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# EVERY HAUSDORFF COMPACTIFICATION OF A LOCALLY COMPACT SEPARABLE SPACE IS A GA COMPACTIFICATION

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**1. Introduction.** In [**4**], De Groot and Aarts constructed Hausdorff compactifications of topological spaces to obtain a new intrinsic characterization of complete regularity. These compactifications were called *GA compactifications* in [**5**] and [**7**]. A characterization of complete regularity was earlier given by Frink [**3**], by means of Wallman compactifications, a method which led to the intriguing problem of whether every Hausdorff compactification is a Wallman compactification. An analogous question was posed by A. B. Paalman de Miranda; can every Hausdorff compactification of a Tychonoff space be obtained as a *GA* compactification? We will give a partial answer to this question, suggesting that the answer will be yes. This paper is organized as follows: in the second section we will recall the definition of *GA* compactifications and we will characterize the class of *GA* compactifications of a given topological space. Using an analogous characterization of Wallman compactifications, given by Steiner [**11**], it then follows that every Wallman compactification is a *GA* compactification. In the third section we will show that every Hausdorff compactification of a locally compact separable space is a *GA* compactification. In fact we have a more general result from which this is a corollary.

**2. GA compactifications.** Let $X$ be a topological space and let $\zeta$ be a subbase for the closed subsets of $X$. Then $\zeta$ is defined to be a

(i) $T_1$-*subbase* if for each $x \in X$ and $S \in \zeta$ such that $x \notin S$ there exists a $T \in \zeta$ with $x \in T$ and $S \cap T = \emptyset$;

(ii) *weakly normal subbase* if for each $S, T \in \zeta$ with $S \cap T = \emptyset$ there exists a finite cover $\mathscr{U}$ of $X$ by elements of $\zeta$ such that each element of $\mathscr{U}$ meets at most one of $S$ and $T$;

(iii) *normal subbase* if for each $S_0, T_0 \in \zeta$ with $S_0 \cap T_0 = \emptyset$ there exist $S_1, T_1 \in \zeta$ with $S_1 \cup T_1 = X$ and $S_1 \cap T_0 = \emptyset = S_0 \cap T_1$.

Note that a normal subbase is a weakly normal subbase.

A subsystem $\mathscr{U} \subset \zeta$ is called a *linked system* if every two of its members meet. A *maximal linked system* is a linked system not properly contained in

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another linked system. By Zorn's lemma, every linked system is contained in at least one maximal linked system. Define

$$\lambda_\zeta(X) = \{\mathscr{U} \subset \zeta | \mathscr{U} \text{ is a maximal linked system}\}.$$

If $A \subset X$ then define $A^+ = \{\mathscr{U} \in \lambda_\zeta(X) | \exists\, M \in \mathscr{U} : M \subset A\}$. The set $\lambda_\zeta(X)$ can be topologized by taking $\zeta^+ = \{S^+ | S \in \zeta\}$ as a closed subbase. With this subbase $\lambda_\zeta(X)$ is called *the superextension of* $X$ *relative to* $\zeta$. Superextensions have been studied in [**6**; **9** and **12**]. It is easy to see that every superextension is compact (in fact, $\lambda_\zeta(X)$ is *supercompact*, i.e. it admits an open subbase $\mathscr{U}$ such that each covering of $\lambda_\zeta(X)$ by elements of $\mathscr{U}$ contains a subcover of two elements of $\mathscr{U}$). The subbase $\zeta^+$ has the property that each linked subsystem of $\mathscr{U} \subset \zeta^+$ has a nonempty intersection, as can easily be seen. A closed subbase with this property is called *binary*. If $\zeta$ is a $T_1$-subbase, then $X$ can be embedded in $\lambda_\zeta(X)$ by the natural embedding $i$ defined by $i(x) = \{S \in \zeta | x \in S\}$. We will always identify $X$ and $i(X)$. The *GA* compactification $\beta_\zeta(X)$ relative to $\zeta$ now is the closure of $X$ in $\lambda_\zeta(X)$. As a matter of fact De Groot and Aarts obtained $\beta_\zeta(X)$ in another way; however in [**6**] it was shown that the compactification $\beta_\zeta(X)$ as defined above is equivalent to the compactification they defined in [**4**].

