{(\lambda)}$ contains no $S_{s,t}$ subgraph.

Proof. Suppose for contradiction that $S_{s,t}$ is a subgraph of $G^{(\lambda)}$. Since G does not contain $S_{s,2t+1}$ as a subgraph, there is no set of s internally disjoint paths of length at least $2t + 2$ in G between any pair of vertices. Hence, since $\lambda \geq 1 + s + st(2t + 1)$ we may greedily replace each edge uv of $S_{s,t}$ with a (u, v) -path in G of length between 2 and $2t + 1$ such that all these paths are internally vertex disjoint from each other, and each of these paths internally avoids $V(S_{s,t})$. This works because the number of vertices to avoid when replacing an edge $uv \in E(S_{s,t})$ is at most $1 + (2t + 1)(|E(S_{s,t}) \setminus \{uv\}|) = 1 + (2t + 1)(st - 1)$, and there is a collection of $\lambda - s \geq 1 + st(2t + 1)$ internally disjoint (u, v) -paths in G of length between 2 and $2t + 1$. Finally, since each leaf of $S_{s,t}$ has degree at least $\lambda \geq 1 + s + st(2t + 1)$ in G , we can extend each leg of the spider thus constructed by one further vertex by adding a distinct neighbour in G of each leaf of $S_{s,t}$. We obtain $S_{s,2t+1}$ as a subgraph of G , a contradiction. \square

Lemma 33. *There exists a function f such that for all $c, k, s, t, \lambda, \gamma \in \mathbb{N}$, if a graph G has treewidth at most k and no $S_{s,t}$ subgraph, and $\text{tpw}_c(G^{(\lambda)}) \leq \gamma$, then $\text{tpw}_{c+1}(G) \leq f(c, k, s, t, \lambda, \gamma)$.*

Proof. Let $d := st$, $\Delta := s \cdot (1 + \lambda^{2t})$, $m := (1 + \Delta^d) \cdot \gamma^{c+1}$ and $n := (k + 1)m$. Let $\beta := (V_h : h \in V(H))$ be an H -partition of $G^{(\lambda)}$ with width at most γ , where $\text{tw}(H) \leq c$. We show that β is a $(c + 1, n)$ -disjointed γ -covering of G . By Corollary 10, G has $(c + 1)$ -tree-partition-width at most $f(c, k, s, t, \lambda, \gamma) := 2(c + 1) \cdot n \cdot \gamma(12k)^c$.

Consider distinct $V_1, \dots, V_{c+1} \in \beta$ and a connected component X of $G - (V_1 \cup \dots \cup V_{c+1})$. Let \mathcal{F} be the collection of connected subgraphs in X that intersect $N_G(V_i)$ for each $i \in \{1, \dots, c + 1\}$. Let \mathcal{R}' be a maximum-sized set of vertex-disjoint elements of \mathcal{F} . By Lemma 16, it suffices to show that $|\mathcal{R}'| < m$.

We make two simplifying assumptions. First, we may assume that each V_i consists of a single vertex at the cost of a factor of γ^{c+1} : by averaging over all $(c + 1)$ -tuples in $V_1 \times \dots \times V_{c+1}$, there is one such tuple (x_1, \dots, x_{c+1}) and a subset $\mathcal{R} \subset \mathcal{R}'$ such that $|\mathcal{R}'| \leq \gamma^{c+1}|\mathcal{R}|$ and $N_G(x_i) \cap V(R) \neq \emptyset$ for each $R \in \mathcal{R}$ and $i \in \{1, \dots, c + 1\}$. Thus it suffices to show that $|\mathcal{R}| \leq 1 + \Delta^d$. Second, we may assume that $\bigcup\{V(R) : R \in \mathcal{R}\} = V(X)$.

Let X' be the quotient graph of \mathcal{R} (with respect to X). If $|V(X')| < \lambda$, then we are done. Thus we may assume that $|V(X')| \geq \lambda$ and so the vertices $\{x_1, \dots, x_{c+1}\}$ form a $(c + 1)$ -clique in $G^{(\lambda)}$. To show that $|\mathcal{R}| = |V(X')| \leq 1 + \Delta^d$, we use the well-known fact that every graph with diameter less than d and maximum degree Δ has at most $1 + \Delta^d$ vertices [55].

Claim 2. *X' contains no path on d vertices and thus has diameter less than d .*

Proof. For the sake of contradiction, suppose that X' contains a path $P' = (R_0, \dots, R_{d-1})$ on d vertices. Consider the vertex-disjoint paths $(P'_j : j \in \{1, \dots, s\})$ in X' where $P'_j := (R_{(j-1)t}, \dots, R_{jt})$. For each $j \in \{1, \dots, s\}$ fix $u_j \in V(R_{(j-1)t}) \cap N_G(x_1)$. Then there is a (u_j, v_j) -path P_j in $X[\bigcup(V(R) : R \in V(P'_j))]$ of length at least $t - 1$ for some vertex $v_j \in V(R_{jt})$. Since the paths $(P_j : j \in \{1, \dots, s\})$ are pairwise vertex-disjoint, it follows that $G[\{x_1\} \cup \bigcup(V(P_j) : j \in \{1, \dots, s\})]$ contains $S_{s,t}$ (see Figure 3a), a contradiction. \square

It remains to show that X' has maximum degree less than Δ . For the sake of contradiction, suppose that X' contains a vertex R' with degree at least Δ . Let R_1, \dots, R_Δ be neighbours of R' and for each $i \in \{1, \dots, \Delta\}$, let $y_i \in V(R_i)$ be adjacent to some $w_i \in V(R')$. Let $L := \{y_i : i \in \{1, \dots, \Delta\}\}$ and let R^* be obtained from R' by adding L and $\{y_i w_i : i \in \{1, \dots, \Delta\}\}$. Let T be a vertex-minimal tree in R^* such that $L \subseteq V(T)$. Then L is precisely the set of leaves of T .

Claim 3. *T has maximum degree less than λ .*

Proof. For the sake of contradiction, suppose there is a vertex $v \in V(T)$ with degree (in T) at least λ . Then there exists a set of λ internally vertex-disjoint paths from v to distinct leaves y_1, \dots, y_λ in T . Then for each $i \in \{1, \dots, c + 1\}$ and $j \in \{1, \dots, \lambda\}$, since R_j is connected, there is an (x_i, y_j) -path in $G[V(R_j) \cup \{x_i\}]$. As such, there exists λ internally disjoint (v, x_i) -paths in G for each $i \in \{1, \dots, c + 1\}$ (see Figure 3b). Thus $v x_i \in E(G^{(\lambda)})$. So $\{x_1, \dots, x_{c+1}, v\}$ is a $(c + 2)$ -clique in $G^{(\lambda)}$. However, as each vertex of this clique belongs to a different part in β , it follows that H contains a $(c + 2)$ -clique, which contradicts $\text{tw}(H) \leq c$. \square

Thus, balls of radius r (in T) have fewer than $1 + \lambda^r$ elements. Since $|L| = \Delta = s \cdot (1 + \lambda^{2t})$, there are vertices $y_1, \dots, y_s \in L$ whose balls of radius t in T are pairwise disjoint. Let $u_i \in V(R_i) \cap N_G(x_1)$ and let $v_i \in V(T)$ be a vertex with distance (in T) exactly t from y_i . By construction, there exists a set of disjoint paths $(P_i : i \in \{1, \dots, s\})$ such that P_i is a (u_i, v_i) -path of length at least t . Then, $G[\{x_1\} \cup \bigcup(V(P_i) : i \in \{1, \dots, s\})]$ contains $S_{s,t}$ (see Figure 3c), a contradiction. Thus X' has maximum degree less than Δ . \square