LEMMA 2.1: *Let $\zeta$ be a closed $T_1$-subbase for the topological space $X$. Then the following properties are equivalent:*

(i) *$\beta_\zeta(X)$ is Hausdorff.*

(ii) *$\zeta$ is weakly normal.*

(iii) *$\{S^+ \cap \beta_\zeta(X) | S \in \zeta\}$ is weakly normal.*

*Proof.* (i) $\Rightarrow$ (ii). Assume that $\beta_\zeta(X)$ is Hausdorff and take $S_0, S_1 \in \zeta$ such that $S_0 \cap S_1 = \emptyset$. As $S_0 \cap S_1 = \emptyset$ it follows that $(S_0^+ \cap \beta_\zeta(X)) \cap (S_1^+ \cap \beta_\zeta(X)) = \emptyset$ and hence there exist open $U_0, U_1 \subset \beta_\zeta(X)$ such that $S_i \subset U_i (i = 0, 1)$ and $U_0 \cap U_1 = \emptyset$. Then $\beta_\zeta(X) \backslash U_i$ is closed in $\beta_\zeta(X)$ and as $\beta_\zeta(X)$ is closed in $\lambda_\zeta(X)$ it is closed in $\lambda_\zeta(X)$ too $(i = 0, 1)$. As $\zeta^+$ is a closed subbase for the compact space $\lambda_\zeta(X)$, there exist $T_{ij} \in \zeta$ and $T_{ij}' \in \zeta$ $(i, j \in \{1, 2, \ldots, n\})$ such that

$$\text{(i)}\ \beta_\zeta(X)\backslash U_0 \subset \bigcup_{i=1}^{n} \bigcap_{j=1}^{n} T_{ij}^+; \quad \beta_\zeta(X)\backslash U_1 \subset \bigcup_{i=1}^{n} \bigcap_{j=1}^{n} T_{ij}'^+.$$

$$\text{(ii)}\ \bigcup_{i=1}^{n} \bigcap_{j=1}^{n} T_{ij}^+ \cap S_0^+ = \emptyset = \bigcup_{i=1}^{n} \bigcap_{j=1}^{n} T_{ij}'^+ \cap S_1^+.$$

Notice that a finite intersection of finite unions of subbase elements also can be represented as a finite union of finite intersections of subbase elements. As $\zeta$ is binary, for each $i \in \{1, 2, \ldots, n\}$ there exists a $j_0 \in \{1, 2, \ldots, n\}$ such that $T_{ij_0}^+ \cap S_0^+ = \emptyset$ and a $j_1 \in \{1, 2, \ldots, n\}$ such that $T_{ij_1}^+ \cap S_1^+ = \emptyset$, and therefore we may assume that there exist $T_i \in \zeta$ $(i \in \{1, 2, \ldots, n\})$ and

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$T_i' \in \zeta$ $(i \in \{1, 2, \ldots, n\})$ such that

(i) $\beta_\zeta(X)\backslash U_0 \subset \bigcup_{i=1}^{n} T_i^+$; $\beta_\zeta(X)\backslash U_1 \subset \bigcup_{i=1}^{n} T_i'^+$.

(ii) $\bigcup_{i=1}^{n} T_i^+ \cap S_0^+ = \emptyset = \bigcup_{i=1}^{n} T_i'^+ \cap S_1^+$.

Then $X \subset \beta_\zeta(X) \subset \bigcup_{i=1}^n T_i^+ \cup \bigcup_{i=1}^n T_i'^+$ and consequently

$$X = \bigcup_{i=1}^n (T_i^+ \cap X) \cup \bigcup_{i=1}^n (T_i'^+ \cap X) = \bigcup_{i=1}^n T_i \cup \bigcup_{i=1}^n T_i'.$$

Moreover it is obvious that $\bigcup_{i=1}^n T_i \cap S_0 = \emptyset = \bigcup_{i=1}^n T_i' \cap S_1$, which implies that $\zeta$ is weakly normal.

(ii) $\Rightarrow$ (i). See De Groot and Aarts [**4**, Lemma 9] or Verbeek [**12**, Theorem II. 2.3].

(ii) $\Rightarrow$ (iii). Choose $S_0^+, S_1^+ \in \zeta^+$ such that $S_0^+ \cap S_1^+ = \emptyset$. As $S_0 \cap S_1 = \emptyset$, there exist $T_i \in \zeta$ and $T_i' \in \zeta$ $(i \in \{1, 2, \ldots, n\})$ such that

(i) $S_0 \cap \bigcup_{i=1}^{n} T_i' = \emptyset = S_1 \cap \bigcup_{i=1}^{n} T_i$.