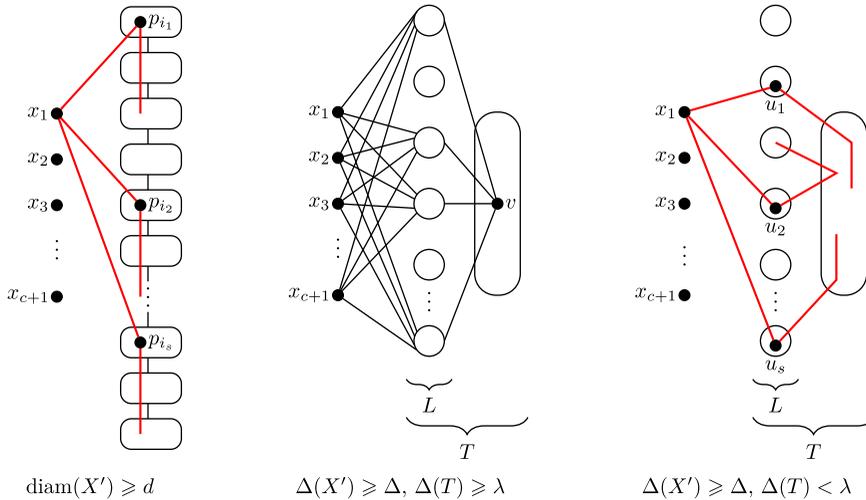


Figure 3. Finding spiders and cliques in the proof of Lemma 33.

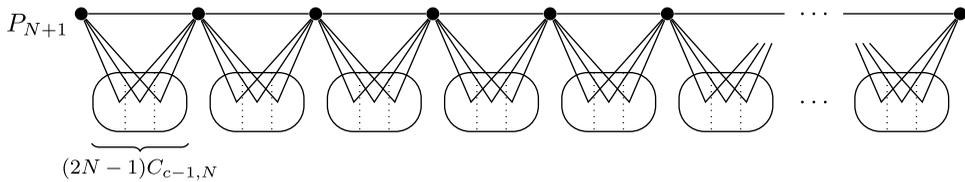


Figure 4. Construction of $J_{c,N}$.

Lemma 34. For each $n \in \mathbb{N}$ with $n \geq 3$, the underlying treewidth of \mathcal{G}_{P_n} equals $\lfloor \log n \rfloor - 1$.

Proof. Let $c := \lfloor \log n \rfloor$. For the lower bound, Lemma 12 (ii) and (vii) imply that $\{C_{c-1,\ell} : \ell \in \mathbb{N}\}$ has underlying treewidth at least $c - 1$ and is a subset of \mathcal{G}_{P_n} . The lower bound follows.

We prove the upper bound by induction on $c \geq 1$ with the hypothesis that there is a non-decreasing function f such that every $P_{2^{c+1}-1}$ -free graph G has $(c - 1)$ -tree-partition-width at most $f(\text{tw}(G), c)$. Since $P_n \subseteq P_{2^{c+1}-1}$, the claim follows. We make no attempt to optimise f . When $c = 1$, G contains no P_3 and so each component has at most two vertices. Therefore, $\text{tpw}_0(G) \leq f(\text{tw}(G), 1) := 2$.

Now assume $c \geq 2$ and the lemma holds for $c - 1$. Let G be a graph with no $P_{2^{c+1}-1}$ and let $\lambda := \max\{3 + (2^c - 2)(2^c - 1), \text{tw}(G) + 1\}$. By Lemmas 31 and 32, $G^{(\lambda)}$ has treewidth at most $\text{tw}(G)$ and contains no P_{2^c-1} . By induction, $\text{tpw}_{c-2}(G^{(\lambda)}) \leq \gamma := f(\text{tw}(G), c - 1)$. By Lemma 33, $\text{tpw}_{c+1}(G) \leq f(\text{tw}(G), c) := \tilde{f}(c, \text{tw}(G), 2, 2^c - 1, \lambda, \gamma)$ where \tilde{f} is from Lemma 33, as required. \square

Lemma 34 proves (ii) in Theorem 29.

To prove (iv) in Theorem 29, we need the following lower bound result.

Lemma 35. For all $c \in \mathbb{N}$, there exists an $S_{3,2^c}$ -free graph class $\{J_{c,N} : N \in \mathbb{N}\}$ with underlying treewidth $c + 1$.

Proof. Let $J_{c,N}$ be the graph obtained from a path $P_{N+1} = (p_1, \dots, p_{N+1})$ where for each edge $p_i p_{i+1} \in E(P_{n+1})$, we add a copy X_i of $(2N - 1)C_{c-1,N}$ and let p_i and p_{i+1} dominate X_i (see Figure 4; this construction also provides a lower bound on the clustered chromatic number [58]). To prove the lemma, it suffices to show that $J_{c,N}$ is $S_{3,2^c}$ -free, $\text{tw}(J_{c,N}) = c + 1$ and $\text{tpw}_c(J_{c,N}) > N$.