(ii) $\bigcup_{i=1}^{n} T_i' \cup \bigcup_{i=1}^{n} T_i = X$.

It then follows that $S_0^+ \cap \bigcup_{i=1}^n T_i'^+ = \emptyset = S_1^+ \cap \bigcup_{i=1}^n T_i^+$ and that $X \subset \beta_\zeta(X) \subset \bigcup_{i=1}^n T_i' \cup \bigcup_{i=1}^n T_i^+$; therefore

$$\beta_\zeta(X) = \bigcup_{i=1}^n (T_i'^+ \cap \beta_\zeta(X)) \cup \bigcup_{i=1}^n (T_i^+ \cap \beta_\zeta(X)).$$

(iii) $\Rightarrow$ (ii). This can be proved in a similar way.

THEOREM 2.2: *A Hausdorff compactification $\alpha X$ of $X$ is a GA compactification if and only if $\alpha X$ possesses a weakly normal closed $T_1$-subbase $\mathscr{T}$ such that for all $T_0, T_1 \in \mathscr{T}$ with $T_0 \cap T_1 \neq \emptyset$ we have $T_0 \cap T_1 \cap X \neq \emptyset$.*

*Proof.* ($\Rightarrow$) This follows from Lemma 2.1 and from the trivial observation that if $\alpha X = \beta_\zeta(X)$, then $\{S^+ \cap \beta_\zeta(X) | S \in \zeta\}$ is a closed $T_1$-subbase for $\beta_\zeta(X)$.

($\Leftarrow$) Suppose that $\alpha X$ possesses a weakly normal closed $T_1$-subbase $\mathscr{T}$ such that for all $T_0, T_1 \in \mathscr{T}$ with $T_0 \cap T_1 \neq \emptyset$ we have $T_0 \cap T_1 \cap X \neq \emptyset$. Define $\mathscr{T}|X = \{T \cap X | T \in \mathscr{T}\}$. We will show that $\alpha X = \beta_{\mathscr{T}|X}(X)$.

(A) Let $x \in \alpha X$ and define $\mathscr{U}(x) = \{T \cap X | T \in \mathscr{T}$ and $x \in T\}$. We claim that $\mathscr{U}(x)$ is a maximal linked system (in $\mathscr{T}|X$). That $\mathscr{U}(x)$ is a linked system is evident. Suppose that $\mathscr{U}(x)$ is not maximal linked. Then there exists a $T \in \mathscr{T}$ such that $\mathscr{U}(x) \cup \{(T \cap X)\}$ is linked and $T \cap X \notin \mathscr{U}(x)$. Then $x \notin T$ and since $\mathscr{T}$ is a closed $T_1$-subbase, there exists a $T_0 \in \mathscr{T}$ such that $x \in T_0$ and $T_0 \cap T = \emptyset$. Then $T_0 \cap X \in \mathscr{U}(x)$ and $(T_0 \cap X) \cap (T \cap X) = \emptyset$, which is a contradiction.

Define a map $f : \alpha X \rightarrow \lambda_{\mathscr{T}|X}(X)$ by $f(x) = \mathscr{U}(x)$. We show the following.

(B) $f$ is continuous. Choose $T \cap X \in \mathscr{T}|X$. Then $x \in f^{-1}((T \cap X)^+) \Leftrightarrow f(x) \in (T \cap X)^+ \Leftrightarrow T \cap X \in f(x) = \mathscr{U}(x) \Leftrightarrow x \in T$, which shows that $f$ is continuous.

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(C) $f$ is one to one. Choose $x, y \in \alpha X$ such that $x \neq y$. Then there exist $T_0, T_1 \in \mathcal{T}$ such that $x \in T_0$ and $y \in T_1$ and $T_0 \cap T_1 = \emptyset$. Now $T_0 \cap X \in \mathcal{U}(x)$ and $T_1 \cap X \in \mathcal{U}(y)$ and therefore, as $(T_0 \cap X) \cap (T_1 \cap X) = \emptyset$ we have $f(x) = \mathcal{U}(x) \neq \mathcal{U}(y) = f(y)$.

(D) $f$ is the identity on $X$. Choose $x \in X$. Then $f(x) = \mathcal{U}(x) = \{T \cap X | T \in \mathcal{T} \text{ and } x \in T\} = x$.