The claimed treewidth of $J_{c,N}$ holds since $\text{tw}((2N - 1)C_{c-1,N}) = c - 1$ (Lemma 12 (i)), and the two dominant vertices for each copy of $(2N - 1)C_{c-1,N}$ give $\text{tw}(J_{c,N}) = c + 1$. Now suppose, for the sake of contradiction, that $S_{3,2^c}$ is a subgraph of $J_{c,N}$ with centre $v \in V(J_{c,N})$. Observe that v has at most two neighbours p_i and p_j in P_N . Moreover, each neighbour of v in $J_{c,N} \setminus \{v, p_i, p_j\}$ is contained in a component that is a subgraph of $C_{c-1,N}$. Since $S_{3,2^c}$ has three legs, one of the legs must be contained in a copy of $C_{c-1,N}$ which contradicts $C_{c-1,N}$ not having a path on 2^c vertices (Lemma 12 (vii)).

It remains to show that $\text{tpw}_c(J_{c,N}) > N$. For the sake of contradiction, suppose that $J_{c,N}$ has a c -tree-partition $\beta := (V_h : h \in V(H))$ with width at most N . Since $|V(P_{N+1})| = N + 1$, there exists an edge $p_i p_{i+1} \in E(P_{N+1})$ such that p_i and p_{i+1} belong to different parts in β . Let V_i and V_{i+1} be such that $p_i \in V_i$ and $p_{i+1} \in V_{i+1}$. Since $|(V_i \cup V_{i+1}) \setminus \{p_i, p_{i+1}\}| \leq 2N - 2$, there exists a copy of $C_{c-1,N}$ in X_i such that $V(C_{c-1,N}) \cap (V_i \cup V_{i+1}) = \emptyset$. By Lemma 12 (ii), $C_{c-1,N}$ contains a c -clique whose vertices are in different parts in β . Together with p_i and p_{i+1} , it follows that $J_{c,N}$ contains a $(c + 2)$ -clique whose vertices are in different parts in β . Thus H has a $(c + 2)$ -clique, a contradiction. \square

Lemma 36. *For all $s, t \in \mathbb{N}$ where $s \geq 3$, the underlying treewidth of $\mathcal{G}_{S_{s,t}}$ is $\lfloor \log t \rfloor + 1$.*

Proof. Let $c := \lfloor \log t \rfloor$. For the lower bound, $\mathcal{G}_{S_{3,2^c}}$ has underlying treewidth at least $c + 1$ by Lemma 35. Since $S_{3,2^c} \subseteq S_{s,t}$, it follows that $\mathcal{G}_{S_{3,2^c}} \subseteq \mathcal{G}_{S_{s,t}}$, which implies the lower bound.

We prove the upper bound by induction on $c \in \mathbb{N}_0$ with the following hypothesis: There is an increasing function f such that for each $s \in \mathbb{N}$, every $S_{s,2^{c+1}-1}$ -free graph G has $(c + 1)$ -tree-partition-width at most $f(\text{tw}(G), c, s)$. Since $S_{s,t} \subseteq S_{s,2^{c+1}-1}$, the claim follows. When $c = 0$, G has maximum degree at most $s - 1$, and G has 1-tree-partition-width at most $f(\text{tw}(G), 0, s) := 24(\text{tw}(G) + 1)(s - 1)$ by Lemma 2.

Now assume $c \geq 1$ and the lemma holds for $c - 1$. Let G be an $(s, 2^{c+1} - 1)$ -spider-free graph and let $\lambda := \max\{1 + s + st(2t + 1), \text{tw}(G) + 1\}$. By Lemmas 31 and 32, $\text{tw}(G^{(\lambda)}) \leq \text{tw}(G)$ and $G^{(\lambda)}$ contains no $S_{s,2^c-1}$. By induction, $\text{tpw}_c(G^{(\lambda)}) \leq \gamma := f(\text{tw}(G), c, s)$. By Lemma 33, $\text{tpw}_{c+1}(G) \leq f(\text{tw}(G), c, s) := \tilde{f}(c, \text{tw}(G), s, 2^{c+1} - 1, \lambda, \gamma)$ where \tilde{f} is from Lemma 33, as required. \square

Lemma 36 proves (iv) in Theorem 29.