(E) $f(\alpha X) = \beta_{\mathcal{T}|X}(X)$. The weak normality of $\mathcal{T}$, together with the property that for all $T_0, T_1 \in \mathcal{T}$ with $T_0 \cap T_1 \neq \emptyset$ we have $T_0 \cap T_1 \cap X \neq \emptyset$, imply that $\mathcal{T}|X$ also is weakly normal and consequently $\beta_{\mathcal{T}|X}(X)$ is Hausdorff (Lemma 2.1). Therefore $f(\alpha X) = \beta_{\mathcal{T}|X}(X)$, since $X$ is dense in $\alpha X$.

It now follows that $f$ is a homeomorphism, which on $X$ is the identity. This completes the proof.

Using an analogous characterization of Wallman compactifications, given by Steiner [**11**], we immediately obtain the following remarkable corollary:

COROLLARY 2.3. *If a Hausdorff compactification $\alpha X$ of $X$ is a Wallman compactification, then it is a GA compactification.*

Many compactifications are Wallman compactifications; therefore it follows that many compactifications are *GA* compactifications.

In the following section we will use Theorem 2.2 to obtain our main result.

## 3. Every Hausdorff compactification of a locally compact separable space is a *GA* compactification.

THEOREM 3.1: *Every Hausdorff compactification $\alpha X$ of a locally compact space $X$ such that* weight $(\alpha X) \leqq 2^{\aleph_0}$ *is a GA compactification.*

*Proof*. Let $\mathcal{B}$ be an open basis for $\alpha X$ with card $(\mathcal{B}) \leqq 2^{\aleph_0}$. Without loss of generality we may assume that $\mathcal{B}$ is closed under finite intersections and finite unions. Define

$$\mathcal{C} = \{(\mathrm{cl}_{\alpha X}(B_0), \mathrm{cl}_{\alpha X}(B_1)) | B_i \in \mathcal{B} \ (i = 0, 1) \text{ and } \mathrm{cl}_{\alpha X}(B_0) \cap \mathrm{cl}_{\alpha X}(B_1) = \emptyset\}.$$

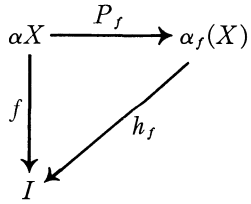
For each pair $(\mathrm{cl}_{\alpha X}(B_0), \mathrm{cl}_{\alpha X}(B_1)) \in \mathcal{C}$ choose an $f \in C(\alpha X, I)$ such that $f(\mathrm{cl}_{\alpha X}(B_0)) = 0$ and $f(\mathrm{cl}_{\alpha X}(B_1)) = 1$. Let $\mathcal{F}$ denote the set of mappings obtained in this way, and assume that $\mathcal{F}$ is most economically well-ordered (denote the order by $<$). Note that card $(\mathcal{F}) \leqq 2^{\aleph_0}$. By transfinite induction we will construct for each $f \in \mathcal{F}$ a $\delta_f \in (0, 1)$ such that

$$(*) \quad \mathrm{cl}_{\alpha X}(f^{-1}[0, \delta_f)) \cap \mathrm{cl}_{\alpha X}(g^{-1}[0, \delta_f)) \neq \emptyset \,) \Rightarrow \mathrm{cl}_{\alpha X}(f^{-1}[0, \delta_f)) \cap \mathrm{cl}_{\alpha X}(g^{-1}[0, \delta_g)) \cap X \neq \emptyset\cdot$$

for all $g < f$ $(g \in \mathcal{F})$.

Let $f \in \mathcal{F}$ and define $\mathcal{M} = \{f^{-1}(p) \backslash X | p \in f(\alpha X \backslash X)\} \cup \{\{x\} | x \in X\}$. Then $\mathcal{M}$ is an upper semicontinuous decomposition of $\alpha X$ such that the decomposition space $\alpha_f(X)$ is a Hausdorff compactification of $X$ with $f(\alpha X \backslash X)$ as remainder. Let $P_f$ denote the projection map. Then $P_f$ is the identity on $X$.

Moreover define $h_f : \alpha_f(X) \to I$ by $h_f = f \circ P_f^{-1}$. Then $h_f$ is continuous and the diagram



commutes. Notice that $h_f$ restricted to $\alpha_f(X)\setminus X$ is a homeomorphism (we will identify $\alpha_f(X)\setminus X$ and $h_f(\alpha_f(X)\setminus X)$).

Let $f_0$ be the first element of $\mathscr{F}$ and define $\delta_f = \frac{1}{2}$. Assume that all $\delta_g$ have been constructed for all $g < f (g \in \mathscr{F})$ such that (*) is satisfied.

(A) Let $U$ be an open subset of $\alpha_f(X)$ and let

$$A = \{\delta \in (0, 1) | \overline{h_f^{-1}[0, \delta)} \cap \bar{U} \cap (\alpha_f(X)\setminus X) \neq \overline{h_f^{-1}[0, \delta) \cap U} \cap (\alpha_f(X)\setminus X)\}.$$

(if $B \subset \alpha_f(X)$, then $\bar{B}$ denotes the closure of $B$ in $\alpha_f(X)$). Then $A$ is a subset of $f(\alpha X\setminus X)$ while moreover $A$ is countable. Indeed, choose $\delta \in I\setminus f(\alpha X\setminus X)$. Then

$$\overline{h_f^{-1}[0, \delta)} \cap \bar{U} \cap (\alpha_f(X)\setminus X) \subset h_f^{-1}[0, \delta] \cap \bar{U} \cap (\alpha_f(X)\setminus X) = h_f^{-1}[0, \delta) \cap \bar{U} \cap (\alpha_f(X)\setminus X) \subset \overline{h_f^{-1}[0, \delta) \cap U} \cap (\alpha_f(X)\setminus X),$$

since $f^{-1}(\delta) \cap (\alpha X\setminus X) = \emptyset$.

To show that $A$ is countable, assume that $A$ were uncountable. Then as $A$ is an uncountable subset of the real numbers it must contain a condensation point. What is more, it is obvious that there even exists a condensation point $\delta_0$ which is a limit point from below. Now, let $O$ be an open neighborhood of $\delta_0$ in $\alpha_f(X)$. Then there exists a $\delta_1 \in O \cap A$ such that $\delta_1 < \delta_0$ and consequently

$$\delta_1 \in h_f^{-1}[0, \delta_0) \cap O \cap \bar{U} \subset O \cap \overline{h_f^{-1}[0, \delta_0) \cap U}.$$

Therefore, it follows that $\delta_0 \in \overline{h_f^{-1}[0, \delta_0) \cap U} \cap (\alpha_f(X)\setminus X)$, which is a contradiction.

(B) There exists a $\delta_0 \in (0, 1)$ such that

$$\overline{g^{-1}[0, \delta_g) \cap X} \cap \overline{h_f^{-1}[0, \delta_0)} \cap (\alpha_f(X)\setminus X) = \overline{g^{-1}[0, \delta_g) \cap h_f^{-1}[0, \delta_0) \cap X} \cap (\alpha_f(X)\setminus X)$$

for all $g < f$ $(g \in \mathscr{F})$. As card $\{g \in \mathscr{F} | g < f\} < 2^{\aleph_0}$, since the well-ordering is most economical, and as $X$ (being locally compact) is open in $\alpha_f(X)$, we conclude from (A) that

$$\operatorname{card}\left(\bigcup_{g<f}\left\{\delta \in (0, 1) | \overline{g^{-1}[0, \delta_g) \cap X} \cap \overline{h_f^{-1}[0, \delta)} \cap (\alpha_f(X)\setminus X) \neq \overline{g^{-1}[0, \delta_g) \cap h_f^{-1}[0, \delta) \cap X} \cap (\alpha_f(X)\setminus X)\right\}\right) < \aleph_0 \cdot 2^{\aleph_0} = 2^{\aleph_0},$$

and therefore such a choice for $\delta_0$ is possible.

(C) Define $\delta_f = \delta_0$. We claim that (*) is satisfied. Take $g \in \mathscr{F}$ such that $g < f$ and assume that $\text{cl}_{\alpha X}(f^{-1}[0, \delta_f)) \cap \text{cl}_{\alpha X}(g^{-1}[0, \delta_g)) \neq \emptyset$. Then $P_f(\text{cl}_{\alpha X}(f^{-1}[0, \delta_f))) \cap P_f(\text{cl}_{\alpha X}(g^{-1}[0, \delta_g))) \neq \emptyset$ and consequently

$$\overline{P_f f^{-1}[0, \delta_f)} \cap \overline{g^{-1}[0, \delta_g) \cap X} \neq \emptyset,$$

since it is easily seen that $P_f(\text{cl}_{\alpha X}(U)) = \overline{U \cap X}$ for each open $U \subset \alpha X$. Therefore

$$\overline{h_f^{-1}[0, \delta_f)} \cap \overline{g^{-1}[0, \delta_g) \cap X} \neq \emptyset.$$