Lemma 37. *For every connected graph H and $\ell \in \mathbb{N}$ where $\ell \geq 2$, if \mathcal{G}_H has underlying treewidth c then $\mathcal{G}_{\ell H}$ has underlying treewidth $c + 1$.*

Proof. We first prove the lower bound. Since \mathcal{G}_H has underlying treewidth c , there exists $k \in \mathbb{N}$ such that for all $j \in \mathbb{N}$, there is a graph $G_j \in \mathcal{G}_H$ with $\text{tw}(G_j) = k$ and $\text{tpw}_{c-1}(G_j) > j$. Consider the graph class $\mathcal{J} := \{\widehat{jG_j} : j \in \mathbb{N}\}$. Observe that every graph in \mathcal{J} is ℓH -free, so $\mathcal{J} \subseteq \mathcal{G}_{\ell H}$. Moreover, since adding a dominant vertex increase the treewidth of a graph by 1, $\text{tw}(j\widehat{G_j}) = k + 1$ for all $j \in \mathbb{N}$. Finally, consider a j -partition $\beta := (V_x : x \in V(J))$ of $j\widehat{G_j}$. At most $j - 1$ copies of G_j contain a vertex in the same part V_y as the dominant vertex v . Thus, some copy G of G_j contains no vertex in V_y . Consider the sub-partition $\beta' := (V_x \cap V(G) : x \in V(J'))$ of G where $J' \subseteq J$. Since β' has width at most j , J' has treewidth at least c and so $J[V(J') \cup \{y\}]$ has treewidth at least $c + 1$. Thus \mathcal{J} has underlying treewidth at least $c + 1$.

We now prove the upper bound. Let G be an ℓH -free graph. Let A_1, \dots, A_m be a maximal set of pairwise disjoint copies of H in G . Then $m < \ell$. Let $B := G - V(A_1 \cup \dots \cup A_m)$. By the maximality of m , B is H -free. Thus B has a c -tree-partition $(V_x : x \in V(J'))$ with width at most $\tilde{f}(\text{tw}(B))$ for some function \tilde{f} . Let J be the graph obtained from J' by adding a dominant vertex y , and let $V_y := V(A_1 \cup \dots \cup A_m)$. Then $(V_x : x \in V(J))$ is a $(c + 1)$ -tree-partition of G with width at most $f(\text{tw}(G)) := \max\{(\ell - 1)|V(H)|, \tilde{f}(1), \dots, \tilde{f}(\text{tw}(G))\}$, as required. \square

Lemmas 34, 36 and 37 prove Theorem 29 (iii) and (iv). Since every spider-forest is a subgraph of $\ell S_{s,t}$ for some $\ell, s, t \in \mathbb{N}$, these results also imply the sufficiency of (i) in Theorem 29.

8. Excluding an induced subgraph

For a graph H , let \mathcal{I}_H be the class of graphs with no induced subgraph isomorphic to H . This section characterises the graphs H such that \mathcal{I}_H has bounded underlying treewidth, and determines the precise underlying treewidth for each such H .

Theorem 38. *For any graph H ,*

- (a) \mathcal{I}_H has bounded underlying treewidth if and only if H is a star-forest;
- (b) if H is a star-forest, then \mathcal{I}_H has underlying treewidth at most 2;
- (c) \mathcal{I}_H has underlying treewidth at most 1 if and only if H is a star or each component of H is a path on at most three vertices;
- (d) \mathcal{I}_H has underlying treewidth 0 if and only if H is a path on at most three vertices, or $E(H) = \emptyset$.

We prove Theorem 38 by a sequence of lemmas.

Lemma 39. *If \mathcal{I}_H has bounded underlying treewidth, then H is a star-forest.*

Proof. Suppose that \mathcal{I}_H has underlying treewidth c . So \mathcal{G}_H has underlying treewidth at most c (since $\mathcal{G}_H \subseteq \mathcal{I}_H$). Lemma 30 implies that H is a spider-forest. By Corollary 14, $\{C_{c,\ell} : c, \ell \in \mathbb{N}\}$ has unbounded underlying treewidth. Hence, H is an induced subgraph of some $C_{c,\ell}$. By Lemma 12, $C_{c,\ell}$ has no induced P_4 and so H contains no induced P_4 . Every spider-forest with no induced P_4 is a star-forest. So H is a star-forest. □

Lemma 39 proves the necessity of (i) in Theorem 38.

Lemma 40. *If H is a star $K_{1,s}$, then \mathcal{I}_H has underlying treewidth at most 1.*

Proof. Say $G \in \mathcal{I}_H$. For each vertex $v \in V(G)$, if $G_v := G[N_G(v)]$ then $\alpha(G_v) \leq s - 1$ and $\chi(G_v) \leq \text{tw}(G_v) + 1 \leq \text{tw}(G)$, implying $\deg_G(v) = |V(G_v)| \leq \chi(G_v)\alpha(G_v) \leq \text{tw}(G)(s - 1)$. By Lemma 2, $\text{tpw}(G) \leq 24\text{tw}(G)^2(s - 1)$. Hence \mathcal{I}_H has underlying treewidth at most 1. □

Lemma 41. *If H is a star-forest, then \mathcal{I}_H has underlying treewidth at most 2.*

Proof. Say H has ℓ components, each with at most s leaves. Let $G \in \mathcal{I}_H$. Let A_1, \dots, A_n be a maximal set of pairwise disjoint induced $K_{1,s}$ subgraphs in G . Define J to be the graph obtained from $G[V(A_1 \cup \dots \cup A_n)]$ by contracting each A_i into a vertex v_i . Then $n \leq \chi(J)\alpha(J) \leq (\text{tw}(J) + 1)(\ell - 1) \leq (\text{tw}(G) + 1)(\ell - 1)$ since J is a minor of G . Thus $|V(A_1 \cup \dots \cup A_n)| \leq (\text{tw}(G) + 1)(\ell - 1)(s + 1)$. Let $B := G - V(A_1 \cup \dots \cup A_n)$. By the maximality of n , B contains no induced $K_{1,s}$. As in the proof of Lemma 40, B has a tree-partition $(V_x : x \in V(T))$ of width at most $24\text{tw}(G)^2(s - 1)$. Let H be the graph obtained from T by adding a dominant vertex y , and let $V_y := V(A_1 \cup \dots \cup A_n)$. Then $(V_h : h \in V(H))$ is a 2-tree-partition of G with width at most $\max\{24\text{tw}(G)^2(s - 1), (\text{tw}(G) + 1)(\ell - 1)(s + 1)\}$. Hence \mathcal{I}_H has underlying treewidth at most 2. □

Lemma 41 proves (ii) and the sufficiency of (i) in Theorem 38.

Lemma 42. *If H is a forest in which each component is a path on at most three vertices, then \mathcal{I}_H has underlying treewidth at most 1.*

Proof. Say H has k components. Let $G \in \mathcal{I}_H$. Let A_1, \dots, A_n be a maximal set of pairwise disjoint induced P_3 subgraphs in G . Let J be the minor of G obtained from $G[V(A_1 \cup \dots \cup A_n)]$ by contracting each A_i into a vertex. Thus $n = |V(J)| \leq \chi(J)\alpha(J) \leq (\text{tw}(J) + 1)(k - 1) \leq (\text{tw}(G) + 1)(k - 1)$. Let B_1, \dots, B_m be the components of $G - V(A_1 \cup \dots \cup A_n)$. By the maximality of n , each B_i has no induced P_3 subgraph, so B_i is a complete subgraph and $|V(B_i)| \leq \text{tw}(G) + 1$. Let T be the star with centre x and leaves y_1, \dots, y_m . Let $V_x := V(A_1 \cup \dots \cup A_n)$ and $V_{y_j} := V(B_j)$. So

$(V_h : h \in V(T))$ is a tree-partition of G with width at most $\max\{3(k-1)(\text{tw}(G)+1), \text{tw}(G)+1\}$. Hence \mathcal{I}_H has underlying treewidth at most 1. \square

Lemma 43. *If H is a star-forest with at least two components and at least one component of H has at least three leaves, then \mathcal{I}_H has underlying treewidth 2.*

Proof. Consider a star S with at least three leaves that is a subgraph of a fan F with dominant vertex v . Since $F - v$ has maximum degree 2, v is in S . Since every vertex of F is adjacent to v , H is not an induced subgraph of F . By Lemma 12, the class of fan graphs has underlying treewidth 2. Thus \mathcal{I}_H has underlying treewidth at least 2, with equality by Lemma 41. \square

Lemmas 40, 42 and 43 prove (iii) in Theorem 38.