Now assume that

$$\overline{h_f^{-1}[0, \delta_f)} \cap \overline{g^{-1}[0, \delta_g) \cap X} \cap X = \emptyset.$$

It then follows that

$$\begin{aligned}\overline{h_f^{-1}[0, \delta_f)} \cap \overline{g^{-1}[0, \delta_g) \cap X} \cap (\alpha_f(X)\backslash X) \\ = \overline{h_f^{-1}[0, \delta_f) \cap g^{-1}[0, \delta_g) \cap X} \cap (\alpha_f(X)\backslash X) \neq \emptyset, \quad \text{(B)}\end{aligned}$$

which implies that $h_f^{-1}[0, \delta_f) \cap g^{-1}[0, \delta_g) \cap X \neq \emptyset$, which is a contradiction. Therefore $\overline{h_f^{-1}[0, \delta_f)} \cap \overline{g^{-1}[0, \delta_g) \cap X} \cap X \neq \emptyset$. Now choose

$$x \in \overline{h_f^{-1}[0, \delta_f)} \cap \overline{g^{-1}[0, \delta_g) \cap X} \cap X;$$

then $x \in \text{cl}_{\alpha X}(f^{-1}[0, \delta_f)) \cap \text{cl}_{\alpha X}(g^{-1}[0, \delta_g))$. Thus (*) holds indeed for $\delta_f$; this completes the construction of the $\delta_f (f \in \mathscr{F})$.

(D) Define $\mathscr{A} = \{\text{cl}_{\alpha X}(f^{-1}[0, \delta_f)) | f \in \mathscr{F}\}$. It is easy to see that $\mathscr{A}$ is a closed base for $\alpha X$. Moreover it is clear that for all $A_0, A_1 \in \mathscr{A}$ with $A_0 \cap A_1 \neq \emptyset$ it follows that $A_0 \cap A_1 \cap X \neq \emptyset$; hence, by Theorem 2.2, it suffices to show that $\mathscr{A}$ is weakly normal and $T_1$ to prove the theorem.

Take $A_0, A_1 \in \mathscr{A}$ such that $A_0 \cap A_1 = \emptyset$. Then, using the fact that $\alpha X$ is compact Hausdorff, there exist closed sets $G_0$ and $G_1$ such that $A_0 \cap G_1 = \emptyset = G_0 \cap A_1$ and $G_0 \cup G_1 = \alpha X$. Now, since $\mathscr{B}$ is closed under finite intersections and finite unions there exist $B_0, B_1 \in \mathscr{B}$ such that $A_0 \subset B_0 \subset \text{cl}_{\alpha X}(B_0)$ and $G_1 \subset B_1 \subset \text{cl}_{\alpha X}(B_1)$ and $\text{cl}_{\alpha X}(B_0) \cap \text{cl}_{\alpha X}(B_0) = \emptyset$. Now choose $A_1' \in \mathscr{A}$ such that $\text{cl}_{\alpha X}(B_1) \subset A_1'$ and $A_1' \cap \text{cl}_{\alpha X}(B_0) = \emptyset$. In the same way we can find an $A_0' \in \mathscr{A}$ such that $G_0 \subset A_0'$ and $A_0' \cap A_1 = \emptyset$. Therefore $A \cap A_1' = \emptyset = A' \cap A_1$ and $A_0' \cup A_1' = \alpha X$. Therefore $\mathscr{A}$ is a normal closed base. That $\mathscr{A}$ is also $T_1$ can be proved in the same way.

COROLLARY 3.2. *Every Hausdorff compactification of a locally compact separable space is a GA compactification.*

*Proof.* Let $\alpha X$ be a Hausdorff compactification of a locally compact separable space $X$. Then $\alpha X$ is also separable and consequently weight $(\alpha X) \leqq 2^{\aleph_0}$ (Juhász [**8**]).

*Remark.* In the proof of Theorem 3.1 we, in fact, implicitly made use of a concept called *strongly* $\aleph_1$ *compact*, which is a very useful concept in compacti-

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fication theory. It was introduced in the theory of Wallman compactifications by E. S. Berney [**2**]. Parts (A) and (B) of the proof of 3.1 are modifications of a technique also due to Berney [**2**].

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