Lemma 44. *If $E(H) = \emptyset$ or H is a path on at most three vertices, then \mathcal{I}_H has underlying treewidth 0.*

Proof. First suppose that $E(H) = \emptyset$. For each $G \in \mathcal{I}_H$, $\{V(G)\}$ is a 0-tree-partition of G with width $|V(G)| \leq \chi(G)\alpha(G) \leq (\text{tw}(G)+1)(|V(H)|-1)$. Hence \mathcal{I}_H has underlying treewidth 0. Now suppose H is a path on at most three vertices. For each $G \in \mathcal{I}_H$, if G_1, \dots, G_k are the connected components of G , then each G_i is a complete graph, implying $\{V(G_1), \dots, V(G_k)\}$ is a 0-tree-partition of G with width $\max_i |V(G_i)| = \text{tw}(G)+1$. \square

Lemma 45. *If H is a star-forest with at least one edge and at least two components, then \mathcal{I}_H has underlying treewidth at least 1.*

Proof. Let $G := K_{2,n}$. Then for every edge vw of G , every vertex $x \in V(G) \setminus \{v, w\}$ is adjacent to v or w . Thus $G \in \mathcal{I}_H$. Since G is connected, $\text{tpw}_0(G) = n+2$. Since $\text{tw}(G) \leq 2$, \mathcal{I}_H has underlying treewidth at least 1. \square

Lemmas 44 and 45 prove (iv) in Theorem 38.

We finish this section with an open problem. For any set \mathcal{X} of graphs, let $\mathcal{I}_{\mathcal{X}}$ be the class of graphs G such that no induced subgraph of G is isomorphic to a graph in \mathcal{X} . For which sets \mathcal{X} does $\mathcal{I}_{\mathcal{X}}$ have bounded underlying treewidth? Consider the case when \mathcal{X} is finite. If some star-forest is in \mathcal{X} , then $\mathcal{I}_{\mathcal{X}}$ has bounded underlying treewidth by Lemma 41. If no star-forest is in \mathcal{X} and $\mathcal{I}_{\mathcal{X}}$ still has bounded underlying treewidth, then some spider-forest is in \mathcal{X} (by Lemma 30), and some graph for which every component is the closure of a rooted tree is also in \mathcal{X} (by Corollary 14). In related work, Lozin and Razgon [54] proved a graph class defined by finitely many excluded induced subgraphs has bounded treewidth if and only if it excludes a complete graph, a complete bipartite graph, a tripod (a forest in which every connected component has at most three leaves), and the line graph of a tripod.

9. Graph drawings

A graph is k -planar if it has a drawing in the plane with at most k crossings on each edge, where we assume that no three edges cross at the same point. Of course, the class of 0-planar graphs is the class of planar graphs, which has underlying treewidth 3 (Theorem 21). However, 1-planar graphs behave very differently. It is well-known that every graph has a 1-planar subdivision: take an arbitrary drawing of G and for each edge e add a subdivision vertex between consecutive crossings on e . Since the class of 1-planar graphs is monotone, Theorem 28 implies that the class of 1-planar graphs has unbounded underlying treewidth.

By restricting the type of drawing, we obtain positive results. A *circular drawing* of a graph G positions each vertex on a circle in the plane, and draws each edge as an arc across the circle, such that no two edges cross more than once. A graph is *outer k -planar* if it has a circular drawing such that each edge is involved in at most k crossings. The outer 0-planar graphs are precisely the outerplanar graphs, which have treewidth 2. We show below that for each $k \in \mathbb{N}$, the class of

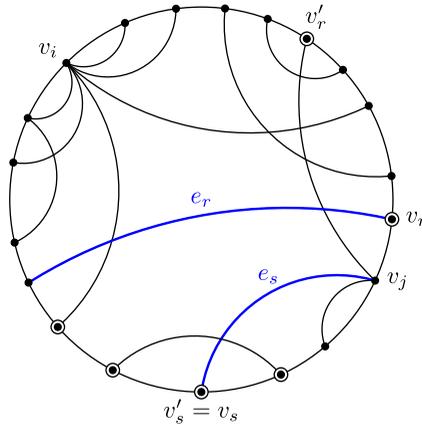


Figure 5. Setup to define Q (circled vertices) with $k = 2$.

outer k -planar graphs has underlying treewidth 2. In fact, we prove a slightly more general result. A graph is *weakly outer k -planar* if it has a circular drawing such that whenever two edges e and f cross, e or f crosses at most k edges. Clearly every outer k -planar graph is weakly outer k -planar.

Theorem 46. *Every weakly outer k -planar graph G has 2-tree-partition-width $O(k^3)$.*

Proof. Wood and Telle [80, Prop. 8.5] proved that any weakly outer k -planar graph G has $\text{tw}(G) \leq 3k + 11$. Thus, by Corollary 10, it suffices to show that the singleton partition of G is $(2, 4k + 4)$ -disjointed. Let v_1, \dots, v_n be the cyclic ordering of $V(G)$ given by the drawing. Addition is taken modulo n in this proof. If $n \leq 2$, then the claim holds trivially, so assume that $n \geq 3$. We may assume that $v_i v_{i+1} \in E(G)$ for all $i \in \{1, \dots, n\}$.

Consider distinct $v_i, v_j \in V(G)$. If $j \neq i + 1$, then let v'_r be the clockwise-closest vertex to v_i that is adjacent to v_j and let v'_s be the anticlockwise-closest vertex to v_i that is adjacent to v_j . If $v'_r v_j$ is involved in more than k crossings, then let v_r be the anticlockwise-closest vertex to v_j that is incident to an edge e_r that crosses $v'_r v_j$. Otherwise $v'_r v_j$ is involved in at most k crossings so let $v_r := v'_r$ and $e_r := v'_r v_j$. Similarly, if $v'_s v_j$ is involved in more than k crossings, then let v_s be the clockwise-closest vertex to v_j that is incident to an edge e_s that crosses $v'_s v_j$. Otherwise let $v_s := v'_s$ and $e_s := v'_s v_j$. Let R be the set of vertices that are incident to edges that cross e_r and let S be the set of vertices incident to edges that cross e_s . Let $Q := (R \cup S \cup \{v_r, v_s, v'_r, v'_s\}) \setminus \{v_i, v_j\}$. The setup so far is illustrated in Figure 5, with vertices in Q circled. We have $|Q| \leq 4k + 4$, since e_r and e_s are involved in at most k crossings.

Let $a \in N(v_i) \setminus (Q \cup \{a, b\})$, $b \in N(v_j) \setminus (Q \cup \{a, b\})$, and P be an (a, b) -path in $G - \{v_i, v_j\}$. Since G is weakly outer k -planar, if P does not contain v_r or v_s as an internal vertex, then it contains an edge that crosses e_r or e_s . Thus $V(P) \cap Q \neq \emptyset$ and hence Q separates $N(v_i)$ from $N(v_j)$ in $G - \{v_i, v_j\}$. As such, G is $(2, 4k + 4)$ -disjointed. \square

Theorem 46 implies the next result, where the lower bound holds since $G_{2,\ell}$ from Lemma 12 is outerplanar.

Theorem 47. *For every fixed $k \in \mathbb{N}$, the underlying treewidth of the class of weakly outer k -planar graphs equals 2, with constant treewidth-binding function.*

A graph D is a *planarisation* of a graph G if G has a drawing in the plane and D is the graph obtained by replacing each crossing point with a dummy vertex. Observe that G is a topological minor of $D \boxtimes K_2$. As such, we have the following consequence of Observation 1, Lemma 27, and Theorem 21.

Corollary 48. *For every graph G , if D is a planarisation of G where $\text{tw}(D) < k$ then G has 3-tree-partition-width $O(k^9)$.*

As a special case of Corollary 48, consider a drawing of a graph G with at most k crossings per edge. If G has radius r , then the planarisation of G has radius at most $r(k+1)$ and thus has treewidth at most $3r(k+1)$ [64, (2.7)]. Thus we have the following:

Corollary 49. *If G is a k -planar graph with radius r , then G has 3-tree-partition-width $O((rk)^9)$.*

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